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HARMONIC MULTIVALENT FUNCTIONS DEFINED BY INTEGRAL OPERATOR

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Abstract. We define and investigate a new class of harmonic multivalent functions defined by integral operator. We obtain coefficient inequalities, extreme points and distortion bounds for the functions in our classes.

1. Introduction

A continuous complex-valued function $f = u+iv$ defined in a complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain we can write $f = h + \overline{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$, $z \in D$. (See Clunie and Sheil-Small [2]).

Denote by H the class of functions $f = h+\overline{g}$ that are harmonic univalent and sense preserving in the unit disc $U = \{z : \ |z| < 1\}$ so that $f = h + \overline{g}$ is normalized by $f(0) = h(0) = f'_z(0) - 1 = 0$.

Recently, Ahuja and Jahangiri [5] defined the class $H_p(n)$ $(p, n \in \mathbb{N})$, consisting of all p-valent harmonic functions $f = h + \overline{g}$ that are sense preserving in U and h and g are of the form

$$h(z) = z^p + \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1}, \quad g(z) = \sum_{k=1}^{\infty} b_{k+p-1} z^{k+p-1}, \quad |b_p| < 1. \tag{1.1}$$

The integral operator I^n is defined (see [4], for $p = 1$) by:

(i) $I^0 f(z) = f(z)$;

(ii) $I^1 f(z) = If(z) = p \int_0^z f(t)t^{-1}dt$;

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(iii) $I^n f(z) = I(I^{n-1} f(z))$, $n \in \mathbb{N}$, $f \in \mathcal{A}$,

where $\mathcal{A} = \{f \in \mathcal{H} : \ f(z) = z + a_2 z^2 + \dots\}$ and $\mathcal{H} = \mathcal{H}(U)$.

For $f = h + \overline{g}$ given by (1.1) the integral operator of f is defined as

$$I^n f(z) = I^n h(z) + (-1)^n \overline{I^n g(z)}, \quad p > n \tag{1.2}$$

where

$$I^n h(z) = z^p + \sum_{k=2}^{\infty} \left(\frac{p}{k+p-1} \right)^n a_{k+p-1} z^{k+p-1}$$

and

$$I^n g(z) = \sum_{k=1}^{\infty} \left(\frac{p}{k+p-1} \right)^n b_{k+p-1} z^{k+p-1}.$$

For $0 \le \alpha < 1$, $n \in \mathbb{N}$, $z \in U$, let $H_p(n, \alpha)$ denote the family of harmonic functions f of the form (1.1) such that

$$\operatorname{Re} \left(\frac{I^n f(z)}{I^{n+1} f(z)} \right) > \alpha, \tag{1.3}$$

where I^n is defined by (1.2).

The families $H_p(m, n, \alpha)$ and $H_p^-(m, n, \alpha)$ include a variety of well-known classes of harmonic functions as well as many new ones. For example $HS(\alpha) = \overline{H_1}(1, 0, \alpha)$ is the class of sense-preserving,harmonic univalent functions f which are starlike of order $\alpha \in U$, and $HK(\alpha) = \overline{H_1}(2, 1, \alpha)$ is the class of sense-preserving,harmonic univalent functions f which are convex of order α in U, and $\overline{H_1}(n+1, n, \alpha) = \overline{H}(n, \alpha)$ is the class of Sălăgean-type harmonic univalent functions.

Let we denote the subclass $H_p^-(n, \alpha)$ consists of harmonic functions $f_n = h + \overline{g}_n$ in $H_p^-(n, \alpha)$ so that h and g_n are of the form

$$h(z) = z^p - \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1} \quad \text{and} \quad g_n(z) = (-1)^{n-1} \sum_{k=1}^{\infty} b_{k+p-1} z^{k+p-1} \tag{1.4}$$

where $a_{k+p-1}, b_{k+p-1} \ge 0$, $|b_p| < 1$.

For the harmonic functions f of the form (1.1) with $b_1 = 0$, Avei and Zlotkiewich in [1] show that if

$$\sum_{k=2}^{\infty} k(|a_k| + |b_k|) \le 1,$$

then $f \in SH(0)$, where $HS(0) = \overline{H_1}(1,0,0)$ and if

$$\sum_{k=2}^{\infty} k^2(|a_k| + |b_k|) \leq 1$$

then $f \in HK(0)$, where $HK(0) = \overline{H_1}(2,1,0)$.

For the harmonic functions f of the form (1.4) with $n = 0$, Jahongiri in [3] showed that $f \in HS(\alpha)$ if and only if

$$\sum_{k=2}^{\infty}(k-\alpha)|a_k| + \sum_{k=1}^{\infty}(k+\alpha)|b_k| \leq 1-\alpha$$

and $f \in \overline{H_1}(2,1,\alpha)$ if and only if

$$\sum_{k=2}^{\infty} k(k-\alpha)|a_k| + \sum_{k=1}^{\infty} k(k+\alpha)|b_k| \leq 1-\alpha.$$

2. **Main results**

In our first theorem, we deduce a sufficient coefficient bound for harmonic functions in $H_p(n,\alpha)$.

Theorem 2.1. *Let $f = h + \overline{g}$ be given by (1.1). If*

$$\sum_{k=1}^{\infty}\{\psi(n,p,k,\alpha)|a_{k+p-1}| + \theta(n,p,k,\alpha)|b_{k+p-1}|\} \leq 2 \tag{2.1}$$

where

$$\psi(n,p,k,\alpha) = \frac{\left(\frac{p}{k+p-1}\right)^n - \alpha\left(\frac{p}{k+p-1}\right)^{n+1}}{1-\alpha}$$

$$\theta(n,p,k,\alpha) = \frac{\left(\frac{p}{k+p-1}\right)^n + \alpha\left(\frac{p}{k+p-1}\right)^{n+1}}{1-\alpha},$$

$$a_p = 1, \quad 0 \leq \alpha < 1, \quad n \in \mathbb{N}.$$

Then f is sense preserving in U and $f \in H_p(n,\alpha)$.

Proof. According to (1.2) and (1.3) we only need to show that

$$\operatorname{Re}\left(\frac{I^n f(z) - \alpha I^{n+1} f(z)}{I^{n+1} f(z)}\right) \geq 0.$$

The case $r = 0$ is obvious.

For $0 < r < 1$, it follows that

$$\operatorname{Re}\left(\frac{I^n f(z) - \alpha I^{n+1} f(z)}{I^{n+1} f(z)}\right)$$

$$= \operatorname{Re}\left\{\frac{z^p(1-\alpha) + \sum_{k=2}^{\infty}\left[\left(\frac{p}{k+p-1}\right)^n - \alpha\left(\frac{p}{k+p-1}\right)^{n+1}\right] a_{k+p-1} z^{k+p-1}}{z^p + \sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1} a_{k+p-1} z^{k+p-1} + (-1)^{n+1}\sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1} \overline{b}_{k+p-1} \overline{z}^{k+p-1}}\right.$$

$$\left. + \frac{(-1)^n \sum_{k=1}^{\infty}\left[\left(\frac{p}{k+p-1}\right)^n + \alpha\left(\frac{p}{k+p-1}\right)^{n+1}\right] \overline{b}_{k+p-1} \overline{z}^{k+p-1}}{z^p + \sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1} a_{k+p-1} z^{k+p-1} + (-1)^{n+1}\sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1} \overline{b}_{k+p-1} \overline{z}^{k+p-1}}\right\}$$

$$= \operatorname{Re}\left\{\frac{(1-\alpha) + \sum_{k=2}^{\infty}\left[\left(\frac{p}{k+p-1}\right)^n - \alpha\left(\frac{p}{k+p-1}\right)^{n+1}\right] a_{k+p-1} z^{k-1}}{1 + \sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1} a_{k+p-1} z^{k-1} + (-1)^{n+1}\sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1} \overline{b}_{k+p-1} \overline{z}^{k+p-1} z^{-p}}\right.$$

$$\left. + \frac{(-1)^n \sum_{k=1}^{\infty}\left[\left(\frac{p}{k+p-1}\right)^n + \alpha\left(\frac{p}{k+p-1}\right)^{n+1}\right] \overline{b}_{k+p-1} \overline{z}^{k+p-1} z^{-p}}{1 + \sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1} a_{k+p-1} z^{k-1} + (-1)^{n+1}\sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1} \overline{b}_{k+p-1} \overline{z}^{k+p-1} z^{-p}}\right\}$$

$$= \operatorname{Re}\left[\frac{(1-\alpha) + A(z)}{1 + B(z)}\right].$$

For $z = re^{i\theta}$ we have

$$A(re^{i\theta}) = \sum_{k=2}^{\infty}\left[\left(\frac{p}{k+p-1}\right)^n - \alpha\left(\frac{p}{k+p-1}\right)^{n+1}\right] a_{k+p-1} r^{k-1} e^{(k-1)\theta i}$$

$$+(-1)^n \sum_{k=1}^{\infty}\left[\left(\frac{p}{k+p-1}\right)^n + \alpha\left(\frac{p}{k+p-1}\right)^{n+1}\right] \overline{b}_{k+p-1} r^{k-1} e^{-(k+2p-1)\theta i};$$

$$B(re^{i\theta}) = \sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1} a_{k+p-1} r^{k-1} e^{(k-1)\theta i}$$

$$+(-1)^{n+1}\sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1} \overline{b}_{k+p-1} r^{k-1} e^{-(k+2p-1)\theta i}.$$

Setting

$$\frac{(1-\alpha) + A(z)}{1 + B(z)} = (1-\alpha)\frac{1 + w(z)}{1 - w(z)},$$

the proof will be complete if we can show that $|w(z)| \leq 1$. This is the case since, by the condition (2.1), we can write

$$|w(z)| = \left| \frac{A(z) - (1-\alpha)B(z)}{A(z) + (1-\alpha)B(z) + 2(1-\alpha)} \right|$$

$$= \left| \frac{\sum_{k=2}^{\infty} \left[\left(\frac{p}{k+p-1} \right)^n - \left(\frac{p}{k+p-1} \right)^{n+1} \right] a_{k+p-1} r^{k-1} e^{(k-1)\theta i}}{2(1-\alpha) + \sum_{k=2}^{\infty} C(n,p,k,\alpha) a_{k+p-1} r^{k-1} e^{(k-1)\theta i} + (-1)^n \sum_{k=1}^{\infty} D(n,p,k,\alpha) \overline{b}_{k+p-1} r^{k-1} e^{-(k+2p-1)\theta i}} \right.$$

$$\left. + \frac{(-1)^n \sum_{k=1}^{\infty} \left[\left(\frac{p}{k+p-1} \right)^n + \left(\frac{p}{k+p-1} \right)^{n+1} \right] \overline{b}_{k+p-1} r^{k-1} e^{-(k+2p-1)\theta i}}{2(1-\alpha) + \sum_{k=2}^{\infty} C(n,p,k,\alpha) a_{k+p-1} r^{k-1} e^{(k-1)\theta i} + (-1)^n \sum_{k=1}^{\infty} D(n,p,k,\alpha) \overline{b}_{k+p-1} r^{k-1} e^{-(k+2p-1)\theta i}} \right|$$

$$\leq \frac{\sum_{k=2}^{\infty} \left[\left(\frac{p}{k+p-1} \right)^n - \left(\frac{p}{k+p-1} \right)^{n+1} \right] |a_{k+p-1}| r^{k-1}}{2(1-\alpha) - \sum_{k=2}^{\infty} C(n,p,k,\alpha) |a_{k+p-1}| r^{k-1} - \sum_{k=1}^{\infty} D(n,p,k,\alpha) |b_{k+p-1}| r^{k-1}}$$

$$+ \frac{\sum_{k=1}^{\infty} \left[\left(\frac{p}{k+p-1} \right)^n + \left(\frac{p}{k+p-1} \right)^{n+1} \right] |b_{k+p-1}| r^{k-1}}{2(1-\alpha) - \sum_{k=2}^{\infty} C(n,p,k,\alpha) |a_{k+p-1}| r^{k-1} - \sum_{k=1}^{\infty} D(n,p,k,\alpha) |b_{k+p-1}| r^{k-1}}$$

$$= \frac{\sum_{k=1}^{\infty} \left[\left(\frac{p}{k+p-1} \right)^n - \left(\frac{p}{k+p-1} \right)^{n+1} \right] |a_{k+p-1}| r^{k-1}}{4(1-\alpha) - \sum_{k=1}^{\infty} \{ C(n,p,k,\alpha) |a_{k+p-1}| + D(n,p,k,\alpha) |b_{k+p-1}| \} r^{k-1}}$$

$$+ \frac{\sum_{k=1}^{\infty} \left[\left(\frac{p}{k+p-1} \right)^n + \left(\frac{p}{k+p-1} \right)^{n+1} \right] |b_{k+p-1}| r^{k-1}}{4(1-\alpha) - \sum_{k=1}^{\infty} \{ C(n,p,k,\alpha) |a_{k+p-1}| + D(n,p,k,\alpha) |b_{k+p-1}| \} r^{k-1}}$$

$$< \frac{\sum_{k=1}^{\infty} \left[\left(\frac{p}{k+p-1} \right)^n - \left(\frac{p}{k+p-1} \right)^{n+1} \right] |a_{k+p-1}|}{4(1-\alpha) - \sum_{k=1}^{\infty} \{ C(n,p,k,\alpha) |a_{k+p-1}| + D(n,p,k,\alpha) |b_{k+p-1}| \}}$$

$$+\frac{\sum_{k=1}^{\infty}\left[\left(\frac{p}{k+p-1}\right)^{n}+\left(\frac{p}{k+p-1}\right)^{n+1}\right]|b_{k+p-1}|}{4(1-\alpha)-\sum_{k=1}^{\infty}\{C(n,p,k,\alpha)|a_{k+p-1}|+D(n,p,k,\alpha)|b_{k+p-1}|\}}\leq 1.$$

where

$$C(n,p,k,\alpha)=\left(\frac{p}{k+p-1}\right)^{n}+(1-2\alpha)\left(\frac{p}{k+p-1}\right)^{n+1}$$

and

$$D(n,p,k,\alpha)=\left(\frac{p}{k+p-1}\right)^{n}+(-1)(1-2\alpha)\left(\frac{p}{k+p-1}\right)^{n+1}$$

The harmonic univalent functions

$$f(z)=z^{p}+\sum_{k=2}^{\infty}\frac{1}{\psi(n,p,k,\alpha)}x_{k}z^{k+p-1}+\sum_{k=1}^{\infty}\frac{1}{\theta(n,p,k,\alpha)}\overline{y_{k}z^{k+p-1}}, \tag{2.2}$$

where $n\in\mathbb{N}$ and $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$, show that the coefficient bound given by (2.1) is sharp. The functions of the form (2.2) are in $H_p(n,\alpha)$ because

$$\sum_{k=1}^{\infty}\{\psi(n,p,k,\alpha)|a_{k+p-1}|+\theta(n,p,k,\alpha)|b_{k+p-1}|\}=1+\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=2.$$

In the following theorem it is show that the condition (2.1) is also necessary for functions $f_n=h+\overline{g}_n$, where h and g_n are of the form (1.4).

Theorem 2.2. *Let $f_n=h+\overline{g}_n$ be given by (1.4). Then $f_n\in H_p^{-}(n,\alpha)$ if and only if*

$$\sum_{k=1}^{\infty}\{\psi(n,p,k,\alpha)a_{k+p-1}+\theta(n,p,k,\alpha)b_{k+p-1}\}\leq 2, \tag{2.3}$$

where $a_p=1$, $0\leq\alpha<1$, $n\in\mathbb{N}$.

Proof. Since $H_p^{-}(n,\alpha)\subset H_p(n,\alpha)$, we only need to prove the "only if" part of the theorem. For functions f_n of the form (1.4), we note that the condition

$$\operatorname{Re}\left\{\frac{I^{n}f_n(z)}{I^{n+1}f_n(z)}\right\}>\alpha$$

is equivalent to

$$\operatorname{Re}\left\{\frac{(1-\alpha)z^p-\sum_{k=2}^{\infty}\left[\left(\frac{p}{k+p-1}\right)^n-\alpha\left(\frac{p}{k+p-1}\right)^{n+1}\right]a_{k+p-1}z^{k+p-1}}{z^p-\sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}a_{k+p-1}z^{k+p-1}+(-1)^{2n}\sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}b_{k+p}\overline{z}^{k+p-1}}\right. \tag{2.4}$$

$$\left.+\frac{(-1)^{2n-1}\sum_{k=1}^{\infty}\left[\left(\frac{p}{k+p-1}\right)^n+\alpha\left(\frac{p}{k+p-1}\right)^{n+1}\right]b_{k+p-1}\overline{z}^{k+p-1}}{z^p-\sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}a_{k+p-1}z^{k+p-1}+(-1)^{2n}\sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}b_{k+p-1}\overline{z}^{k+p-1}}\right\}\geq 0.$$

The above required condition (2.4) must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0\leq z=r<1$, we must have

$$\frac{(1-\alpha)-\sum_{k=2}^{\infty}\left[\left(\frac{p}{k+p-1}\right)^n-\alpha\left(\frac{p}{k+p-1}\right)^{n+1}\right]a_{k+p-1}r^{k-1}}{1-\sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}a_{k+p-1}r^{k-1}+\sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}b_{k+p-1}r^{k-1}}$$

$$+\frac{-\sum_{k=1}^{\infty}\left[\left(\frac{p}{k+p-1}\right)^n+\alpha\left(\frac{p}{k+p-1}\right)^{n+1}\right]b_{k+p-1}r^{k-1}}{1-\sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}a_{k+p-1}r^{k-1}+\sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^{n+1}b_{k+p-1}r^{k-1}}\geq 0. \tag{2.5}$$

If the condition (2.3) does not hold, then the expression in (2.5) is negative for r sufficiently close to 1. Hence there exist $z_0=r_0$ in $(0,1)$ for which the quotient in (2.5) is negative.

This contradicts the required condition for $f_n\in H_p^-(n,\alpha)$. So the proof is complete.

Next we determine the extreme points of the closed convex hull of $H_p^-(n,\alpha)$, denoted by $clcoH_p^-(n,\alpha)$.

Theorem 2.3. *Let f_n be given by (1.4). Then $f_n\in H_p^-(n,\alpha)$ if and only if*

$$f_n(z)=\sum_{k=1}^{\infty}[x_{k+p-1}h_{k+p-1}(z)+y_{k+p-1}g_{n_{k+p-1}}(z)],$$

where

$$h_p(z) = z^p, \quad h_{k+p-1}(z) = z^p - \frac{1}{\psi(n,p,k,\alpha)} z^{k+p-1}, \quad k = 2, 3, \dots$$

and

$$g_{n_{k+p-1}}(z) = z^p + (-1)^{n-1} \cdot \frac{1}{\theta(n,p,k,\alpha)} \overline{z}^{k+p-1}, \quad k = 1, 2, 3, \dots$$

$$x_{k+p-1} \geq 0, \quad y_{k+p-1} \geq 0, \quad x_p = 1 - \sum_{k=2}^{\infty} x_{k+p-1} - \sum_{k=1}^{\infty} y_{k+p-1}.$$

In particular, the extreme points of $H_p^-(n,\alpha)$ are $\{h_{k+p-1}\}$ and $\{g_{n_{k+p-1}}\}$.

Proof. For functions f_n of the form (2.1),

$$f_n(z) = \sum_{k=1}^{\infty} [x_{k+p-1} h_{k+p-1}(z) + y_{k+p-1} g_{n_{k+p-1}}(z)]$$

$$= \sum_{k=1}^{\infty} (x_{k+p-1} + y_{k+p-1}) z^p - \sum_{k=2}^{\infty} \frac{1}{\psi(n,p,k,\alpha)} x_{k+p-1} z^{k+p-1}$$

$$+ (-1)^{n-1} \sum_{k=1}^{\infty} \frac{1}{\theta(n,p,k,\alpha)} y_{k+p-1} \overline{z}^{k+p-1}.$$

Then

$$\sum_{k=2}^{\infty} \psi(n,p,k,\alpha) \left(\frac{1}{\psi(n,p,k,\alpha)} x_{k+p-1} \right) + \sum_{k=1}^{\infty} \theta(n,p,k,\alpha) \left(\frac{1}{\theta(n,p,k,\alpha)} y_{k+p-1} \right)$$

$$= \sum_{k=2}^{\infty} x_{k+p-1} + \sum_{k=1}^{\infty} y_{k+p-1} = 1 - x_p \leq 1,$$

and so $f_n(z) \in clcoH_p^-(n,\alpha)$.

Conversely, suppose $f_n(z) \in clcoH_p^-(n,\alpha,\beta)$. Letting

$$x_p = 1 - \sum_{k=2}^{\infty} x_{k+p-1} - \sum_{k=1}^{\infty} y_{k+p-1},$$

let

$$x_{k+p-1} = \psi(n,p,k,\alpha) a_{k+p-1}$$

and

$$y_{k+p-1} = \theta(n,p,k,\alpha) b_{k+p-1}, \quad k = 2, 3, \dots$$

We obtain the required representation, since

$$f_n(z) = z^p - \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1} + (-1)^{n-1} \sum_{k=1}^{\infty} b_{k+p-1} \overline{z}^{k+p-1}$$

$$= z^p - \sum_{k=2}^{\infty} \frac{1}{\psi(n,p,k,\alpha)} x_{k+p-1} z^{k+p-1} + (-1)^{n-1} \sum_{k=1}^{\infty} \frac{1}{\theta(n,p,k,\alpha)} y_{k+p-1} \overline{z}^{k+p-1}$$

$$= z^p - \sum_{k=2}^{\infty} [z^p - h_{k+p-1}(z)] x_{k+p-1} - \sum_{k=1}^{\infty} [z^p - g_{n_{k+p-1}}(z)] y_{k+p-1}$$

$$= \left[1 - \sum_{k=2}^{\infty} x_{k+p-1} - \sum_{k=1}^{\infty} y_{k+p-1}\right] z^p + \sum_{k=2}^{\infty} x_{k+p-1} h_{k+p-1}(z)$$

$$+ \sum_{k=1}^{\infty} y_{k+p-1} g_{n_{k+p-1}}(z) = \sum_{k=1}^{\infty} [x_{k+p-1} h_{k+p-1}(z) + y_{k+p-1} g_{n_{k+p-1}}(z)].$$

The following theorem gives the distortion bounds for functions in $H_p^-(n,\alpha)$ which yields a covering results for this class.

Theorem 2.4. *Let $f_n \in H_p^-(n,\alpha)$. Then for $|z| = r < 1$ we have*

$$|f_n(z)| \leq (1 + b_p) r^p + \{\phi(n,p,k,\alpha) - \Omega(n,p,k,\alpha) b_p\} r^{p+1}$$

and

$$|f_n(z)| \geq (1 - b_p) r^p - \{\phi(n,p,k,\alpha) - \Omega(n,p,k,\alpha) b_p\} r^{p+1},$$

where

$$\phi(n,p,k,\alpha) = \frac{1-\alpha}{\left(\frac{p}{p+1}\right)^n - \alpha \left(\frac{p}{p+1}\right)^{n+1}},$$

$$\Omega(n,p,k,\alpha) = \frac{1+\alpha}{\left(\frac{p}{p+1}\right)^n - \alpha \left(\frac{p}{p+1}\right)^{n+1}}.$$

Proof. We prove the right hand side inequality for $|f_n|$. The proof for the left hand inequality can be done using similar arguments. Let $f_n \in H_p^-(n,\alpha)$. Taking the absolute value of f_n then by Theorem 2.2, we obtain:

$$|f_n(z)| = \left| z^p - \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1} + (-1)^{n-1} \sum_{k=1}^{\infty} b_{k+p-1} \overline{z}^{k+p-1} \right|$$

$$\leq r^p + \sum_{k=2}^{\infty} a_{k+p-1} r^{k+p-1} + \sum_{k=1}^{\infty} b_{k+p-1} r^{k+p-1}$$

$$= r^p + b_p r^p + \sum_{k=2}^{\infty}(a_{k+p-1} + b_{k+p-1})r^{k+p-1}$$

$$\leq r^p + b_p r^p + \sum_{k=2}^{\infty}(a_{k+p-1} + b_{k+p-1})r^{p+1}$$

$$= (1 + b_p)r^p + \phi(n,p,k,\alpha)\sum_{k=2}^{\infty}\frac{1}{\phi(n,p,k,\alpha)}(a_{k+p-1} + b_{k+p-1})r^{p+1}$$

$$\leq (1 + b_p)r^p + \phi(n,p,k,\alpha)r^{p+1}\left[\sum_{k=2}^{\infty}\psi(n,p,k,\alpha)a_{k+p-1} + \theta(n,p,k,\alpha)b_{k+p-1}\right]$$

$$\leq (1 + b_p)r^p + \{\phi(n,p,k,\alpha) - \Omega(n,p,k,\alpha)b_p\}r^{p-1}.$$

The following covering result follows from the left hand inequality in Theorem 2.4.

Corollary 2.1. *Let* $f_n \in H_p^-(n,\alpha)$*, the for* $|z| = r < 1$ *we have*

$$\{w: \ |w| < 1 - b_p - [\phi(n,p,k,\alpha) - \Omega(n,p,k,\alpha)b_p] \subset f_b(U)\}.$$

Similar results was obtained in [6] by Bilal Şekel and Sevtap Sümer Eker for the differential operator of Sălăgean defined in [4].

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