STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 1, March 2009

ANALYSIS OF A ELECTRO-ELASTIC CONTACT PROBLEM WITH FRICTION AND ADHESION

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Abstract. We consider a mathematical model which describes the quasistatic frictional contact between a piezoelectric body and an obstacle, the so-called foundation. A nonlinear electro-elastic constitutive law is used to model the piezoelectric material. The contact is modelled with Signorini's conditions and the associated with a regularized Coulomb's law of dry friction in witch the adhesion of contact surfaces is taken into account. The evolution of the bonding field is described by a first order differential equation. We derive a variational formulation for the model, in the form of a coupled system for the displacements, the electric potential and the adhesion. Under a smallness assumption on the coefficient of friction, we prove the existence of a unique weak solution of the model. The proof is based on arguments of time-dependent quasi-variational inequalities, differential equations and Banach's fixed point theorem.

1. Introduction

The piezoelectric effect is characterized by the coupling between the mechanical and electrical properties of the materials. Indeed, the apparition of electric charges on some crystals submitted to the action of body forces and surface tractions was observed and their dependence on the deformation process was underlined. Conversely, it was proved experimentally that the action of electric field on the crystals may generate strain and stress. A deformable material which presents such a behavior is called a piezoelectric material. Piezoelectric materials are used extensively as switches and

Received by the editors: 01.03.2008.

2000 *Mathematics Subject Classification.* 74H10, 74H10, 74M15, 74F25, 49J40.

Key words and phrases. piezoelectric material, electro-elastic, erictional contact, nonlocal Coulomb's law, adhesion; quasi-variational inequality, weak solution, fixed point.

actuary in many engineering systems, in radioelectronics, electroacoustics, and measuring equipments. General models for electro-elastic materials can be found in [3], [5] and in [17]. A static frictional contact problem for electro-elastic materials was considered in [4] and in [20]. A slip-dependent frictional contact problem for electro-elastic materials was studied in [26] and a frictional problem with normal compliance for electroviscoelastic materials was considered in [27], [19] and in [18]. In the last two references the variational formulations of the corresponding problems were derived and existence and uniqueness results for the weak solutions were obtained.

The adhesive contact between deformable bodies, when a glue is added to prevent relative motion of the surfaces, has received recently increased attention in the mathematical literature. Basic modelling can be found in [13], [15] and in [9]. Analysis of models for adhesive contact can be found in [2]-[7], [16] and in the recent monographs [24] and [25]. An application of the theory of adhesive contact in the medical field of prosthetic limbs was considered in [22] and in [23]; there, the importance of the bonding between the bone-implant and the tissue was outlined, since debonding may lead to decrease in the persons ability to use the artificial limb or joint.

Contact problems for elastic and elastic-viscoelastic bodies with adhesion and friction appear in many applications of solids mechanics such as the fiber-matrix interface of composite materials. A consistent model coupling unilateral contact, adhesion and friction is proposed by Raous, Cangémi and Cocu in [21]. Adhesive problems have been the subject of some recent publications (see for instance [12], [1], [6] and [9]). The novelty in all the above papers is the introduction of a surface internal variable, the bonding field, denoted in this paper by β; it describes the pointwise fractional density of active bonds on the contact surface, and sometimes referred to as the intensity of adhesion. Following [13], [14], the bonding field satisfies the restrictions $0 \le \beta \le 1$; when $\beta = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active; when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion; when $0 < \beta < 1$ the adhesion is partial and only a fraction β of the bonds is active. We refer the reader to the extensive bibliography on the subject in [15] and in [22]. Such models contain a new internal variable β which represents the adhesion intensity over the contact surface, it takes values between 0 and 1, and describes the fractional density of active bonds on the contact surface.

The aim of this paper is to continue the study of problems begun in [19], [27] and in [18]. The novelty of the present paper is to extend the result when the contact and friction are modelled by Signorini's conditions and a non local Coulomb's friction law, respectively. Moreover, the adhesion is taken into account at the interface and the material behavior is assumed to be electro-elastic.

The paper is structured as follows. In Section 2 we present the electro-elastic contact model with friction and adhesion and provide comments on the contact boundary conditions. In Section 3 we list the assumptions on the data and derive the variational formulation. In Sections 4, we present our main existence and uniqueness results, Theorems 4.1, which states the unique weak solvability of the Signorini's adhesive contact electro-elastic problem with non local Coulomb's friction law conditions.

2. **Problem statement**

We consider the following physical setting. An electro-elastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a smooth boundary $\partial\Omega = \Gamma$. The body is submitted to the action of body forces of density f_0 and volume electric charges of density q_0. It is also submitted to mechanical and electric constraints on the boundary. To describe them, we consider a partition of Γ into three measurable parts Γ_1, Γ_2, Γ_3 on one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand., such that $meas(\Gamma_1) > 0$, $meas(\Gamma a) > 0$. We assume that the body is clamped on Γ_1 and surface tractions of density f_2 act on Γ_2. On Γ_3 the body is in adhesive contact with an insulator obstacle, the so-called foundation. We also assume that the electrical potential vanishes on Γ_a and a surface electric charge of density q_2 is prescribed on Γ_b. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ and we use $\cdot$ and $||\cdot||$ for the inner product and the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$, respectively. Also, below ν represents the unit outward normal on Γ. With these assumptions, the classical formulation of the electro-elastic contact problem coupling friction and adhesion is the following.

Problem 2.1 $(\mathcal{P})$**.** *Find a displacement field* $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^d$*, a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$*, an electric potential* $\varphi : \Omega \times [0,T] \to \mathbb{R}$*, an electric displacement field* $D : \Omega \times [0,T] \to R^d$ *and a bonding field* $\beta : \Omega \times [0,T] \to R$ *such that*

$$\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathcal{E}^* E(\varphi) \quad in\ \Omega \times (0,T), \tag{2.1}$$

$$D = \mathcal{B}E(\varphi) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad in\ \Omega \times (0,T), \tag{2.2}$$

$$Div\boldsymbol{\sigma} + f_0 = 0 \quad in\ \Omega \times (0,T), \tag{2.3}$$

$$div\ D = q_0 \quad on\ \Omega \times (0,T), \tag{2.4}$$

$$\boldsymbol{u} = 0 \quad on\ \Gamma_1 \times (0,T), \tag{2.5}$$

$$\boldsymbol{\sigma}\nu = f_2 \quad on\ \Gamma_2 \times (0,T), \tag{2.6}$$

$$\boldsymbol{u}_\nu \le 0,\ \boldsymbol{\sigma}_\nu - \gamma_\nu \beta^2 R_\nu(\boldsymbol{u}_\nu) \le 0,\ \boldsymbol{u}_\nu(\boldsymbol{\sigma}_\nu - \gamma_\nu \beta^2 R_\nu(\boldsymbol{u}_\nu)) = 0\ on\ \Gamma_3 \times (0,T), \tag{2.7}$$

$$\begin{cases} |\boldsymbol{\sigma}_\tau + \gamma_\tau \beta^2 R_\tau(\boldsymbol{u}_\tau)| \le \mu p(|R(\boldsymbol{\sigma}_\nu) - \gamma_\nu \beta^2 R_\nu(\boldsymbol{u}_\nu)|), & \\ |\boldsymbol{\sigma}_\tau + \gamma_\tau \beta^2 R_\tau(\boldsymbol{u}_\tau)| < \mu p(|R(\boldsymbol{\sigma}_\nu) - \gamma_\nu \beta^2 R_\nu(\boldsymbol{u}_\nu)|) \Rightarrow \boldsymbol{u}_\tau = 0, & on\ \Gamma_3 \times (0,T), \\ |\boldsymbol{\sigma}_\tau + \gamma_\tau \beta^2 R_\tau(\boldsymbol{u}_\tau) = \mu p(|R(\boldsymbol{\sigma}_\nu) - \gamma_\nu \beta^2 R_\nu(\boldsymbol{u}_\nu)|) \Rightarrow \exists \lambda \ge 0, & \\ such\ that\ \boldsymbol{\sigma}_\tau + \gamma_\tau \beta^2 R_\tau(\boldsymbol{u}_\tau) = -\lambda \boldsymbol{u}_\tau, & \end{cases} \tag{2.8}$$

$$\dot{\beta} = -(\beta(\gamma_\nu R_\nu(\boldsymbol{u}_\nu)^2 + \gamma_\tau \|R_\tau(\boldsymbol{u}_\tau)\|^2) - \epsilon_a) \quad on\ \Gamma_3 \times (0,T), \tag{2.9}$$

$$\varphi = 0 \quad on\ \Gamma_a \times (0,T), \tag{2.10}$$

$$D \cdot \nu = q_2 \quad on\ \Gamma_b \times (0,T), \tag{2.11}$$

$$D \cdot \nu = 0 \quad on\ \Gamma_3 \times (0,T), \tag{2.12}$$

$$\beta(0) = \beta_0 \quad on\ \Gamma_3. \tag{2.13}$$

We now provide some comments on equations and conditions (2.1)-(2.13).

Equations (2.1) and (2.2) represent the electro-elastic constitutive law in which $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the linearized strain tensor, $\boldsymbol{E}(\varphi) = -\nabla \varphi$ is the electric field, where φ is the electric potential, $\mathcal{F}$ is a given nonlinear function, $\mathcal{E}$ represents the piezoelectric operator, $\mathcal{E}^*$ is its transposed, $\mathcal{B}$ denotes the electric permittivity operator, and $\mathbf{D} = (D_1, \ldots, D_d)$ is the electric displacement vector. Details on the constitutive equations of the form (2.1) and (2.2) can be find, for instance, in [3] and in [4]. Next, equations (2.3) and (2.4) are the equilibrium equations for the stress and electric-displacement fields, respectively, in which "Div" and "div" denote the divergence operator for tensor and vector valued functions, respectively. Equations (2.5) and (2.6) represent the displacement and traction boundary conditions Conditions (2.10) and (2.11) represent the electric boundary conditions.

Conditions (2.7) represent he Signorini's contact condition with adhesion where $\boldsymbol{u}_\nu$ is the normal displacement $\boldsymbol{\sigma}_\nu$ represents the normal stress, γ_ν denote a given adhesion coefficient and R_ν is the truncation operator define by

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L, \\ -s & \text{if } -L \leq s \leq 0, \\ 0 & \text{if } s > 0, \end{cases}$$

where $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction. The introduction of operator R_ν, together with the operator R_τ defined below , is motivated by the mathematical arguments but it is not restrictive for physical point of view, since no restriction on the size of the parameter L is made in what follows. Thus, by choosing L very large, we can assume that $R_\nu(\boldsymbol{u}_\nu) = \boldsymbol{u}_\nu$ and, therefore, from (2.7) we recover the contact conditions

$$\boldsymbol{u}_\nu \leq 0,\ \boldsymbol{\sigma}_\nu - \gamma_\nu \beta^2 \boldsymbol{u}_\nu \leq 0,\ \boldsymbol{u}_\nu(\boldsymbol{\sigma}_\nu - \gamma_\nu \beta^2 \boldsymbol{u}_\nu) = 0 \text{ on } \Gamma_3 \times (0, T),$$

It follows from (2.7) that there is no penetration between the body and the foundation, since $\boldsymbol{u}_\nu \leq 0$ during the process.

Conditions (2.8) are a non local Coulomb's friction law conditions coupled with adhesion, where $\boldsymbol{u}_\tau$ and $\boldsymbol{\sigma}_\tau$ denote tangential components of vector $\boldsymbol{u}$ and tensor

$\boldsymbol{\sigma}$ respectively. R_τ is the truncation operator given by

$$R_\tau(\boldsymbol{v}) = \begin{cases} \boldsymbol{v} & \text{if } \|\boldsymbol{v}\| \leq L, \\ L\dfrac{\boldsymbol{v}}{\|\boldsymbol{v}\|} & \text{if } \|\boldsymbol{v}\| > L. \end{cases}$$

This condition shows that the magnitude of the shear on the contact surface depends on the bonding field and on the tangential displacement, but as long as it does not exceed the bond length L.

R will represent a normal regularization operator that is , linear and continues operator $R: H^{-\frac{1}{2}}(\Gamma) \to L^2(\Gamma)$. We shall need it to regularize the normal trace of the stress witch is too rough on Γ. p is a non-negative function, the so-called friction bound, $\mu \geq 0$ is the coefficient of friction. The friction law was used in some studies with $p\,(r) = r_+$ where $r_+ = \max\{0, r\}$. Recently, from thermodynamic considerations, a new version of *Coulomb's* law is proposed; its consists to take

$$p(r) = r(1 - \alpha r)_+, \tag{2.14}$$

where α is a small positive coefficient related to the hardness and the wear of the contact surface.

Also, note that when the bonding field vanishes, then the contact conditions (2.7) and (2.8) become the classic Signorini's contact with a non local Coulomb's friction law conditions were used in ([11]), that is

$$\boldsymbol{u}_\nu \leq 0,\ \boldsymbol{\sigma}_\nu \leq 0,\ \boldsymbol{u}_\nu \boldsymbol{\sigma}_\nu = 0 \text{ on } \Gamma_3 \times (0, T),$$

$$\begin{cases} |\boldsymbol{\sigma}_\tau| \leq \mu p(|R(\boldsymbol{\sigma}_\nu)|), \\ |\boldsymbol{\sigma}_\tau| < \mu p(|R(\boldsymbol{\sigma}_\nu)|) \Rightarrow \boldsymbol{u}_\tau = 0, & \text{on } \Gamma_3 \times (0, T), \\ |\boldsymbol{\sigma}_\tau| = \mu p(|R(\boldsymbol{\sigma}_\nu)|) \Rightarrow \exists \lambda \geq 0, \text{ such that } \boldsymbol{\sigma}_\tau = -\lambda \boldsymbol{u}_\tau. \end{cases}$$

The evolution of the bonding field is governed by the differential equation (2.9) with given positive parameters γ_ν, γ_τ and ϵ_a, where $r_+ = \max\{0, r\}$. Here and below in this paper, a dot above a function represents the derivative with respect to the time variable. We note that the adhesive process is irreversible and, indeed, once debonding occurs bonding cannot be reestablished, since $\dot{\beta} \leq 0$. Finally, (2.13) is the initial condition in which β_0 is a given bonding field.

3. Variational formulation and preliminaries

In this section, we list the assumptions on the data and derive a variational formulation for the contact problem. To this end we need to introduce some notation and preliminary material.

Here and below $\mathbb{S}^d$ represents the space of second order symmetric tensors on $\mathbb{R}^d$. We recall that the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}_i \boldsymbol{v}_i \ , \qquad \|\boldsymbol{v}\| = (\boldsymbol{v} \cdot \boldsymbol{v})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d,$$

$$\boldsymbol{\sigma} \cdot \tau = \boldsymbol{\sigma}_{ij} \tau_{ij} \ , \qquad \|\tau\| = (\tau \cdot \tau)^{\frac{1}{2}} \qquad \forall\, \boldsymbol{\sigma}, \tau \in \mathbb{S}^d.$$

Here and everywhere in this paper, i, j, k, l run from 1 to d, summation over repeated indices is applied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g. $\boldsymbol{u}_{i,j} = \frac{\partial \boldsymbol{u}_i}{\partial x_j}$.

Everywhere below, we use the classical notation for L^p and Sobolev spaces associated to Ω and Γ. Moreover, we use the notation $L^2(\Omega)^d$, $H^1(\Omega)^d$ and $\mathcal{H}$ and $\mathcal{H}_1$ for the following spaces :

$$L^2(\Omega)^d = \{\ \boldsymbol{v} = (\boldsymbol{v}_i) \mid \boldsymbol{v}_i \in L^2(\Omega)\ \}, \quad H^1(\Omega)^d = \{\ \boldsymbol{v} = (\boldsymbol{v}_i) \mid \boldsymbol{v}_i \in H^1(\Omega)\ \},$$

$$\mathcal{H} = \{\ \tau = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega)\ \}, \quad \mathcal{H}_1 = \{\ \tau \in \mathcal{H} \mid \tau_{ij,j} \in L^2(\Omega)\ \}.$$

The spaces $L^2(\Omega)^d$, $H^1(\Omega)^d$, $\mathcal{H}$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{u}, \boldsymbol{v})_{L^2(\Omega)^d} = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v}\, dx, \quad (\boldsymbol{u}, \boldsymbol{v})_{H^1(\Omega)^d} = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v}\, dx + \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v}\, dx,$$

$$(\boldsymbol{\sigma}, \tau)_{\mathcal{H}} = \int_\Omega \boldsymbol{\sigma} \cdot \tau\, dx, \quad (\boldsymbol{\sigma}, \tau)_{\mathcal{H}_1} = \int_\Omega \boldsymbol{\sigma} \cdot \tau\, dx + \int_\Omega \operatorname{Div} \boldsymbol{\sigma} \cdot \operatorname{Div} \tau\, dx,$$

and the associated norms $\|\cdot\|_{L^2(\Omega)^d}$, $\|\cdot\|_{H^1(\Omega)^d}$, $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively. Here and below we use the notation

$$\nabla \boldsymbol{v} = (\boldsymbol{v}_{i,j}), \quad \boldsymbol{\varepsilon}(\boldsymbol{v}) = (\boldsymbol{\varepsilon}_{ij}(\boldsymbol{v})), \quad \boldsymbol{\varepsilon}_{ij}(\boldsymbol{v}) = \frac{1}{2}(\boldsymbol{v}_{i,j} + \boldsymbol{v}_{j,i}) \quad \forall\, \boldsymbol{v} \in H^1(\Omega)^d,$$

$$\mathrm{Div}\,\tau = (\tau_{ij,j}) \quad \forall\, \tau \in \mathcal{H}_1.$$

For every element $\boldsymbol{v} \in H^1(\Omega)^d$ we also write $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on Γ and we denote by $\boldsymbol{v}_\nu$ and $\boldsymbol{v}_\tau$ the normal and tangential components of $\boldsymbol{v}$ on Γ given by $\boldsymbol{v}_\nu = \boldsymbol{v} \cdot \nu$, $\boldsymbol{v}_\tau = \boldsymbol{v} - \boldsymbol{v}_\nu \nu$.

Let now consider the closed subspace of $H^1(\Omega)^d$ defined by

$$V = \{\ \boldsymbol{v} \in H^1(\Omega)^d \mid \boldsymbol{v} = 0 \text{ on } \Gamma_1\ \}.$$

Since $meas\,(\Gamma_1) > 0$, the following Korn's inequality holds:

$$\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}} \geq c_K\, \|\boldsymbol{v}\|_{H^1(\Omega)^d} \quad \forall\, \boldsymbol{v} \in V, \tag{3.1}$$

where $c_K > 0$ is a constant which depends only on Ω and Γ_1. Over the space V we consider the inner product given by

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \tag{3.2}$$

and let $\|\cdot\|_V$ be the associated norm. It follows from Korn's inequality (3.1) that $\|\cdot\|_{H^1(\Omega)^d}$ and $\|\cdot\|_V$ are equivalent norms on V and, therefore, $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, (3.1) and (3.2), there exists a constant c_0 depending only on the domain Ω, Γ_1 and Γ_3 such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \leq c_0 \|\boldsymbol{v}\|_V \qquad \forall \boldsymbol{v} \in V. \tag{3.3}$$

We also introduce the following spaces.

$$W = \{\ \psi \in H^1(\Omega) \mid \psi = 0 \text{ on } \Gamma_a\ \}, \quad \mathcal{W}_1 = \{\ D = (D_i) \mid D_i \in L^2(\Omega),\ D_{i,i} \in L^2(\Omega)\ \}.$$

Since $meas\,(\Gamma_a) > 0$, the following Friedrichs-Poincaré inequality holds:

$$\|\nabla\psi\|_{L^2(\Omega)^d} \geq c_F\, \|\psi\|_{H^1(\Omega)} \quad \forall\, \psi \in W, \tag{3.4}$$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a and $\nabla\psi = (\psi_{,i})$. Over the space W, we consider the inner product given by

$$(\varphi, \psi)_W = \int_\Omega \nabla\varphi \cdot \nabla\psi \, dx$$

and let $\|\cdot\|_W$ be the associated norm. It follows from (3.4) that $\|\cdot\|_{H^1(\Omega)}$ and $\|\cdot\|_W$ are equivalent norms on W and therefore $(W, \|\cdot\|_W)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant c_0, depending only on Ω, Γ_a and Γ_C, such that

$$\|\zeta\|_{L^2(\Gamma_C)} \leq \tilde{c}_0 \|\zeta\|_W \quad \forall \zeta \in W. \tag{3.5}$$

The space $\mathcal{W}_1$ is real Hilbert space with the inner product

$$(D, \boldsymbol{E})_{\mathcal{W}_1} = \int_\Omega D \cdot \boldsymbol{E}\, dx + \int_\Omega \operatorname{div} D \cdot \operatorname{div} \boldsymbol{E}\, dx,$$

where $\operatorname{div} = (D_{i,i})$, and the associated norm $\|\cdot\|_{\mathcal{W}_1}$.

For every real Hilbert space X we use the classical notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, $1 \leq p \leq \infty$, $k \geq 1$ and we also introduce the set

$$\mathcal{Q} = \{\, \theta \in L^\infty(0,T;L^2(\Gamma_3)) \mid 0 \leq \theta(t) \leq 1 \ \ \forall\, t \in [0,T], \text{ a.e. on } \Gamma_3 \,\}.$$

Finally, if X_1 and X_2 are two Hilbert spaces endowed with the inner products $(\cdot,\cdot)_{X_1}$ and $(\cdot,\cdot)_{X_2}$ and the associated norms $\|\cdot\|_{X_1}$ and $\|\cdot\|_{X_2}$, respectively, we denote by $X_1 \times X_2$ the product space together with the canonical inner product $(\cdot,\cdot)_{X_1 \times X_2}$ and the associated norm $\|\cdot\|_{X_1 \times X_2}$.

In the study of the problem $\mathcal{P}$, we consider the following assumptions on the problem data.

The elasticity operator $\mathcal{F}$, the piezoelectric operator $\mathcal{E}$ and the electric permittivity operator $\mathcal{B}$ satisfy

$$\left\{\begin{array}{l}\text{(a) } \mathcal{F} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d,\\[2mm] \text{(b) there exists } L_{\mathcal{F}} > 0 \text{ such that}\\ \quad \|\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{F}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{a.e. } \boldsymbol{x} \in \Omega,\\[2mm] \text{(c) there exists } m > 0 \text{ such that}\\ \quad (\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2), \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{F}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{a.e. } \boldsymbol{x} \in \Omega,\\[2mm] \text{(d) the mapping } \boldsymbol{x} \mapsto \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is Lebesgue measurable in } \Omega, \text{ for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d,\\[2mm] \text{(e) the mapping } \boldsymbol{x} \mapsto \mathcal{F}(\boldsymbol{x}, 0) \in \mathcal{H} \end{array}\right. \tag{3.6}$$

$$\left\{\begin{array}{l}\text{(a) } \mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{R}^d.\\[2mm] \text{(b) } \mathcal{E}(\boldsymbol{x}, \boldsymbol{\tau}) = (e_{ijk}(\boldsymbol{x})\tau_{jk}) \quad \forall \boldsymbol{\tau} = (\boldsymbol{\tau}_{ij}) \in \mathbb{S}^d, \quad \text{a.e. } \boldsymbol{x} \in \Omega.\\[2mm] \text{(c) } e_{ijk} = e_{ikj} \in L^\infty(\Omega). \end{array}\right. \tag{3.7}$$

$$\left\{\begin{array}{l}\text{(a) } \mathcal{B} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d.\\[2mm] \text{(b) } \mathcal{B}(\boldsymbol{x}, \boldsymbol{E}) = (b_{ij}(\boldsymbol{x})E_j) \quad \forall \boldsymbol{E} = (E_i) \in \mathbb{R}^d, \quad \text{a.e. } \boldsymbol{x} \in \Omega.\\[2mm] \text{(c) } b_{ij} = b_{ji} \in L^\infty(\Omega).\\[2mm] \text{(d) There exists } m_{\mathcal{B}} > 0 \text{ such that } b_{ij}(\boldsymbol{x})E_iE_j \ge m_{\mathcal{B}}\|\boldsymbol{E}\|^2\\ \qquad \forall \boldsymbol{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \end{array}\right. \tag{3.8}$$

From the assumptions (3.7) and (3.8), we deduce that the piezoelectric operator $\mathcal{E}$ and the electric permittivity operator $\mathcal{B}$ are linear, have measurable bounded components denoted e_{ijk} and b_{ij}, respectively, and moreover, $\mathcal{B}$ is symmetric and positive definite.

Recall also that the transposed operator $\mathcal{E}^*$ is given by $\mathcal{E}^* = (e^*_{ijk})$ where $e^*_{ijk} = e_{kij}$, and the following equality holds :

$$\mathcal{E}\boldsymbol{\sigma} \cdot \boldsymbol{v} = \boldsymbol{\sigma} \cdot \mathcal{E}^*\boldsymbol{v} \qquad \forall\, \boldsymbol{\sigma} \in \mathbb{S}^d,\ \boldsymbol{v} \in \mathbb{R}^d. \tag{3.9}$$

The friction function satisfies :

$$\left\{\begin{array}{l} \quad p:\,\Gamma_3\times\mathbb{R}\rightarrow\mathbb{R}_+ \text{ verifies} \\ (a)\ \text{there exists } M>0 \text{ such that :} \\ \quad |p\,(x,r_1)-p\,(x,r_2)|\leq M\,|r_{1-}r_2| \\ \quad \text{for every } r_1,r_2\in\mathbb{R},\quad \text{a.e. } x\in\Gamma_3; \\ (b)\ x\mapsto p\,(x,r)\ \text{ is measurable on } \Gamma_3,\ \text{for every } r\in\mathbb{R}; \\ (c)\ p\,(x,0)=0,\quad \text{a.e. } x\in\Gamma_3. \end{array}\right. \tag{3.10}$$

We note that (3.10) is satisfied in the case of function p given by (2.14).

We also suppose that the body forces and surface tractions have the regularity

$$f_0\in W^{1,\infty}(0,T;L^2(\Omega)^d),\quad f_2\in W^{1,\infty}(0,T;L^2(\Gamma_2)^d), \tag{3.11}$$

and the densities of electric charges satisfy

$$q_0\in W^{1,\infty}(0,T;L^2(\Omega)),\quad q_2\in W^{1,\infty}(0,T;L^2(\Gamma_b)), \tag{3.12}$$

Note that we need to impose assumption (3.12) for physical reasons; indeed, the foundation is supposed to be insulator and therefore the electric boundary conditions on Γ_3 do not have to change in function of the status of the contact, are the same on the contact and on the separation zone, and are included in the boundary condition (2.11).

We define the function $f:[0,T]\rightarrow V$ and $q:[0,T]\rightarrow W$ by

$$(f(t),\boldsymbol{v})_V=\int_\Omega f_0(t)\cdot\boldsymbol{v}\,dx+\int_{\Gamma_2} f_2(t)\cdot\boldsymbol{v}\,da, \tag{3.13}$$

$$(q(t),\psi)_W=\int_\Omega q_0(t)\psi\,dx-\int_{\Gamma_b} q_2(t)\psi\,da,$$

for all $\boldsymbol{u},\boldsymbol{v}\in V$, $\psi\in W$ and $t\in[0,T]$, and note that conditions (3.11) and (3.12) imply that

$$f\in W^{1,\infty}(0,T;V),\qquad q\in W^{1,\infty}(0,T;W). \tag{3.14}$$

The adhesion coefficients γ_ν, γ_τ and the limit bound ϵ_a satisfy the conditions

$$\gamma_\nu,\ \gamma_\tau\in L^\infty(\Gamma_3),\quad \epsilon_a\in L^2(\Gamma_3),\quad \gamma_\nu,\,\gamma_\tau,\,\epsilon_a\geq 0\quad \text{a.e. on } \Gamma_3 \tag{3.15}$$

while the friction coefficient μ is such that

$$\mu \in L^\infty(\Gamma_3), \quad \mu(x) \geq 0 \quad \text{a.e. on } \Gamma_3 \tag{3.16}$$

and finally, the initial condition β_0 satisfies

$$\beta_0 \in L^2(\Gamma_3), \quad 0 \leq \beta_0 \leq 1 \ \text{ a.e. on } \Gamma_3. \tag{3.17}$$

We denote by U_{ad} the convex subset of admissible displacements fields given by

$$U_{ad} = \{\boldsymbol{v} \in H_1 \mid \ \boldsymbol{v} = 0 \text{ on } \Gamma_1\,,\ \boldsymbol{v}_\nu \leq 0 \text{ on } \Gamma_3\}\,. \tag{3.18}$$

We define the adhesion functional $j_{ad} : L^2(\Gamma_3) \times V \times V \to \mathbb{R}$ by

$$j_{ad}(\beta, \boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} \big(-\gamma_\nu \beta^2 R_\nu(\boldsymbol{u}_\nu)\boldsymbol{v}_\nu + \gamma_\tau \beta^2 R_\tau(\boldsymbol{u}_\tau) \cdot \boldsymbol{v}_\tau \big)\, da, \tag{3.19}$$

and the friction functional $j_{fr} : L^2(\Gamma_3) \times \mathcal{H}_1 \times V \times V \to \mathbb{R}$ by

$$j_{fr}(\beta, \boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} \mu p(|R(\boldsymbol{\sigma}_\nu) - \gamma_\nu \beta^2 R_\nu(\boldsymbol{u}_\nu)|) \cdot |\boldsymbol{v}_\tau|\, da. \tag{3.20}$$

By a standard procedure based on Green's formula we can derive the following variational formulation of the contact problem (2.1)-(2.13).

Problem 3.1 ($\mathcal{P}^V$)**.** *Find a displacement field $\boldsymbol{u} : [0,T] \longrightarrow V$, an electric potential field $\varphi : [0,T] \to W$ and a bonding field $\beta : [0,T] \to L^2(\Gamma_3)$ such that*

$$\begin{gathered} \boldsymbol{u}(t) \in U_{ad} \quad (\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u}(t)))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi(t), \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u}(t))_{\mathcal{H}} + \\ + j_{ad}(\beta(t), \boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)) + j_{fr}(\beta(t), \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mathcal{E}^*\nabla\varphi(t), \boldsymbol{u}(t), \boldsymbol{v}) - \\ - j_{fr}(\beta(t), \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mathcal{E}^*\nabla\varphi(t), \boldsymbol{u}(t), \boldsymbol{u}(t)) \geq (f(t), y - \boldsymbol{u}(t))_V \quad \forall \boldsymbol{v} \in U_{ad},\ t \in [0,T], \end{gathered} \tag{3.21}$$

$$(\mathcal{B}\nabla\varphi(t), \nabla\psi)_{L^2(\Omega)^d} - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \nabla\psi)_{L^2(\Omega)^d} = (q(t), \psi)_W, \quad \forall \psi \in W, \forall t \in [0,T], \tag{3.22}$$

$$\dot\beta(t) = -(\beta(t)\,(\gamma_\nu R_\nu(\boldsymbol{u}_\nu(t))^2 + \gamma_\tau \|R_\tau(\boldsymbol{u}_\tau(t))\|^2) - \epsilon_a)_+ \ \textit{a.e. } t \in (0,T), \tag{3.23}$$

$$\beta(0) = \beta_0. \tag{3.24}$$

In the rest of this section, we derive some inequalities involving the functionals j_{ad}, and j_{fr} which will be used in the following sections. Below in this section β, β_1, β_2 denote elements of $L^2(\Gamma_3)$ such that $0 \leq \beta,\ \beta_1,\ \beta_2 \leq 1$ a.e. on Γ_3, $\boldsymbol{u}_1,\ \boldsymbol{u}_2, \boldsymbol{v}_1, \boldsymbol{v}_2$, $\boldsymbol{u}$ and $\boldsymbol{v}$ represent elements of V; $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$ denote elements of $\mathcal{H}_1$ and c is a generic positive constants which may depend on Ω, Γ_1, Γ_3, p, γ_ν, γ_τ and L, whose value may change from place to place. For the sake of simplicity, we suppress in what follows the explicit dependence on various functions on $x \in \Omega \cup \Gamma_3$.

First, we remark that the j_{ad} is linear with respect to the last argument and therefore

$$j_{ad}(\beta, \boldsymbol{u}, -\boldsymbol{v}) = -j_{ad}(\beta, \boldsymbol{u}, \boldsymbol{v}). \tag{3.25}$$

Next, using (3.19) and the inequalities $|R_\nu(\boldsymbol{u}_{1\nu})| \leq L,\ \|R_\tau(\boldsymbol{u}_\tau)\| \leq L$,

$|\beta_1| \leq 1,\ |\beta_2| \leq 1$, for the previous inequality, we deduce that

$$j_{ad}(\beta_1, \boldsymbol{u}_1, \boldsymbol{u}_2 - \boldsymbol{u}_1) + j_{ad}(\beta_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \leq c \int_{\Gamma_3} |\beta_1 - \beta_2| \, \|\boldsymbol{u}_1 - \boldsymbol{u}_2\| \, da,$$

then, we combine this inequality with (3.3), to obtain

$$j_{ad}(\beta_1, \boldsymbol{u}_1, \boldsymbol{u}_2 - \boldsymbol{u}_1) + j_{ad}(\beta_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \leq c \, \|\beta_1 - \beta_2\|_{L^2(\Gamma_3)} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V. \tag{3.26}$$

Next, we choose $\beta_1 = \beta_2 = \beta$ in (3.26) to find

$$j_{ad}(\beta, \boldsymbol{u}_1, \boldsymbol{u}_2 - \boldsymbol{u}_1) + j_{ad}(\beta, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \leq 0. \tag{3.27}$$

Similar manipulations, based on the Lipschitz continuity of operators R_ν, R_τ show that

$$|j_{ad}(\beta, \boldsymbol{u}_1, \boldsymbol{v}) - j_{ad}(\beta, \boldsymbol{u}_2, \boldsymbol{v})| \leq c \, \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V \|\boldsymbol{v}\|_V. \tag{3.28}$$

Also, we take $\boldsymbol{u}_1 = \boldsymbol{v}$ and $\boldsymbol{u}_2 = 0$ in (3.27), then we use the equalities $R_\nu(0) = 0$, $R_\tau(0) = 0$ and (3.26) to obtain

$$j_{ad}(\beta, \boldsymbol{v}, \boldsymbol{v}) \geq 0. \tag{3.29}$$

Next, we use (3.20), (3.10)(a), keeping in mind (3.3), propriety of a normal regularization operator and the inequalities $|R_\nu(\boldsymbol{u}_\nu)| \leq L,\ |\beta_1| \leq 1,\ |\beta_2| \leq 1$ and the regularity of the operator R we obtain

$$j_{fr}(\beta_1,\boldsymbol{\sigma}_1,\boldsymbol{u}_1,\boldsymbol{v}_2)-j_{fr}(\beta_1,\boldsymbol{\sigma}_1,\boldsymbol{u}_1,\boldsymbol{v}_1)+j_{fr}(\beta_2,\boldsymbol{\sigma}_2,\boldsymbol{u}_2,\boldsymbol{v}_1)-j_{fr}(\beta_2,\boldsymbol{\sigma}_2,\boldsymbol{u}_2,\boldsymbol{v}_2)\le$$
$$\le c_0^2M\|\mu\|_{L^\infty(\Gamma_3)}(||\boldsymbol{u}_2-\boldsymbol{u}_1\|_V+c(\|\beta_2-\beta_1\|_{L^2(\Gamma_3)}+\|\boldsymbol{\sigma}_{2-}\boldsymbol{\sigma}_1||_{\mathcal{H}_1}))||\boldsymbol{v}_2-\boldsymbol{v}_1\|_V. \tag{3.30}$$

now, by using (3.10)(a) and (3.16), it follows that the integral in (3.20) is well defined. Moreover, we have

$$j_{fr}(\beta,\boldsymbol{\sigma},\boldsymbol{u},\boldsymbol{v})\le c_0^2M\|\mu\|_{L^\infty(\Gamma_3)}\|(||\boldsymbol{u}\|_V\|+c(\|\boldsymbol{\sigma}||_{\mathcal{H}_1}+\|\beta\|_{L^2(\Gamma_3)}))\boldsymbol{v}\|_V. \tag{3.31}$$

The inequalities (3.26)-(3.31) combined with equalities (3.25) will be used in various places in the rest of the paper.

4. Existence and uniqueness result

Our main result which states the unique solvability of Problem $\mathcal{P}^V$, is the following.

Theorem 4.1. *Assume that* (3.6)-(3.8), (3.10) *and* (3.15)-(3.17) *hold. Then, there exists* $\mu_0>0$ *depending only on* $\Omega,\Gamma_1,\Gamma_3,F,\ B$ *and* p *such that, if* $\|\mu\|_{L^\infty(\Gamma_3)}<\mu_0$, *then Problem* $\mathcal{P}^V$ *has a unique solution* $(\boldsymbol{u},\varphi,\beta)$. *Moreover, the solution satisfies*

$$\boldsymbol{u}\in W^{1,\infty}(0,T;V), \tag{4.1}$$

$$\varphi\in W^{1,\infty}(0,T;W). \tag{4.2}$$

$$\beta\in W^{1,\infty}(0,T;L^2(\Gamma_3))\cap\mathcal{Q}. \tag{4.3}$$

A “quintuple” of functions ($\boldsymbol{u}$, $\boldsymbol{\sigma}$, φ, D, β) which satisfy (2.1), (2.2), (3.21)-(3.24) is called a *weak solution* of the contact problem $\mathcal{P}$. We conclude by Theorem 4.1 that, under the stated assumptions, Problem $\mathcal{P}$ has a unique weak solution. To precise the regularity of the weak solution we note that the constitutive relations (2.1) and (2.2), the assumptions (3.6), (3.8) and the regularities (4.1), (4.2) show that $\boldsymbol{\sigma}\in W^{1,\infty}([0,T];\mathcal{H})$, $\boldsymbol{D}\in W^{1,\infty}([0,T];L^2(\Omega)^d)$; moreover, (3.21), (3.22) combined with the definitions of $\boldsymbol{f}$ q and functionals j_{ad} and j_{fr} yield

$$\operatorname{Div}\boldsymbol{\sigma}(t)+\boldsymbol{f}_0(t)=\boldsymbol{0},\quad \operatorname{div}\boldsymbol{D}(t)=q_0(t)\quad \forall\, t\in[0,T].$$

It follows now from the regularities (3.11), (3.9) that $\operatorname{Div}\boldsymbol{\sigma} \in W^{1,\infty}(0,T;L^2(\Omega)^d)$ and $\operatorname{div}\boldsymbol{D} \in W^{1,\infty}(0,T;L^2(\Omega))$, which shows that

$$\boldsymbol{\sigma} \in W^{1,\infty}(0,T;\mathcal{H}_1), \tag{4.4}$$

$$\boldsymbol{D} \in W^{1,\infty}(0,T;\mathcal{W}_1). \tag{4.5}$$

We conclude that the weak solution $(\boldsymbol{u},\boldsymbol{\sigma},\varphi,\boldsymbol{D},\beta)$ of the piezoelectric contact problem $\mathcal{P}$ has the regularity (4.1), (4.2), (4.3), (4.4) and (4.5).

The proof of Theorem 4.1 is carried out in several steps and is based on the following abstract result for variational inequalities.

Let X be a real Hilbert space with the inner product $(\cdot,\cdot)_X$ and the associated norm $\|\cdot\|_X$, and consider the problem of finding $u \in X$ such that

$$(Au, v-u)_X + j(u,v) - j(u,u) \geqslant (f, v-u) \quad \forall v \in X. \tag{4.6}$$

To study problem (4.6) we need the following assumptions: The operator $A : X \to X$ is strongly monotone and Lipschitz continuous, i.e.,

$$\begin{cases} \text{(a) There exists } m_A > 0 \text{ such that} \\ \qquad (Au_1 - Au_2, u_1 - u_2)_X \geq m_A \|u_1 - u_2\|_X^2 \quad \forall\, u_1, u_2 \in X. \\ \text{(b) There exists } L_A > 0 \text{ such that} \\ \qquad \|Au_1 - Au_2\|_X \leq L_A \|u_1 - u_2\|_X \quad \forall\, u_1, u_2 \in X. \end{cases} \tag{4.7}$$

The functional $j : X \times X \to \mathbb{R}$ satisfies:

$$\begin{cases} \text{(a) } j(u,\cdot) \text{ is convex and l.s.c. on } X \text{ for all } u \in X. \\ \text{(b) There exists } m > 0 \text{ such that} \\ \qquad j(u_1,v_2) - j(u_1,v_1) + j(u_2,v_1) - j(u_2,v_2) \\ \qquad\quad \leq m\,\|u_1 - u_2\|_X\,\|v_1 - v_2\|_X \quad \forall\, u_1,u_2,v_1,v_2 \in X. \end{cases} \tag{4.8}$$

Finally, we assume that

$$f \in X \tag{4.9}$$

The following existence, uniqueness was proved in [28].

Theorem 4.2. *Assume that (4.7), (4.8) and (4.9) hold. Then, if $m < m_A$, for all $f \in X$, there exists a unique solution $u \in Y$ of Problem 4.6.*

We return now to proof of theorem 4.1. To this end, we assume in the following that (3.6)-(3.8), (3.10)-(3.12) and (3.15)-(3.17) hold; below, c is a generic positive constants which may depend on Ω, Γ_1, Γ_3, $\mathcal{F}$, p, γ_ν, γ_τ and L, whose value may change from place to place. For the sake of simplicity, we suppress in what follows the explicit dependence on various functions on $\boldsymbol{x} \in \Omega \cup \Gamma_3$.

Let $\mathcal{L}$ denotes the closed set of the space $C([0,T]; L^2(\Gamma_3))$ defined by

$$\mathcal{L} = \left\{ \beta \in C([0,T]; L^2(\Gamma_3)) \cap \mathcal{Q} \mid \beta(0) = \beta_0 \right\} \tag{4.10}$$

and let $\beta \in \mathcal{L}$ and $g \in W^{1,\infty}(0,T; \mathcal{H}_1)$ are given. In the first step, we consider the following variational problem.

Problem 4.3 ($\mathcal{P}_{\beta g}$)**.** *Find a displacement field* $\boldsymbol{u} : [0,T] \to V$*, an electric potential field* $\varphi : [0,T] \to W$ *such that*

$$\boldsymbol{u}_{\beta g}(t) \in U_{ad}, \quad (\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{\beta g}(t))), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{u}_{\beta g}(t))_{\mathcal{H}} + (\mathcal{E}^* \nabla \varphi_{\beta g}(t), \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u}_{\beta g}(t)))_{\mathcal{H}} +$$

$$+ j_{ad}(\beta(t), \boldsymbol{u}_{\beta g}(t), \boldsymbol{v} - \boldsymbol{u}_{\beta g}(t)) + j_{fr}(\beta(t), g(t), \boldsymbol{u}_{\beta g}(t), \boldsymbol{v}) -$$

$$- j_{fr}(\beta(t), g(t), \boldsymbol{u}_{\beta g}(t), \boldsymbol{u}_{\beta g}(t)) \geq (f(t), \boldsymbol{v} - \boldsymbol{u}_{\beta g}(t))_V \quad \forall \boldsymbol{v} \in U_{ad}, \tag{4.11}$$

$$(\mathcal{B} \nabla \varphi_{\beta g}(t), \nabla \psi)_{L^2(\Omega)^d} - (\mathcal{E} \boldsymbol{\varepsilon}(\boldsymbol{u}_{\beta g}(t)), \nabla \psi)_{L^2(\Omega)^d} = (q(t), \psi)_W \quad \forall\, \psi \in W. \tag{4.12}$$

In order to solve Problem $\mathcal{P}_{\beta g}$ we consider the product space $X = V \times W$ endowed with the inner product

$$(x, y)_X = (\boldsymbol{u}, \boldsymbol{v})_V + (\varphi, \psi)_W \qquad \forall x = (\boldsymbol{u}, \varphi),\ y = (\boldsymbol{v}, \psi) \in X \tag{4.13}$$

and the associated norm $||.||_X$. We also introduce the set $K \subset X$ and the function $A_{\beta g} : [0,T] \times X \to X$, $\mathbf{f}$ $[0,T] \to X$, defined by

$$K = U_{ad} \times W, \tag{4.14}$$

$$(A_{\beta g}(t)x, y)_X = (\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t))), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (\mathcal{B}\nabla\varphi(t), \nabla\psi)_{L^2(\Omega)^d} + \tag{4.15}$$

$$(\mathcal{E}^*\nabla\varphi(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \nabla\psi)_{L^2(\Omega)^d} \tag{4.16}$$

$$+j_{ad}(\beta(t), \boldsymbol{u}(t), \boldsymbol{v}) \quad \forall x = (\boldsymbol{u}, \varphi)_V,\ y = (\boldsymbol{v}, \psi)_W \in X, \quad t \in [0, T],$$

$$j_{\beta g}(x, y) = j_{fr}(\beta(t), g(t), \boldsymbol{u}(t), \boldsymbol{u}(t)) \quad \forall x = (\boldsymbol{u}, \varphi),\ y = (\boldsymbol{v}, \psi) \in X \tag{4.17}$$

$$\mathbf{f}{=}(f(t), q(t)) \quad \forall t \in [0, T]. \tag{4.18}$$

We start with the following equivalence result.

Lemma 4.4. *The couple* $(x_{\beta g}, \varphi_{\beta g}) : [0, T] \to V \times W$ *is a solution to Problem* $\mathcal{P}_{\beta g}$ *if and only if* $x_{\beta g} : [0, T] \to X$ *satisfies*

$$x_{\beta g} \in K, \quad (A_{\beta g}(t)x_{\beta}(t), y - x_{\beta g}(t))_X + j_{\beta g}(x_{\beta g}(t), y(t)) - \tag{4.19}$$

$$-j_{\beta g}(x_{\beta g}(t), x_{\beta g}(t)) \geqslant (\mathbf{f}(\mathbf{t}), y - x_{\beta g}(t))_X \quad \forall y \in K, \textit{ for all } t \in [0, T].$$

Proof. Let $x_{\beta g} = (\boldsymbol{u}_{\beta g}, \varphi_{\beta g}) : [0, T] \to V \times W$ be a solution to Problem $\mathcal{P}_{\beta g}$. Let $y = (\boldsymbol{v}, \psi) \in K$ and let $t \in [0, T]$. We use the test function $\psi - \varphi_{\beta g}(t)$ in (4.12), add the corresponding inequality to (4.11), and use (4.13)-(4.18) to obtain (4.19). Conversely, assume that $x_{\beta g} = (\boldsymbol{u}_{\beta g}, \varphi_{\beta g}) : [0, T] \to X$ satisfies (4.19) and let $t \in [0, T]$. For any $\boldsymbol{v} \in U_{ad}$, we take $y = (\boldsymbol{v}, \varphi_{\beta g}(t))$ in (4.19) to obtain (4.11). Then, for any $\psi \in W$, we take successively $y = (\boldsymbol{u}_{\beta g}, \varphi_{\beta g}(t) + \psi)$ and $y = (\boldsymbol{u}_{\beta g}, \varphi_{\beta g}(t) - \psi)$ in (4.19) to obtain (4.12). □

We use now Lemma 4.4 to obtain the following existence and uniqueness result.

Lemma 4.5. *There exists* $\mu_0 > 0$ *depending only on* $\Omega, \Gamma_1, \Gamma_3, \mathcal{F}$, $\mathcal{B}$ *and* p *such that, if* $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$, *Problem* $\mathcal{P}_{\beta g}$ *has a unique solution* $(\boldsymbol{u}_{\beta g}, \varphi_{\beta g}) \in C([0, T]; V \times W)$.

Proof. We apply Theorem 4.2 where $X = V \times W$ and $Y = K = U_{ad} \times W$. Let $t \in [0, T]$.We use (3.6)-(3.9), (3.28), and (3.29) to see that $A_{\beta g}(t)$ is a strongly monotone Lipschitz continuous operator on X and it satisfies

$$(A_{\beta g}(t)x_1(t) - A_{\beta g}(t)x_2(t), x_1(t) - x_2(t))_X \geqslant \min(m_{\mathcal{F}}, m_{\mathcal{B}})||x_1(t) - x_2(t)||_X. \tag{4.20}$$

Using (3.20), we can easily check that $j_{\beta g}(x,.)$ is a continuous seminorm on X and moreover, it satisfies (3.30) and(3.31) which shows that the functional $j_{\beta g}$ satisfies condition (4.8) on $.X$. By (3.14) and (4.18) it is easy to see that the function $\mathbf{f}$ defined by (4.18) satisfies $\mathbf{f}(t) \in X$.

Let

$$\mu_0 = \frac{\min(m_{\mathcal{F}}, m_{\mathcal{B}})}{c_0^2 M},$$

where μ, $m_{\mathcal{F}}$, $m_{\mathcal{B}}$, c_0 and M are given in (2.8), (3.6), (3.8), (3.3) and (3.10), respectively. We note that μ_0 depends on $\Omega, \Gamma_1, \Gamma_3, \mathcal{F}$, $\mathcal{B}$ *and* p. Assume that $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$, then

$$c_0^2 M \|\mu\|_{L^\infty(\Gamma_3)} < \min(m_{\mathcal{F}}, m_{\mathcal{B}}), \tag{4.21}$$

and note that this smallness assumption involves only the geometry and the electrical part, and does not depend on the mechanical data of problem.

Using (3.30), (3.31), 4.20, the existence and uniqueness part in Lemma 4.5 is now a consequence of Lemma 4.4 and theorem 4.2.

For $t_1, t_2 \in [0, T]$, an argument based on (3.6), (3.28) and (3.30) shows that

$$\begin{aligned} ||x_{\beta g}(t_2) - x_{\beta g}(t_1)||_X \leq \frac{c}{\min(m_{\mathcal{F}}, m_{\mathcal{B}}) - c_0^2 \|\mu\|_{L^\infty(\Gamma_3)} M} (||\beta(t_2) - \beta(t_1)||_{L^2(\Gamma_3)} + \\ + ||g(t_2) - g(t_1)||_{\mathcal{H}_1} + ||\mathbf{f}(t_2) - \mathbf{f}(t_1)||_X). \end{aligned} \tag{4.22}$$

The last inequality implies that

$$\begin{aligned} ||u(t_2) - u(t_1)||_V \quad &\leq \quad \frac{c}{m_{\mathcal{F}} - c_0^2 \|\mu\|_{L^\infty(\Gamma_3)} M} (||\beta(t_2) - \beta(t_1)||_{L^2(\Gamma_3)} + \\ &+ ||g(t_2) - g(t_1)||_{\mathcal{H}_1} + ||\mathbf{f}(t_2) - \mathbf{f}(t_1)||_X). \end{aligned} \tag{4.23}$$

Keeping in mind that $\mathbf{f} \in W^{1,\infty}(0, T; X)$ and recall that $\beta \in C([0, T]; X)$, $g \in W^{1,\infty}(0, T; \mathcal{H}_1)$, it follows now from (4.22) that the mapping $t \to x_{\beta g} = (\boldsymbol{u}_{\beta g}, \varphi_{\beta g}) : [0, T] \to X$ is continuous. □

We assume in what follows that $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$ and therefore (4.21) is valid. In the next step, we use the displacement field $\boldsymbol{u}_{\beta g}$ obtained in Lemma 4.5, denote by $\boldsymbol{u}_{\beta g \nu}$, $\boldsymbol{u}_{\beta g \tau}$ its normal and tangential components, and we consider the following initial value problem.

Problem 4.6 ($\mathcal{P}^{\theta}_{\beta g}$). *Find a bonding field* $\theta_{\beta g}$: $[0,T] \to L^2(\Gamma_3)$ *such that*

$$\dot{\theta}_{\beta g}(t) = -\Big(\theta_{\beta g}(t)\big(\gamma_\nu R_\nu(\boldsymbol{u}_{\beta g\nu}(t))^2 + \gamma_\tau \|R_\tau(\boldsymbol{u}_{\beta g\tau}(t))\|^2\big) - \epsilon_a\Big)_+$$

$$\text{a.e. } t \in (0,T), \tag{4.24}$$

$$\theta_{\beta g}(0) = \beta_0. \tag{4.25}$$

We obtain the following result.

Lemma 4.7. *There exists a unique solution to Problem* $\mathcal{P}^{\theta}_{\beta g}$ *and it satisfies* $\theta_{\beta g} \in W^{1,\infty}(0,T,L^2(\Gamma_3)) \cap \mathcal{Q}$

Proof. Consider the mapping $F_{\beta g} : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ defined by

$$F_{\beta g}(t,\theta) = -(\theta(t)(\gamma_\nu R_\nu((\boldsymbol{u}_{\beta g})_\gamma(t))^2 + \gamma_\tau \|R_\tau((\boldsymbol{u}_{\beta g})_\tau(t))\|^2) - \epsilon_a)_+, \tag{4.26}$$

for all $t \in [0,T]$ and $\theta \in L^2(\Gamma_3)$. It follows from the properties of the truncation operators R_ν and R_τ that F_β is Lipschitz continuous with respect to the second argument, uniformly in time. Moreover, for any $\theta \in L^2(\Gamma_3)$, the mapping $t \mapsto F_{\beta g}(t,\theta)$ belongs to $L^\infty(0,T;L^2(\Gamma_3))$. Using now a version of Cauchy-Lipschitz theorem, we obtain the existence of a unique function $\theta_{\beta g} \in W^{1,\infty}(0,T,L^2(\Gamma_3))$ which solves (4.24), 4.25). We note that the restriction $0 \le \beta \le 1$ is implicitly included in the variational problem $\mathcal{P}_{\mathbf{V}}$. Indeed, (3.23) and (3.24) guarantee that $\beta(t) \le \beta_0$ and, therefore, assumption (3.17) shows that $\beta(t) \le 1$ for $t \ge 0$, a.e. on Γ_3. On the other hand, if $\beta(t_0) = 0$ at $t = t_0$, then it follows from (3.23) and (3.24) that $\dot{\beta}(t) = 0$ for all $t \ge t_0$ and therefore, $\beta(t) = 0$ for all $t \ge 0$, a.e. on Γ_3. We conclude that $0 \le \beta(t) \le 1$ for all $t \in [0,T]$, a.e. on Γ_3. Therefore, from the definition of the set $\mathcal{Q}$, we find that $\theta_{\beta g} \in \mathcal{Q}$, which concludes the proof of Lemma. □

It follows from Lemma 4.7 that for all $\beta \in \mathcal{L}$ and $g \in W^{1,\infty}(0,T,\mathcal{H}_1)$ the solution $\theta_{\beta g}$ of Problem $\mathcal{P}^{\theta}_{\beta g}$ belongs to $\mathcal{L} \times W^{1,\infty}(0,T,L^2(\Gamma_3))$, see (4.10).

We denote now by $\sigma_{\beta g}$ the tensor given by

$$\sigma_{\beta g} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_{\beta g}) + \mathcal{E}^*\nabla(\varphi_{\beta g}). \tag{4.27}$$

From see (3.6), (3.6) and Lemma 4.5, its follows that $\sigma_{\beta g} \in C(0,T,\mathcal{H}_1)$. Therefore, we may consider the operator $\Lambda : \mathcal{L} \times C(0,T,L^2(\Gamma_3) \times \mathcal{H}_1) \to \mathcal{L} \times C(0,T,L^2(\Gamma_3) \times \mathcal{H}_1)$

given by

$$\Lambda(\beta, g) = (\theta_{\beta g}, \sigma_{\beta g}). \tag{4.28}$$

The third step consists in the following result.

Lemma 4.8. *There exists a unique element* $(\beta^*, g^*) \in \mathcal{L} \times C(0, T, L^2(\Gamma_3) \times \mathcal{H}_1)$ *such that* $\Lambda(\beta^*, g^*) = (\beta^*, g^*)$.

Proof. Suppose that (β_i, g_i) are two couples of functions in $\mathcal{L} \times W^{1,\infty}(0, T, L^2(\Gamma_3) \times \mathcal{H}_1)$ and denote by $(\boldsymbol{u}_i, \varphi_i)$, θ_i the functions obtained in Lemmas 4.5 and 4.7, respectively, for (β, g)=(β_i, g_i), $i = 1, 2$. Let $t \in [0, T]$. We use arguments similar to those used in the proof of (4.22) to deduce that

$$||x_{\beta_1 g_1}(t_2) - x_{\beta_2 g_2}(t_1)||_X \leq$$
$$\leq \frac{c}{\min(m_{\mathcal{F}}, m_{\mathcal{B}}) - c_0^2\|\mu\|_{L^\infty(\Gamma_3)} M}(||\beta_2(t) - \beta_1(t)||_{L^2(\Gamma_3)} + ||g_2(t) - g_1(t)||_{\mathcal{H}_1}), \tag{4.29}$$

which implies

$$||\boldsymbol{u}_2(t) - \boldsymbol{u}_1(t))||_V \leq \frac{c}{m_{\mathcal{F}} - c_0^2\|\mu\|_{L^\infty(\Gamma_3)} M}(||\beta_2(t) - \beta_1(t)||_{L^2(\Gamma_3)} + ||g_2(t) - g_1(t)||_{\mathcal{H}_1}). \tag{4.30}$$

On the other hand, it follows from (4.24) and (4.25) that

$$\theta_i(t) = \beta_0 - \int_0^t (\theta_i(s)(\gamma_\nu R_\nu(\boldsymbol{u}_{i\nu}(s))^2 + \gamma_\tau \|R_\tau(\boldsymbol{u}_{i\tau}(s))\|^2) - \epsilon_a)_+ ds \tag{4.31}$$

and then

$$||\theta_2(t) - \theta_1(t)||_{L^2(\Gamma_3)} \leq c(\int_0^t ||\theta_2(s) R_\nu(\boldsymbol{u}_{2\nu}(s))^2 - \theta_1(s) R_\nu(\boldsymbol{u}_{1\nu}(s))^2||_{L^2(\Gamma_3)} ds +$$
$$+ \int_0^t ||\theta_2(s)||R_\tau(\boldsymbol{u}_{2\tau}(s))^{||2} - \theta_1(s)||R_\tau(\boldsymbol{u}_{1\tau}(s))||^2||_{L^2(\Gamma_3)} ds). \tag{4.32}$$

Using the definition of R_ν and R_τ and writing , $\theta_1 = \theta_1 - \theta_2 + \theta_2$ we get

$$||\theta_2(t) - \theta_1(t)||_{L^2(\Gamma_3)} \leq c(\int_0^t ||\theta_2(s) - \theta_1(s)||_{L^2(\Gamma_3)} ds + \int_0^t ||\boldsymbol{u}_2(s)) - \boldsymbol{u}_1(s))||_{L^2(\Gamma_3)} ds). \tag{4.33}$$

By Gronwall's inequality, it follows that

$$||\theta_2(t) - \theta_1(t)||_{L^2(\Gamma_3)} \leq c \int_0^t ||\boldsymbol{u}_2(s)) - \boldsymbol{u}_1(s))||_{L^2(\Gamma_3)} ds \tag{4.34}$$

and, using (3.3), we obtain

$$||\theta_2(t) - \theta_1(t)||_{L^2(\Gamma_3)} \leq c \int_0^t ||\boldsymbol{u}_2(s)) - \boldsymbol{u}_1(s))||_V ds. \tag{4.35}$$

We now combine (4.30) and (4.35) to see that

$$||\theta_2(t) - \theta_1(t)||_{L^2(\Gamma_3)} \leq$$

$$\leq \frac{c}{\min(m_{\mathcal{F}}, m_{\mathcal{B}}) - c_0^2\|\mu\|_{L^\infty(\Gamma_3)}M} \int_0^t (||\beta_2(t) - \beta_1(t)||_{L^2(\Gamma_3)} + ||g_2(t) - g_1(t)||_{\mathcal{H}_1})ds. \tag{4.36}$$

Using now (3.6), (3.7) and (4.27) (4.29) it is easy to see that

$$\|\sigma_{\beta_1 g_1}(t) - \sigma_{\beta_2 g_2}(t)|| \leq \frac{c}{\min(m_{\mathcal{F}}, m_{\mathcal{B}}) - c_0^2\|\mu\|_{L^\infty(\Gamma_3)}M}(||\beta_2(t) - \beta_1(t)\| + \|g_2(t) - g_1(t)\|). \tag{4.37}$$

From (4.28), (4.30) and the last inequality, it results that

$$||\Lambda(\beta_1, g_1)(t) - \Lambda(\beta_2, g_2)(t)||_{L^2(\Gamma_3)\times\mathcal{H}_1} \leq N||(\beta_1, g_1)(t) - (\beta_2, g_2)(t)||_{L^2(\Gamma_3)\times\mathcal{H}_1} +$$

$$+c \int_0^t ||(\beta_1, g_1)(t) - (\beta_2, g_2)(t)||_{L^2(\Gamma_3)\times\mathcal{H}_1} ds. \tag{4.38}$$

such that :

$$N = \frac{c}{\min(m_{\mathcal{F}}, m_{\mathcal{B}}) - c_0^2\|\mu\|_{L^\infty(\Gamma_3)}M}. \tag{4.39}$$

Using the following notations

$$\begin{aligned} I_0(t) &= ||(\beta_1, g_1)(t) - (\beta_2, g_2)(t)||_{L^2(\Gamma_3)\times\mathcal{H}_1}, \qquad (4.40)\\ I_1(t) &= \int_0^t ||(\beta_1, g_1)(t) - (\beta_2, g_2)(t)||_{L^2(\Gamma_3)\times\mathcal{H}_1} ds,\\ I_k(t) &= \int_0^t \int_0^{s_{k-1}} ... \int_0^{s_1} ||(\beta_1, g_1)(t) - (\beta_2, g_2)(t)||_{L^2(\Gamma_3)\times\mathcal{H}_1} dr ds_1 ... ds_{k-1}, \ \forall k \geqslant 2, \end{aligned}$$

and denoting now by Λ^p the powers of operator Λ, (4.38) and (4.40) imply by recurrence that

$$\|\Lambda^p(\beta_1, g_1)(t) - \Lambda^p(\beta_2, g_2)(t)\| \leq (\sum_{k=0}^{p} C_p^k \frac{N^{p-k} M^p T^p}{p!})) \|(\beta_1, g_1)(t) - (\beta_2, g_2)(t)\|$$

$$\leq \frac{(Np + MT)^p}{p!} \|(\beta_1, g_1)(t) - (\beta_2, g_2)(t)\|_{L^2(\Gamma_3)\times\mathcal{H}_1}. \tag{4.41}$$

Using the Stirling's formula, we obtain under the condition $N \leq \dfrac{1}{e}$ that

$$\lim_{p\to\infty} \frac{(Np+MT)^p}{p!} = 0,$$

which shows that for p sufficiently large $\Lambda^p : \mathcal{L} \times C(0,T,\mathcal{H}_1) \rightarrow \mathcal{L} \times C(0,T,\mathcal{H}_1)$ is a contraction. Then, we conclude by using the Banach fixed point theorem that Λ has a unique fixed point $(\beta^*, g^*) \in \mathcal{L} \times C(0,T,\mathcal{H}_1)$ such that $\Lambda(\beta^*, g^*) = (\beta^*, g^*)$. Hence, from (4.28) it results for all $t \in [0,T]$,

$$(\beta^*, g^*)(t) = (\theta_{\beta^* g^*}(t), \sigma_{\beta^* g^*}(t)). \tag{4.42}$$

□

Now, we have all the ingredients to provide the proof of Theorem 4.1.

Proof of Theorem 4.1. Existence. Let $(\beta^*, g^*) \in \mathcal{L} \times C(0,T,\mathcal{H}_1)$ be the fixed point of Λ and let $(\boldsymbol{u}^*, \varphi^*)$ be the solution of Problem $\mathcal{P}_{\beta g}$V for $(\beta, g) = (\beta^*, g^*)$, that is, $\boldsymbol{u}^* = \boldsymbol{u}_{\beta^* g^*}$ and $\varphi^* = \varphi_{\beta^* g^*}$. Since $\theta_{\beta^* g^*} = \beta^*$, we conclude by (4.11)), (4.12), (4.24) and (4.25) that $(\boldsymbol{u}^*, \varphi^*, \beta^*)$ is a solution of Problem $\mathcal{P}^V$ and, moreover, β^* satisfies the regularity (4.3). Also, since $\beta^* = \theta_{\beta^*} \in W^{1,\infty}(0,T; L^2(\Gamma_3))$, $\sigma_{\beta^* g^*} \in W^{1,\infty}(0,T,\mathcal{H}_1)$ and $\mathbf{f} \in W^{1,\infty}(0,T;X)$, inequality (4.22) implies that the function $x^* = (\boldsymbol{u}^*, \varphi^*) : [0,T] \rightarrow X$ is Lipschitz continuous; therefore, x^* belongs to $W^{1,\infty}(0,T;X)$, which shows that the functions x^* and φ^* have the regularity expressed in (4.1), (4.2).

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of operator Λ defined by (4.28). Indeed, let $(\boldsymbol{u}, \varphi, \beta)$ be a another solution of Problem $\mathcal{P}^V$ which satisfies (4.1)-(4.3).
We denote by $(\beta, g) \in W^{1,\infty}(0,T, L^2(\Gamma_3) \times \mathcal{H}_1)$ the couple of function defined by

$$\dot{\beta}(t) = -(\beta(t)\,(\gamma_\nu R_\nu(\boldsymbol{u}_\nu(t))^2 + \gamma_\tau \|R_\tau(\boldsymbol{u}_\tau(t))\|^2) - \epsilon_a)_+ \text{ a.e. } t \in (0,T), \tag{4.43}$$

$$\beta(0) = \beta_0, \tag{4.44}$$

$$g(t) = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mathcal{E}^*\nabla(\varphi(t)). \tag{4.45}$$

It follows from (4.11), (4.12) that $(\boldsymbol{u}, \varphi)$ is a solution to Problem $\mathcal{P}_{\beta g}$ and, since by Lemma 4.5 this problem has a unique solution denoted by $(\boldsymbol{u}_{\beta g}, \varphi_{\beta g})$, we obtain

$$\boldsymbol{u} = \boldsymbol{u}_{\beta g}, \tag{4.46}$$

$$\varphi = \varphi_{\beta g}. \tag{4.47}$$

Then, we replace $\boldsymbol{u} = \boldsymbol{u}_{\beta g}$ in (3.23) and use the initial condition (3.24) to see that β is a solution to Problem $\mathcal{P}^{\theta}_{\beta g}$. Since by Lemma 4.7 this last problem has a unique solution denoted by $\theta_{\beta g}$, we find

$$\beta = \theta_{\beta g}. \tag{4.48}$$

We use now (4.28), (4.48) and Lemma 4.8, it follows that

$$\beta = \beta^*. \tag{4.49}$$

On a other hand, it follows from (4.46), (4.45), (4.46), (4.46) and Lemma 4.8 that

$$g = g^* \tag{4.50}$$

The uniqueness part of the theorem is now a consequence of (4.46), (4.47), (4.49) and the last inequality. □

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