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ABOUT THE UNIVALENCE OF THE BESSEL FUNCTIONS

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Abstract. The authors of [1] and [3] deduced univalence criteria concerning Bessel functions. In [3] the author used the theory developed in [2] to obtain the desired result. In this paper we will extend a few results obtained in [3] employing elementary methods.

1. **Introduction**

Let

$$U(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

be the disc with center z_0 and of the radius r, the particular case $U(0, 1)$ will be denoted by U. The Bessel function of the first kind is defined by

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n+\nu}.$$

The series, which defines J_ν is everywhere convergent and the function defined by the series is generally not univalent in any disc $U(0, r)$. We will study the univalence of the following normalized form:

$$f_\nu(z) = 2^\nu \Gamma(1+\nu) z^{-\frac{\nu}{2}} J_\nu(z^{\frac{1}{2}}), \quad g_\nu(z) = z f_\nu(z). \tag{1}$$

2. **Preliminaries**

In order to prove our main result we need the following lemmas.

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Lemma 1 ([3], equality (6))**.** *The function f_ν satisfies the equality:*

$$f'_\nu(z) = -\frac{1}{2} f_{\nu+1}(z).$$

Lemma 2. *Let R be the function defined by the equality*

$$R(\theta) = \sum_{n=3}^{\infty} \frac{(-1)^n (\nu+1)^n \cos n\theta}{n!(\nu+1)...(\nu+n)}, \ \theta \in \mathbb{R}, \nu \in (-1, \infty).$$

The following inequality holds

$$\left|R(\theta)\right| \le \frac{(\nu+1)^2}{4(\nu+2)(\nu+3)}, \ \theta \in \mathbb{R}.$$

Proof. Since

$$R(\theta) = \frac{\nu+1}{\nu+2} \sum_{n=3}^{\infty} \frac{(-1)^n (\nu+1)^{n-2} \cos n\theta}{n!(\nu+3)...(\nu+n)}$$

it follows that

$$\left|R(\theta)\right| \le \frac{\nu+1}{\nu+2} \sum_{n=3}^{\infty} \left| \frac{(-1)^n (\nu+1)^{n-2} \cos n\theta}{n!(\nu+3)...(\nu+n)} \right| \le$$

$$\frac{\nu+1}{\nu+2} \sum_{n=3}^{\infty} \frac{(\nu+1)^{n-2}}{n!(\nu+3)...(\nu+n)} \le \frac{(\nu+1)^2}{(\nu+2)(\nu+3)} \sum_{n=3}^{\infty} \frac{1}{n!} \le \frac{(\nu+1)^2}{4(\nu+2)(\nu+3)}.$$

□

Lemma 3. *If $z \in U$ then*

$$|g'_\nu(z) - \frac{g_\nu(z)}{z}| \le \frac{2+\nu}{(1+\nu)(4\nu+7)}, \tag{2}$$

$$|f_\nu(z)| = |\frac{g_\nu(z)}{z}| \ge \frac{4\nu^2+10\nu+5}{(1+\nu)(4\nu+7)}, \tag{3}$$

$$\left|f'_\nu(z)\right| \le \frac{\nu+2}{(\nu+1)(4\nu+7)}. \tag{4}$$

Proof. If $z \in U$ then the triangle inequality implies that:

$$\left|g'_\nu(z) - \frac{g_\nu(z)}{z}\right| = \left| \sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n n!(\nu+1)...(\nu+n)} z^n \right| \le$$

$$\sum_{n=1}^{\infty} \frac{n}{4^n n!(\nu+1)...(\nu+n)}.$$

Since

$$\sum_{n=1}^{\infty}\frac{n}{4^n n!(\nu+1)...(\nu+n)} \le \frac{1}{4(\nu+1)}\sum_{n=0}^{\infty}\left(\frac{1}{4(\nu+2)}\right)^n = \frac{2+\nu}{(1+\nu)(4\nu+7)}$$

we obtain (2).

By using again the triangle inequality, we deduce that

$$|\frac{g_\nu(z)}{z}| \ge 1-\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{4^n n!(\nu+1)...(\nu+n)}z^n\right| \ge 1-\sum_{n=1}^{\infty}\frac{1}{4^n n!(\nu+1)...(\nu+n)}$$

and so the inequality

$$1-\sum_{n=1}^{\infty}\frac{1}{4^n n!(\nu+1)...(\nu+n)} \ge 1-\frac{1}{4(\nu+1)}\sum_{n=1}^{\infty}\frac{1}{[4(\nu+2)]^{n-1}} = \frac{4\nu^2+10\nu+5}{(1+\nu)(4\nu+7)}$$

leads to (3). Using similar ideas we obtain the following inequality chain

$$\left|f'_\nu(z)\right| \le \sum_{n=1}^{\infty}\left|\frac{(-1)^n z^n}{4^n(n-1)!(1+\nu)...(n+\nu)}\right| \le \sum_{n=1}^{\infty}\frac{1}{4^n(n-1)!(1+\nu)...(n+\nu)} \le \frac{1}{4(1+\nu)}\sum_{n=0}^{\infty}\left(\frac{1}{4(2+\nu)}\right)^n = \frac{\nu+2}{(\nu+1)(4\nu+7)}.$$

This means that (4) also holds. □

3. **The main result**

Theorem 4. *If* $\nu > -1$ *then*

$$Ref_\nu(z) > 0,\ \textit{for all}\ \ z \in U(0,4(1+\nu)).$$

Proof. The minimum principle for harmonic functions implies that

$$\mathrm{Re}f_\nu(z) \ge \inf_{\theta\in\mathbb{R}}\mathrm{Re}f_\nu(r_\nu e^{i\theta}),\ \text{for all}\ \ z \in U(0,4(1+\nu))$$

where $r_\nu = 4(1+\nu)$. According to the definition of f_ν, we have

$$f_\nu(z) = 1+\sum_{n=1}^{\infty}\frac{(-1)^n z^n}{4^n n!(\nu+1)...(\nu+n)}$$

and

$$\mathrm{Re}f_\nu(r_\nu e^{i\theta}) = 1 + \mathrm{Re}\sum_{n=1}^{\infty}\frac{(-1)^n(\nu+1)^n e^{in\theta}}{n!(\nu+1)...(\nu+n)}$$
$$= 1 - \cos\theta + \frac{\nu+1}{2(\nu+2)}\cos 2\theta + \sum_{n=3}^{\infty}\frac{(-1)^n(\nu+1)^n\cos n\theta}{n!(\nu+1)...(\nu+n)}.$$

If we let

$$P(\theta) = 1 - \cos\theta + \frac{\nu+1}{2(\nu+2)}\cos 2\theta \ \text{ and } \ R(\theta) = \sum_{n=3}^{\infty}\frac{(-1)^n(\nu+1)^n\cos n\theta}{n!(\nu+1)...(\nu+n)}$$

then

$$\mathrm{Re}f_\nu(r_\nu e^{i\theta}) = P(\theta) + R(\theta). \tag{5}$$

A study of the behaviour of the function

$$P:\mathbb{R}\to\mathbb{R},\ P(\theta) = 1 - \cos\theta + \frac{\nu+1}{2(\nu+2)}\cos 2\theta$$

leads to the inequalities

$$P(\theta) \geq \frac{\nu+1}{2(\nu+2)},\ \theta\in\mathbb{R},\ \nu\in(-1,0) \ \text{ and}$$
$$P(\theta) \geq \frac{\nu^2+4\nu+2}{4(\nu+1)(\nu+3)},\quad \theta\in\mathbb{R},\ \nu\in(0,\infty). \tag{6}$$

From (5), Lemma 1 and (6) it follows that

$$\mathrm{Re}f_\nu(r_\nu e^{i\theta}) \geq \min_{\theta\in\mathbb{R}} P(\theta) - \max_{\theta\in\mathbb{R}} R(\theta) \geq 0.$$

□

Now Lemma 1 and Theorem 1 imply the following result:

Theorem 5. *If $\nu > -2$ then $\mathrm{Re}f'_\nu(z) < 0$ for $z\in U(0,4(\nu+2))$ and hence f_ν is univalent in $U(0,4(\nu+2))$.*

Remark 6. *Theorem 1 and Theorem 2 improves Lemma 1 and Theorem 1 from* [3].

Theorem 7. *If $\nu > \frac{-17+\sqrt{33}}{8}$ then the function f_ν is convex in U.*

Proof. We introduce the notation $p_1(z) = 1 + \frac{zf''_\nu(z)}{f'_\nu(z)}$. The function f_ν is convex if and only if

$$\mathrm{Re}p_1(z) > 0,\ z\in U. \tag{7}$$

It is simple to prove that if

$$\big|p_1(z) - 1\big| < 1,\ \ z \in U \tag{8}$$

then results (7).

Lemma 1 leads to the equality

$$\big|p_1(z) - 1\big| = \left|\frac{zf'_{\nu+1}(z)}{f_{\nu+1}(z)}\right|.$$

In (3) and (4) replacing ν by $\nu + 1$, we deduce that if $z \in U$ then

$$\left|\frac{zf'_{\nu+1}(z)}{f_{\nu+1}(z)}\right| \leq \frac{\nu+3}{4\nu^2+18\nu+19}.$$

Now to prove (7) it is enough to show that $\frac{\nu+3}{4\nu^2+18\nu+19} < 1$, but this is immediately using the condition $\nu > \frac{-17+\sqrt{33}}{8}$. □

Theorem 8. *If $\nu > \frac{\sqrt{3}}{2} - 1$ then the function g_ν defined by (1) is starlike of order $\frac{1}{2}$ in U.*

Proof. Let p be the function defined by the equality $p_2(z) = \frac{2zg'_\nu(z)}{g_\nu(z)} - 1$. Since $\frac{g_\nu(z)}{z} \neq 0,\ z \in U$ the function p_2 is analytic in U and $p_2(0) = 1$. The assertion of Theorem 2 is equivalent to

$$\mathrm{Re} p_2(z) > 0,\ \ z \in U. \tag{9}$$

It is simple to prove that if

$$\big|p_2(z) - 1\big| < 1,\ \ z \in U \tag{10}$$

then results (9).

On the other hand inequalities (2) and (3) lead to

$$\big|p_2(z) - 1\big| = 2\left|\frac{g'_\nu(z) - \frac{g_\nu(z)}{z}}{\frac{g_\nu(z)}{z}}\right| < \frac{2(2+\nu)}{4\nu^2+10\nu+5},\ \ z \in U.$$

This means that if $\frac{2(2+\nu)}{4\nu^2+10\nu+5} < 1$ then (8) holds, but this inequality is a consequence of the condition $\nu > \frac{\sqrt{3}}{2} - 1$. □

Corollary 9. *If $\nu > \frac{\sqrt{3}}{2} - 1$ then the function h_ν defined by the equality $h_\nu(z) = z^{1-\nu}J_\nu(z)$ is starlike in U.*

The proof of this result is based on Theorem 3 and is similar to the proof of Corollary 2 in [3], hence we do not reproduce it here again.

Remark 10. *Theorem 3, Theorem 4 and Corollary 1 improves the results of Theorem 2, Theorem 3 and Corollary 2 in* [3].

References

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