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ON MIXED NONLINEAR INTEGRAL EQUATIONS OF VOLTERRA-FREDHOLM TYPE WITH MODIFIED ARGUMENT

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Abstract. In the present paper we consider the following mixed Volterra-Fredholm nonlinear integral equation with modified argument:

$$u(t,x)=g\big(t,x,u(t,x)\big)+\int_0^t H\left(t,x,s,u(s,x)\right)ds$$
$$+\int_0^t\int_a^b K\Big(t,x,s,y,u(s,y),u\big(\varphi_1(s,y),\varphi_2(s,y)\big)\Big)dyds$$

For this equation, we will study: the existence and the uniqueness of the solution, the data dependence of the solution and the differentiability of the solution with respect to parameters.

1. **Introduction**

Let $(X,\|\cdot\|_X)$ be a Banach space.

In this paper we consider the following nonlinear integral equation of Volterra-Fredholm type:

$$u(t,x)=g\big(t,x,u(t,x)\big)+\int_0^t H\left(t,x,s,u(s,x)\right)ds$$
$$+\int_0^t\int_a^b K\Big(t,x,s,y,u(s,y),u\big(\varphi_1(s,y),\varphi_2(s,y)\big)\Big)dyds \tag{1}$$

for all $(t,x)\in[0,T]\times[a,b]:=\overline{D}$; $u\in C(\overline{D},\mathbb{R}^m)$, where $b>a>0$ and $T>0$.

Volterra-Fredholm (VF on short) integral equations often arise from the mathematical modelling of the spreading, in space and time, of some contagious diseases, in the theory of nonlinear parabolic boundary value problem and in many physical and

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biological models.

Most results for VF equation establish numerical approximation of the solutions; e.g. [8], [9], [22], [2], [11], [3], [7].

In [21] H. R. Thieme considered a model for the spatial spread of an epidemic consisting of a nonlinear integral equation of Volterra-Fredholm type which has an unique solution. The author showed that this solution has a temporally asymptotic limit which describes the final state of the epidemic and is the minimal solution of another nonlinear integral equation.

In [4] O. Diekmann described, derived and analysed a model of spatio-temporal development of an epidemic. The model considered leads (see [13]) to the following nonlinear integral equation of Volterra-Fredholm type:

$$u(t,x) = g(t,x) + \int_0^t \int_\Omega g(u(t-\tau,\xi))S_0(\xi)A(\tau,x,\xi)d\xi d\tau \tag{2}$$

for all $(t,x) \in [0,\infty] \times \Omega$, where Ω is a bounded domain in $\mathbb{R}^n$.

In [13] B. G. Pachpatte considered the integral equation

$$u(t,x) = g(t,x) + \int_0^t \int_\Omega g(t,x,s,y,u(s,y))dyds \tag{3}$$

for all $(t,x) \in [0,T] \times \Omega = D$, where Ω is a bounded domain in $\mathbb{R}^n$. Using Contraction Principle, the author proved that, under appropriate assumptions, (3) has a unique solution in a subset S of $C(D,\mathbb{R}^n)$. The result was then applied to show the existence and uniqueness of solutions to certain nonlinear parabolic differential equations and mixed Volterra-Fredholm integral equations occurring in specific physical and biological problems (e.g. a reliable treatment of the Diekmann's model mentioned above is given).

In [10], D. Mangeron and L. E. Krivošein obtained existence, uniqueness and stability conditions for the solutions of a class of boundary problems for linear and nonlinear heat equation with delay. Under certain conditions, this problem is equivalent with the following nonlinear VF equation:

$$u(t,x) = n(t,x) + \int_0^t \int_0^a \Big[G(x,\xi,t-\alpha)g\big(\xi,\alpha,u(\xi,\alpha),u\big(\xi,\alpha-r_1(\alpha)\big)\big)$$

$$+\int_0^a\int_0^\alpha K(\xi,\alpha,s,y)g\big(s,y,u(s,y),u\big(s,y-r_2(s)\big)\big)dyds\Big]d\xi d\alpha$$

where

$$n(t,x)=\int_0^a\left[\frac{2}{a}\sum_{i=1}^{\infty}e^{-(\frac{\pi i}{a})^2t}\cdot sin\frac{\pi ix}{a}\cdot sin\frac{\pi i\xi}{a}\cdot\varphi_0(\xi)\right]d\xi$$

Applying Contraction Principle, an existence and uniqueness theorem is obtained.
In [14], the following problem is considered:

$$\begin{cases} u_t(t,x)=a^2u_{xx}(t,x)+g\big(u(t,x),u(x,[t])\big)\\ u(x,0)=\varphi(x)\quad t\in\mathbb{R}\end{cases}$$

where $[t]$ means the integer part of t. Using integration by parts twice for the equation above, in appropriate conditions, the problem is equivalent with a VF equation and the successive approximation method is applied.

The purpose of the present paper is to give results concerning the following problems related to equation (1): the existence and the uniqueness of the solution, the data dependence of the solution and the differentiability of the solution with respect to parameters.

Because the tool used in the present paper is the Picard operators theory, for the convenient of the reader, we present some basic notions and results concerning this important class of operators.

2. **Picard operators**

Let (X,d) be a metric space and $A:X\to X$ an operator. In this paper we will use the following notations:

$$F_A:=\{x\in X:A(x)=x\};$$

$$A^0:=1_X\ ,\ A^{n+1}:=A\circ A^n\text{ for all }n\in\mathbb{N}.$$

Definition 2.1. *(Rus [15]) The operator A is said to be:*
(i) ***weakly Picard operator (wPo)*** *if $A^n(x_0)\to x_0^*$ for any $x_0\in X$ and the limit x_0^* is a fixed point of A, which may depend on x_0.*

(ii) ***Picard operator (Po)*** *if* $F_A = \{x^*\}$ *and* $A^n(x_0) \to x^*$ *for any* $x_0 \in X$ *.*

For a weakly Picard operator A, the operator A^∞ *is defined as follows:*

$$A^\infty : X \to X, \quad A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

Notice that $A^\infty(X) = F_A$*.*

If A is Picard operator, then $A^\infty(x) = x^*$ *for all* $x \in X$*, where* x^* *is the unique fixed point of A.*

Example 2.1. *Any* α*-contraction on a complete metric space* (X, d) *is a Picard operator.*

The following abstract theorem is needed in the study of data dependence of the solution:

Theorem 2.1. *(Rus [17]) Let* (X, d) *a complete metric space and* $A, B : X \to X$ *two operators. Assume that:*

(i) there exists $\alpha \in [0, 1[$ *such that A is* α*-contraction; let* $F_A = \{x_A^*\}$

(ii) $F_B \neq \emptyset$*; let* $x_B^* \in F_B$*;*

(iii) there exists $\eta > 0$ *such that* $d(A(x), B(x)) \leq \eta$ *for all* $x \in X$*.*

Then

$$d(x_A^*, x_B^*) \leq \frac{\eta}{1-\alpha}.$$

In order to study the differentiability of the solution with respect to a parameter, we need the following theorem, due to I. A. Rus:

Theorem 2.2. *(Fiber Contraction Principle, Rus [16]) Let* (X, d)*,* (Y, ρ) *be two metric spaces and* $B : X \to X,\ C : X \times Y \to Y$ *two operators such that:*

(i) (Y, ρ) *is complete;*

(ii) B is a Picard operator, $F_B = \{x^*\}$*;*

(iii) $C(\cdot, y) : X \to Y$ *is continuous for all* $y \in Y$*;*

(iv) there exists $\alpha \in]0, 1[$ *such that the operator* $C(x, \cdot) : Y \to Y$ *is* α*-contraction for all* $x \in X$*; let* y^* *be the unique fixed point of* $C(x^*, \cdot)$*.*

Then

$$A : X \times Y \to X \times Y, \quad A(x, y) := (B(x), C(x, y))$$

is a Picard operator and $F_A = \{(x^*, y^*)\}$.

For Picard operators theory applied in the study of differential or integral equations see [19], [18], [17], [12], [20], [6], [5].

3. Existence and uniqueness theorem

Consider the equation (1).

Theorem 3.1. *If the following conditions are satisfied:*

(c1) $g \in C(\overline{D} \times X, X)$, $H \in C(\overline{D} \times [0,T] \times X, X)$ $K \in C(\overline{D}^2 \times X^2, X)$, $\varphi_1 \in C(\overline{D}, [0,T])$ *and* $\varphi_2 \in C(\overline{D}, [a,b])$*;*

(c2) *there exists* $L_g > 0$ *such that:*

$$\|g(t,x,u) - g(t,x,v)\|_X \leq L_g \|u - v\|_X \tag{4}$$

for all $(t,x) \in \overline{D}$ *and* $u, v \in X$

(c3) *there exists* $L_H > 0$ *such that:*

$$\|H(t,x,s,u) - H(t,x,s,v)\|_X \leq L_H \|u - v\|_X \tag{5}$$

for all $(t,x,s) \in \overline{D} \times [0,T]$ *and* $u, v \in X$

(c4) *there exists* $L_K > 0$ *such that:*

$$\|K(t,x,s,y,u,\overline{u}) - K(t,x,s,y,v,\overline{v})\|_X \leq L_K \big(\|u - v\|_X + |\overline{u} - \overline{v}\|_X\big) \tag{6}$$

for all $(t,x,s,y) \in \overline{D}^2$ *and* $u, v, \overline{u}, \overline{v} \in X$

(c5) $L_g < 1$

(c6) *there exists* $\tau > 0$ *such that:*

$$\alpha := L_g + \frac{1}{\tau} L_H + \frac{b-a}{\tau} L_K + \max\Big\{ \int_0^t \int_a^b e^{\tau[\varphi_1(s,y) - t]} dyds : t \in [0,T] \Big\} L_K < 1 \tag{7}$$

Then (1) has an unique solution $u^* \in C(\overline{D}, X)$.

Proof. Let the space $C(\overline{D}, X)$ be endowed with a Bielecki-Chebysev suitable norm

$$\|u\|_{BC} := \sup\{\|u(t,x)\|_X \, e^{-\tau t} : t \in [0,T], x \in [a,b]\}, \quad \tau > 0 \tag{8}$$

Consider the operator $A : C(\overline{D}, X) \to C(\overline{D}, X)$ defined by:

$$A(u)(t,x) := g(t,x) + \int_0^t \int_a^b K\Big(t,x,s,y,u\big(\varphi_1(s,y),\varphi_2(s,y)\big)\Big)dyds \qquad (9)$$

for all $u \in C(\overline{D})$, for all $(t,x) \in \overline{D}$.

For any $u, v \in C(\overline{D}, X)$ we have (see [1]):

$$\|A(u)(t,x) - A(v)(t,x)\|_X$$

$$\leq L_g\|u(t,x) - v(t,x)\|_X + L_H \int_0^t \|u(s,x) - v(s,x)\|_X ds + L_K \frac{b-a}{\tau} \|u - v\|_{BC} \cdot e^{\tau t}$$

$$+L_K \max\Big\{ \int_0^t \int_a^b e^{\tau[\varphi_1(s,y)-t]} dyds : t \in [0,T]\Big\} \cdot \|u - v\|_{BC} \cdot e^{\tau t}$$

so:

$$\|A(u) - A(v)\|_{BC} \leq \alpha \|u - v\|_{BC}.$$

From **(c6)** there exists $\tau > 0$ such that $A : C(\overline{D}, X) \to C(\overline{D}, X)$ is α-contraction and, by Contraction Principle, A is a Picard operator, i.e. the equation has a unique solution in $C(\overline{D}, X)$.

Remark 3.1. *Condition **(c6)** from Theorem 3.1 can be replaced by the next simpler condition:*

$$\textbf{\textit{(c7)}} \quad \varphi_1(t,x) \leq t \quad \textit{for all } (t,x) \in \overline{D}$$

In this case the operator A given by (9) is $\overline{\alpha}$-contraction, with

$$\overline{\alpha} = L_g + \frac{L_H + 2(b-a)L_K}{\tau} < 1 \qquad (10)$$

for a suitable chosen τ.

4. **Data dependence of the solution**

In order to prove the dependence of the solution on data, let us consider two mixed VF equations:

$$u(t,x) = g_i\big(t,x,u(t,x)\big) + \int_0^t H_i\,(t,x,s,u(s,x))\,ds$$

$$+ \int_0^t \int_a^b K_i\Big(t,x,s,y,u(s,y),u\big(\varphi_1(s,y),\varphi_2(s,y)\big)\Big)dyds \qquad (11)$$

for all $u \in C(\overline{D}, X)$ and $(t,x) \in \overline{D}$, with $g_i \in C(\overline{D} \times X, X)$, $H_i \in C(\overline{D} \times [0,T] \times X, X)$ and $K_i \in C(\overline{D}^2 \times X^2, X)$ for $i = \overline{1,2}$.

Theorem 4.1. *Assume that the first equation from (11) satisfies conditions* ***(c1)****-****(c5)*** *and* ***(c7)****; let u^* be its unique solution. Assume that the second equation from (11) has at least one solution; let v^* be a such solution.*
If there exist $\eta_1, \eta_2, \eta_3 > 0$ such that:

$$\|g_1(t,x,u) - g_2(t,x,u)\|_X \leq \eta_1 \quad \text{for all } (t,x,u) \in \overline{D} \times X$$

$$\|H_1(t,x,s,u) - H_2(t,x,s,u)\|_X \leq \eta_2 \quad \text{for all } (t,x,s,u) \in \overline{D} \times [0,T] \times X$$

$$\|K_1(t,x,s,y,u) - K_2(t,x,s,y,u)\|_X \leq \eta_3 \quad \text{for all } (t,x,s,y,u) \in \overline{D}^2 \times X$$

Then:

$$\|u^* - v^*\|_{BC} \leq \frac{\eta_1 + T\eta_2 + T(b-a)\eta_3}{1-\overline{\alpha}}$$

where $\overline{\alpha} = L_g + \dfrac{L_H + 2(b-a)L_K}{\tau} < 1$ *for suitable chosen* τ.

Proof. Consider the operators $A_1, A_2 : C(\overline{D}, X) \to C(\overline{D}, X)$ given by:

$$A_i(u)(t,x) := g_i\big(t,x,u(t,x)\big) + \int_0^t H_i\left(t,x,s,u(s,x)\right) ds$$
$$+ \int_0^t \int_a^b K_i\Big(t,x,s,y,u(s,y),u\big(\varphi_1(s,y),\varphi_2(s,y)\big)\Big) dyds$$

for all $u \in C(\overline{D})$ and $(t,x) \in \overline{D}$, $i = \overline{1,2}$.
For any $u \in C(\overline{D})$ we have:

$$\|A_1(u)(t,x) - A_2(u)(t,x)\|_X \leq \eta_1 + T\eta_2 + T(b-a)\eta_3 \quad \text{for all } (t,x) \in \overline{D}$$

Applying $sup_{(t,x)\in\overline{D}}$, we obtain:

$$\|A_1(u) - A_2(u)\|_C \leq \eta_1 + T\eta_2 + T(b-a)\eta_3$$

where $\|\cdot\|_C$ is Chebysev norm:

$$\|u\|_C := \sup\{\|u(t,x)\|_X : (t,x) \in \overline{D}\}, \quad \text{for all } u \in C(\overline{D}, X)$$

But $\|\cdot\|_{BC} \leq \|\cdot\|_C$, so:

$$\|A_1(u) - A_2(u)\|_{BC} \leq \eta_1 + T\eta_2 + T(b-a)\eta_3 \tag{12}$$

Consider the operators A_1 and A_2 defined above, on the space $\big(C(\overline{D}, X), \|\cdot\|_{BC}\big)$. By Theorem 3.1, A_1 is $\overline{\alpha}$-contraction for suitable chosen τ , so $F_{A_1} = \{u^*\}$. Taking account of (12), we are in the conditions of Theorem 2.1 and the conclusion follows.

5. **Differentiability of the solution with respect to parameters**

In order to study the differentiability of the solution with respect to parameters a and b, let us consider the same equation (1):

$$u(t,x) = g\big(t,x,u(t,x)\big) + \int_0^t H\left(t,x,s,u(s,x)\right) ds$$

$$+\int_0^t \int_a^b K\Big(t,x,s,y,u(s,y),u\big(\varphi_1(s,y),\varphi_2(s,y)\big)\Big) dyds$$

for all $t \in [0,T]$, for all $x \in [\alpha,\beta]$, where $0 < \alpha < a < b < \beta$.

Theorem 5.1. *Assume that:*

(i) $g \in C([0,T] \times [\alpha,\beta] \times \mathbb{R})$,

$H \in C([0,T] \times [\alpha,\beta] \times [0,T] \times \mathbb{R})$, $K \in C([0,T] \times [\alpha,\beta] \times [0,T] \times [\alpha,\beta] \times \mathbb{R}^2)$,

$\varphi_1 \in C([0,T] \times [\alpha,\beta],[0,T])$ *and* $\varphi_2 \in C([0,T] \times [\alpha,\beta],[\alpha,\beta])$*;*

(ii) $g(t,x,\cdot) \in C^1(\mathbb{R})$ *for all* $(t,x) \in [0,T] \times [\alpha,\beta]$ *and there exists* $M_g > 0$ *such that:*

$$\left|\frac{\partial g(t,x,u)}{\partial u}\right| \le M_g \tag{13}$$

for all $(t,x,u) \in [0,T] \times [\alpha,\beta] \times \mathbb{R}$*;*

(iii) $H(t,x,s,\cdot) \in C^1(\mathbb{R})$ *for all* $(t,x,s) \in [0,T] \times [\alpha,\beta] \times [0,T]$ *and there exists* $M_H > 0$ *such that:*

$$\left|\frac{\partial H(t,x,s,u)}{\partial u}\right| \le M_H \tag{14}$$

for all $(t,x,s,u) \in [0,T] \times [\alpha,\beta] \times [0,T] \times \mathbb{R}$*;*

(iv) $K(t,x,s,y,\cdot,\cdot) \in C^1(\mathbb{R}^2)$ *for all* $(t,x,s,y) \in [0,T] \times [\alpha,\beta] \times [0,T] \times [\alpha,\beta]$ *and there exists* $M_K > 0$ *such that:*

$$\left|\frac{\partial K(t,x,s,y,u,\overline{u})}{\partial u}\right| \le M_K \quad \text{and} \quad \left|\frac{\partial K(t,x,s,y,u,\overline{u})}{\partial \overline{u}}\right| \le M_K \tag{15}$$

for all $(t,x,s,y,u,\overline{u}) \in [0,T] \times [\alpha,\beta] \times [0,T] \times [\alpha,\beta] \times \mathbb{R}^2$*;*

(v) $M_g < 1$*;*

(vi) $\varphi_1(t,x) \le t$ *for all* $(t,x) \in [0,T] \times [\alpha,\beta]$.

Then:

a) for all $a < b \in [\alpha, \beta]$*, the equation (1) has a unique solution* $u^*(\cdot, \cdot, a, b) \in C([0, T] \times [\alpha, \beta])$*;*

b) for all $u_0 \in C([0, T] \times [\alpha, \beta])$*, the sequence* $(u_n)_{n\geq 0}$ *defined by:*

$$u_n(t, x, a, b) = g(t, x, u_{n-1}(t, x, a, b)) + \int_0^t H\left(t, x, s, u_{n-1}(s, x)\right) ds$$

$$+ \int_0^t \int_a^b K\left(t, x, s, y, u_{n-1}(s, y, a, b), u_{n-1}\big(\varphi_1(s, y), \varphi_2(s, y), a, b\big)\right) dyds$$

converges uniformly to u^* *on* $[0, T] \times [\alpha, \beta] \times [\alpha, \beta] \times [\alpha, \beta]$*;*

c) $u^* \in C([0, T] \times [\alpha, \beta] \times [\alpha, \beta] \times [\alpha, \beta])$*;*

d) $u^*(t, x, \cdot, \cdot) \in C^1([\alpha, \beta] \times [\alpha, \beta])$*, for all* $(t, x) \in [0, T] \times [\alpha, \beta]$*.*

Proof. Let $X := C([0, T] \times [\alpha, \beta] \times [0, T] \times [\alpha, \beta])$ and $B : X \to X$ defined by:

$$B(u)(t, x, a, b) := g(t, x, u(t, x, a, b)) + \int_0^t H\left(t, x, s, u(s, x)\right) ds$$

$$+ \int_0^t \int_a^b K\Big(t, x, s, y, u(s, y, a, b), u\big(\varphi_1(s, y), \varphi_2(s, y), a, b\big)\Big) dyds.$$

The boundedness conditions (13) and (15) implies that f and K are Lipschitz, with Lipschitz constants M_g and M_K. B satisfies **(c1)-(c5)** and **(c7)**, so a), b) and c) result. Let $u^* \in C(X)$ be the unique fixed point of B.

Obviously we have:

$$u^*(t, x, a, b) = g(t, x, u^*(t, x, a, b)) + \int_0^t H\left(t, x, s, u^*(s, x, a, b)\right) ds$$

$$+ \int_0^t \int_a^b K\Big(t, x, s, y, u^*\big(s, y, a, b\big), u^*\big(\varphi_1(s, y), \varphi_2(s, y), a, b\big)\Big) dyds. \tag{16}$$

Let us prove that $\dfrac{\partial u^*(t, x, a, b)}{\partial a}$ and $\dfrac{\partial u^*(t, x, a, b)}{\partial b}$ exist and are continuous.

1. Assume that $\dfrac{\partial u^*(t, x, a, b)}{\partial a}$ exists. Differentiate (16) with respect to a we have:

$$\frac{\partial u^*(t, x, a, b)}{\partial a} = \frac{\partial g\big(t, x, u^*(t, x, a, b)\big)}{\partial u} \cdot \frac{\partial u^*(t, x, a, b)}{\partial a}$$

$$+ \int_0^t \frac{\partial H\big(t, x, s, u^*(s, x, a, b)\big)}{\partial u} \cdot \frac{\partial u^*(s, x, a, b)}{\partial a} ds$$

$$-\int_0^t K(t,x,s,a,u^*(s,a,a,b),u^*\big(\varphi_1(s,a),\varphi_2(s,a),a,b\big))ds$$
$$+\int_0^t\int_a^b \frac{\partial K\Big(t,x,s,y,u^*\big(s,y,a,b\big),u^*\big(\varphi_1(s,y),\varphi_2(s,y),a,b\big)\Big)}{\partial u}\cdot\frac{\partial u^*(s,y,a,b)}{\partial a}dyds$$
$$+\int_0^t\int_a^b \frac{\partial K\Big(t,x,s,y,u^*\big(s,y,a,b\big),u^*\big(\varphi_1(s,y),\varphi_2(s,y),a,b\big)\Big)}{\partial \overline{u}}$$
$$\cdot\frac{\partial u^*((\varphi_1(s,y),\varphi_2(s,y),a,b)\Big)}{\partial a}dyds.$$

This last relationship suggests us to consider the operator $C : X\times X \to X$ defined by:

$$C(u,v)(t,x,a,b) := \frac{\partial g\big(t,x,u(t,x,a,b)\big)}{\partial u}\cdot v(t,x,a,b)$$
$$+\int_0^t \frac{\partial H\big(t,x,s,u(s,x,a,b)\big)}{\partial u}\cdot v(s,x,a,b)ds$$
$$-\int_0^t K\Big(t,x,s,a,u(s,a,a,b),u\big(\varphi_1(s,a),\varphi_2(s,a),a,b\big)\Big)ds$$
$$+\int_0^t\int_a^b \frac{\partial K\Big(t,x,s,y,u(s,y,a,b),u\big(\varphi_1(s,y),\varphi_2(s,y),a,b\big)\Big)}{\partial u}\cdot v(s,y,a,b)dyds$$
$$+\int_0^t\int_a^b \frac{\partial K\Big(t,x,s,y,u(s,y,a,b),u\big(\varphi_1(s,y),\varphi_2(s,y),a,b\big)\Big)}{\partial \overline{u}}$$
$$\cdot\, v\big(\varphi_1(s,y),\varphi_2(s,y),a,b\big)dyds.$$

From the hypotheses, the operator $C(u,\cdot)$ is a contraction, for any $u\in X$. Let v^* be the unique fixed point of $C(u^*,\cdot)$.

Now consider the operator $A : X\times X \to X\times X$ defined by

$$A(u,v)(t,x,a,b) := (B(u)(t,x,a,b), C(u,v)(t,x,a,b)),$$

which is in the hypotheses of Theorem 2.2. So A is a Picard operator and $F_A = \{(u^*,v^*)\}$.

Consider the sequences $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ defined by:

$$u_n(t,x,a,b) := B(u_{n-1}(t,x,a,b))$$

$$= g(t,x,u_{n-1}(t,x,a,b)) + \int_0^t H\left(t,x,s,u_{n-1}(s,x)\right)ds$$

$$+ \int_0^t \int_a^b K\Big(t,x,s,y,u_{n-1}(s,y,a,b),u_{n-1}\big(\varphi_1(s,y),\varphi_2(s,y),a,b\big)\Big)dyds$$

for all $n \geq 1$ and

$$v_n(t,x,a,b) := C(u_{n-1}(t,x,a,b),v_{n-1}(t,x,a,b))$$

$$= \frac{\partial g\big(t,x,u_{n-1}(t,x,a,b)\big)}{\partial u} \cdot v_{n-1}(t,x,a,b)$$

$$+ \int_0^t \frac{\partial H\big(t,x,s,u_{n-1}(s,x,a,b)\big)}{\partial u} \cdot v_{n-1}(s,x,a,b)ds$$

$$- \int_0^t K\Big(t,x,s,a,u_{n-1}(s,a,a,b),u_{n-1}\big(\varphi_1(s,a),\varphi_2(s,a),a,b\big)\Big)ds$$

$$+ \int_0^t \int_a^b \frac{\partial K\Big(t,x,s,y,u_{n-1}(s,y,a,b),u_{n-1}\big(\varphi_1(s,y),\varphi_2(s,y),a,b\big)\Big)}{\partial u} v_{n-1}(s,y,a,b)dyds$$

$$+ \int_0^t \int_a^b \frac{\partial K\Big(t,x,s,y,u_{n-1}(s,y,a,b),u_{n-1}\big(\varphi_1(s,y),\varphi_2(s,y),a,b\big)\Big)}{\partial \overline{u}}$$

$$\cdot\, v_{n-1}\big(\varphi_1(s,y),\varphi_2(s,y),a,b\big)dyds,$$

for all $n \geq 1$.

Obviously, we have:

$$u_n \to u^* \text{ for } n \to \infty \quad \text{and} \quad v_n \to v^* \text{ for } n \to \infty$$

uniformly with respect to $(t,x,a,b) \in [0,T] \times [\alpha,\beta] \times [\alpha,\beta] \times [\alpha,\beta]$, for any $u_0, v_0 \in C([0,T] \times [\alpha,\beta] \times [\alpha,\beta] \times [\alpha,\beta])$.

Choosing $u_0 = v_0 := 0$ we have $v_1 = \dfrac{\partial u_1}{\partial a}$.

By induction we can prove that $v_n = \dfrac{\partial u_n}{\partial a}$ for any positive integer n, so

$$\frac{\partial u_n}{\partial\, a} \to v^* \text{ for } n \to \infty$$

From Weierstrass theorem, it follows that $\dfrac{\partial u^*}{\partial\, a}$ exists and

$$\frac{\partial u^*(t,x,a,b)}{\partial a} = v^*(t,x,a,b).$$

2. The differentiability with respect to b can be proved in the same way.

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