STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 2, June 2009

# THE GLOBAL BEHAVIOR OF THE DIFFERENCE EQUATION

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**Abstract**. In this paper, we investigate the global stability and the periodic nature of the positive solutions of the difference equation

$$y_{n+1} = \alpha + \frac{y_{n-1}}{y_n} - \frac{y_n}{y_{n-1}}, n = 0, 1, 2, ...,$$

where $\alpha > 0$ and the initial conditions $y_0, y_{-1}$ are arbitrary positive real numbers.

## 1. **Introduction**

Consider the difference equation

$$y_{n+1} = \alpha + \frac{y_{n-1}}{y_n} - \frac{y_n}{y_{n-1}}, \ n = 0, 1, 2, ... \tag{1.1}$$

where $\alpha > 0$ and the initial conditions $y_0, y_{-1}$ are arbitrary positive real numbers.We investigate the asymptotic stability and the periodic character of the solutions of Eq. (1.1).

We prove that the positive equilibrium point of Eq. (1.1) is local asymptotic stable or a saddle point under specified conditions of the parameter and show that the solution of the subtraction of two difference equations in [1] and [3], which solutions are globally asymptotically stable, are also asymptotically stable.

The global asymptotic stability, the boundedness character and the periodic nature of the positive solutions of the following difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}, \ n = 0, 1, 2, ... \tag{1.2}$$

Received by the editors: 30.10.2006.

2000 *Mathematics Subject Classification.* 39A10, 39A11.

*Key words and phrases.* Difference equations, period two solutions, global stability.

was investigated in [1], where $\alpha \in [0,\infty)$ and the initial conditions $x_{-1}$ and $x_0$ are arbitrary positive real numbers. H. M. El- Owaidy et al. [2] studied the global stability and the periodic character of positive solutions of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-k}}{x_n},\ n = 0, 1, 2, ... \tag{1.3}$$

where $\alpha \in [1,\infty)$ , $k \in \{1, 2, ...\}$ and the initial conditions $x_{-k}, ..., x_0, x_{-1}$ are arbitrary positive real numbers.

R. M. Abu-Saris and R. De Vault find conditions for the global asymptotic stability of the unique positive equilibrium

$$\bar{y} = A + 1$$

of the equation

$$x_{n+1} = A + \frac{y_n}{y_{n-k}},\ n = 0, 1, 2, ... \tag{1.4}$$

where A,$y_{-k}, ..., y_0, y_{-1} \in (0,\infty)$ and $k \in \{2, 3, ...\}$ [3].

Here, we recall some definitions and results which will be useful in the sequel.

Let $I \subset R$ and let $f : I \times I \to I$ be a continuous function. Consider the difference equation

$$y_{n+1} = f(y_n, y_{n-1}), n = 0, 1, 2, ... \tag{1.5}$$

where the initial conditions $y_0, y_{-1} \in I$. We say that $\bar{y}$ is an equilibrium of Eq. (1.5) if

$$y_{n+1} = f(\bar{y}, \bar{y}),\ n = 0, 1, 2, ... \tag{1.6}$$

Let

$$s = \frac{\partial f}{\partial u}(\bar{y}, \bar{y})\ \ and\ \ t = \frac{\partial f}{\partial v}(\bar{y}, \bar{y})$$

denote the partial derivatives of $f(u, v)$ evaluated at an equilibrium $\bar{y}$ of Eq. (1.5). Then the equation

$$x_{n+1} = sx_n + tx_{n-1}$$

is called the linearized equation associated with Eq. (1.5) about the equilibrium point ȳ [4].

The sequence $\{y_n\}$ is said to be periodic with period p if

$$y_{n+p} = y_n$$

for $n = 0, 1, ...$ [5].

**Theorem 1.1.** [4] **(Linearized Stability)**

$$x_{n+1} = sx_n + tx_{n-1} \tag{1.7}$$

is the linearized equation associated with the difference equation

$$y_{n+1} = f(y_n, y_{n-1}), n = 0, 1, 2, ... \tag{1.8}$$

about the equilibrium point $\bar{y}$. The characteristic equation associated with (1.7) is

$$\lambda^2 - s\lambda - t = 0. \tag{1.9}$$

**(i)** If both roots of the quadratic equation (1.9) lie in the unit disk $|\lambda| < 1$, then the equilibrium $\bar{y}$ of Eq. (1.8) is locally asymptotically stable.

**(ii)** If at least one of the roots of Eq. (1.9) has absolute value greater than one, then the equilibrium of Eq. (1.8) is unstable.

**(iii)** A necessary and sufficient condition for both roots of Eq. (1.9) to lie in the open unit disk $|\lambda| < 1$, is

$$|s| < 1 - t < 2.$$

In this case the locally asymptotically stable equilibrium point $\bar{y}$ is also called a sink.

**(iv)** A necessary and sufficient condition for both roots of Eq. (1.9) to have absolute value greater than one is

$$|t| > 1 \ and \ |s| < |1 - t| .$$

In this case $\bar{y}$ is called a repeller.

**(v)** A necessary and sufficient condition for one root of Eq. (1.9) to have absolute value greater than one and for the other to have absolute value less than one is

$$s^2 + 4t > 0 \ \ and \ \ |s| > |1 - t| .$$

In this case the unstable equilibrium point is called a saddle point.

**Theorem 1.2** [6] Assume that p,q$\in R$ and $k \in \{0, 1, ...\}$. Then

$$|p| + |q| \ < 1$$

is a sufficient condition for asymptotic stability of the difference equation

$$x_{n+1} - px_n + qx_{n-k} = 0. \tag{1.10}$$

Suppose in addition that one of the following two cases holds:

**(i)** $k$ odd and $q < 0$

**(ii)** $k$ even and $pq < 0$.

Then (1.10) is also a necessary condition for asymptotic stability of Eq. (1.10).

**Theorem 1.3.** [7] Consider the difference equation

$$y_{n+1} = f(y_n, y_{n-k}), n = 0, 1, 2, ... \tag{1.12}$$

where $k \in \{1, 2, ...\}$. Let I=[a,b] be some interval of real numbers, and assume that f:[a,b]x[a,b]$\rightarrow [a, b]$ is a continuous function satisfying the following properties:

**(i)** f(u,v) is non-increasing in each arguments.

**(ii)** If (m,M)$\in [a, b]x[a, b]$ is a solution of the system

$$M = f(m, m), m = f(M, M) \tag{1.13}$$

then m=M. From this, Eq. (1.12) has a unique positive equilibrium point and every solution of Eq. (1.12) converges to $\bar{y}$.

## 2. **Linearized stability and period two solutions**

In this section, we consider Eq. (1.1) and show that unique positive equilibrium point $\bar{y}$ =$\alpha$ of Eq. (1.1) is asymptotically stable with basin which depends on certain conditions posed on the coefficient.

The linearized equation associated with Eq. (1.1) about the equilibrium $\bar{y}$ is

$$x_{n+1} + \frac{2}{\alpha}x_n - \frac{2}{\alpha}x_{n-1} = 0. \tag{2.1}$$

Its characteristic equation is

$$\lambda^2 + \frac{2}{\alpha}\lambda - \frac{2}{\alpha} = 0. \tag{2.2}$$

By Theorem 1.1. and Theorem 1.2. we have the following results.

**Theorem 2.1.** **(i)** The equilibrium point $\bar{y}$ of Eq. (1.1) is locally asymptotically stable iff $\alpha > 4$.

**(ii)** The equilibrium point $\bar{y}$ of Eq. (1.1) is unstable ( and in fact is a saddle point ) if $0 < \alpha < 4$.

**Proof.** (i) The inequality (1.10) can be written as

$$\left|\frac{2}{\alpha}\right| + \left|\frac{-2}{\alpha}\right| < 1 . \tag{2.3}$$

This inequality holds if $\alpha > 4$. By using Theorem 1.2., we can also see that q=$\frac{-2}{\alpha} < 0$. These results give us necessary and sufficient conditions for the asymptotic stability of Eq. (2.1) .

**(ii)** From Theorem 1.1./(v) we have,

$$\left(\frac{-2}{\alpha}\right)^2 + 4\left(\frac{2}{\alpha}\right) > 0 \;\; and \;\; \left|\frac{-2}{\alpha}\right| > \left|1 - \frac{2}{\alpha}\right|.$$

Easy computations give

$$\left(\frac{-2}{\alpha}\right)^2 + 4\left(\frac{2}{\alpha}\right) = \frac{4}{\alpha^2} + \frac{8}{\alpha} > 0$$

and

$$\left|\frac{-2}{\alpha}\right| > \left|1 - \frac{2}{\alpha}\right|.$$

Then we have the inequality

$$2 > |\alpha - 2| .$$

This implies that by Theorem 1.1./(v), the equilibrium point is unstable (and is a saddle point).

**Theorem 2.2.** Suppose that $\{y_n\}_{n=-1}^{\infty} \neq 2$ is a solution of Eq. (1.1). The following statements are true.

**(i)** If $0 < \alpha \leq 4$, then Eq. (1.1) has no real period two solutions. Suppose k is odd.

**(ii)** If $\alpha > 4$, then Eq. (1.1) has real period two solutions.

**Proof.** Let

$$..., \phi, \psi, \phi, \psi, ...$$

be a period-2 solution of Eq. (1.1). Then,

$$\phi = \alpha + \frac{\phi}{\psi} - \frac{\psi}{\phi} \tag{2.4}$$

$$\psi = \alpha + \frac{\psi}{\phi} - \frac{\phi}{\psi}. \tag{2.5}$$

Subtracting above two statements, we get

$$\psi = \frac{2\phi}{\phi - 2}. \tag{2.6}$$

From (2.6), we have

$$\phi^2 - 2\alpha\phi + 4\alpha = 0. \tag{2.7}$$

We consider (2.7) under two cases, where $\Delta$ indicates the discriminant of (2.7).

**(i)** Let $\Delta = 0$. Under this condition we have $\alpha = 0$ and $\alpha = 4$.If Eq. (1.1) has period 2 solutions then it must be $\Delta \neq 0$. This implies that if $\alpha \in (0, 4]$, then Eq. (1.1) has no period 2 solutions.

**(ii)** Let $\Delta > 0$. In this case we have $\alpha > 4$. While $\alpha > 4$, Eq. (1.1) has period 2 solutions. These solutions are

$$\phi_1 = \alpha + \sqrt{\alpha(\alpha - 4)} \quad \text{and} \quad \phi_2 = \alpha - \sqrt{\alpha(\alpha - 4)}$$

and they must be of the form

$$..., \alpha - \sqrt{\alpha(\alpha - 4)}, \alpha + \sqrt{\alpha(\alpha - 4)}, ...$$

**Theorem 2.3.** Suppose $\alpha > 4$. Let be $\{y_n\}_{n=-1}^{\infty} \neq 2$ be a solution of Eq. (1.1). If $\{y_n\}_{n=-1}^{\infty} \neq 2$ is periodic with period-2, then y$_0$ is

$$y_0 = \frac{-(y_{-1} - \alpha) y_{-1} \pm y_{-1}\sqrt{(y_{-1} - \alpha)^2 + 4}}{2}. \tag{2.8}$$

**Proof.** If the solution of Eq. (1.1) is periodic with period- 2, we can write Eq.(1.1) as

$$y_{-1} = \alpha + \frac{y_{-1}}{y_0} - \frac{y_0}{y_{-1}}.$$

Computations give

$$y_0^2 + y_{-1}y_0\left(y_{-1} - \alpha\right) - y_{-1}^2 = 0,$$

and we have $\Delta = y_{-1}^2\left[\left(y_{-1} - \alpha\right)^2 + 4\right] > 0$. So, we obtain

$$y_0 = \frac{-(y_{-1}-\alpha)y_{-1} \pm y_{-1}\sqrt{(y_{-1}-\alpha)^2+4}}{2}.$$

**Theorem 2.4.** Let $\{y_n\}_{n=-1}^{\infty}$ be a solution of Eq. (1.1). Then the following statements are true.

**1.** Let $\alpha = 2\sqrt{L-1}$ and $L > 1$.

**(i)** If $\lim_{n\to\infty} y_{2n} = L$, then $\lim_{n\to\infty} y_{2n+1} = \frac{L}{\sqrt{L-1}}$.

**(ii)** If $\lim_{n\to\infty} y_{2n+1} = L$, then $\lim_{n\to\infty} y_{2n} = \frac{L}{\sqrt{L-1}}$.

**2.** Let $\alpha > 2\sqrt{L-1}$ and $L > 1$.

**(i)** If $\lim_{n\to\infty} y_{2n} = L$, then $\lim_{n\to\infty} y_{2n+1} = \frac{L\left[\alpha \pm \sqrt{\alpha^2 - 4(L-1)}\right]}{2(L-1)}$.

**(ii)** If $\lim_{n\to\infty} y_{2n+1} = L$, then $\lim_{n\to\infty} y_{2n} = \frac{L\left[\alpha \pm \sqrt{\alpha^2 - 4(L-1)}\right]}{2(L-1)}$.

**Proof.** **1.** **(i)** Let $\lim_{n\to\infty} y_{2n} = L$ and $\lim_{n\to\infty} y_{2n+1} = x$. By Eq (1.1) we have

$$x = \alpha + \frac{x}{L} - \frac{L}{x}$$

and so we get

$$\left(\frac{L-1}{L}\right)x^2 - \alpha x + L = 0. \tag{2.9}$$

Since $\Delta = \alpha^2 - 4\left(L-1\right)$, the discriminant is $\Delta = 0$. So, (2.9) has only one root, and that is

$$x = \lim_{n\to\infty} y_{2n+1} = \frac{L}{\sqrt{L-1}}.$$

**(ii)** The proof is similar and will be omitted.

2. **(i)** Let $\lim_{n\to\infty} y_{2n} = L$ and $\lim_{n\to\infty} y_{2n+1} = x$. While $\alpha > 2\sqrt{L-1}$, then from (2.9) we have $\Delta > 0$. So,

$$x = \lim_{n\to\infty} y_{2n+1} = \frac{L\left[\alpha \pm \sqrt{\alpha^2 - 4\left(L-1\right)}\right]}{2\left(L-1\right)}.$$

**(ii)** The proof follows in the same way.

## 3. **Analysis of the semi-cycles of eq.(1.1)**

In this section, we give some results about the semi-cycles of Eq. (1.1).

Let $\{y_n\}_{n=-1}^{\infty}$ be a positive solution of Eq. (1.1). A positive semi-cycle of $\{y_n\}_{n=-1}^{\infty}$ consists of a "string" of terms $\{y_p, y_{p+1}, ..., y_m\}$, all greater than or equal to $\bar{y}$, with p$\geq -1$ and m$\leq \infty$ and such that either p=-1 or $p > -1$ and $y_{p-1} < \bar{y}$ and either m=$\infty$ or m$<\infty$ and $y_{m+1}<\bar{y}$.

A negative semi-cycle of consists of $\{y_n\}_{n=-1}^{\infty}$ consists of a "string" of terms $\{y_q, y_{q+1}, ..., y_l\}$, all less than with and and such that either q=-1 or q>-1 and $y_{q-1} \geq \bar{y}$ and either l=$\infty$ or l$<\infty$ and $y_{l+1} \geq \bar{y}$.

A solution $\{y_n\}_{n=-1}^{\infty}$ of Eq. (1.1) is non-oscillatory if there exists N$\geq -1$ such that either

$y_n > \bar{y}$ for all n$\geq N$ or

$y_n < \bar{y}$ for all n$\geq N$.

$\{y_n\}_{n=-1}^{\infty}$ is called oscillatory if it is not non-oscillatory.

**Theorem 3.1.** Let $\{y_n\}_{n=-1}^{\infty}$ be a positive solution of Eq. (1.1) which consists of a single semi-cycle. Then $\{y_n\}_{n=-1}^{\infty}$converges monotonically to $\bar{y} = \alpha$ .

**Proof.** Suppose 0<$y_{n-1} < \alpha$ for all n$\geq$ 0. Note that for all n$\geq$ 0,

$$0 < \alpha + \frac{y_{n-1}}{y_n} - \frac{y_n}{y_{n-1}} < \alpha$$

and so

$$0 < y_{n-1} < y_n < \alpha.$$

From this it is clear that the positive solutions converge monotonically to $\bar{y}$.

**Theorem 3.2.** Let be $\{y_n\}_{n=-1}^{\infty}$a positive solution of Eq. (1.1) which consists at least two semi-cycles. Then $\{y_n\}_{n=-1}^{\infty}$is oscillatory.

**Proof.** We consider the following two cases.

*Case I.* Suppose that $y_{-1} < \alpha \leq y_0$. Then

$$y_1 = \alpha + \frac{y_{-1}}{y_0} - \frac{y_0}{y_{-1}} < \alpha$$

and

$$y_2 = \alpha + \frac{y_0}{y_1} - \frac{y_1}{y_0} > \alpha.$$

*Case II.* Suppose that $y_0 < \alpha \leq y_{-1}$. Then

$$y_1 = \alpha + \frac{y_{-1}}{y_0} - \frac{y_0}{y_{-1}} > \alpha$$

and

$$y_2 = \alpha + \frac{y_0}{y_1} - \frac{y_1}{y_0} < \alpha.$$

Hence the proof is complete.

## 4. Global asymptotically stability of eq. (1.1)

In this section, we find a global asymptotic stability result for Eq. (1.1).

**Lemma 4.1.** Let $\alpha \in (0,\infty)$ and $f(u,v) = \alpha + \frac{v}{u} - \frac{u}{v}$. If u,v$\in (0,\infty)$, then f(u,v) is nonincreasing in each arguments.

**Proof.** The proof is simple and will be omitted.

**Theorem 4.1**. Let $\alpha > 4$. Then the unique positive equilibrium $\bar{y}$ of Eq. (1.1) is globally asymptotically stable.

**Proof.** For u,v$\in (0,\infty)$, set $f(u,v) = \alpha + \frac{v}{u} - \frac{u}{v}$. Then f:IxI$\rightarrow I$ is a continuous function and is non-increasing in each arguments. Let (m,M)$\in IxI$ is a solution of the system

$$M = f(m,m)$$

$$m = f(M,M),$$

then

$$M = \alpha + \frac{m}{m} - \frac{m}{m}$$

and

$$m = \alpha + \frac{M}{M} - \frac{M}{M}.$$

Since M-m=0, we get m=M. By using Theorem 1.3, we have which shows that is globally asymptotically stable equilibrium point of Eq. (1.1).

$$\lim_{n\rightarrow\infty} y_n = \bar{y}$$

which shows that $\bar{y} = \alpha$ is globally asymptotically stable equilibrium point of Eq. (1.1).

**Acknowledgement.** The first-named author would like to thank the Scientific and Technical research Council of Turkey (TUBITAK) for financial helps (code:2211).

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