

ENCOURAGING THE GRAND COALITION IN CONVEX COOPERATIVE GAMES

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Abstract. A solution function for convex transferable utility games encourages the grand coalition if no player prefers (in a precise sense defined in the text) any coalition to the grand coalition. We show that the Shapley value encourages the grand coalition in all convex games and the τ -value encourages the grand coalitions in convex games up to three (but not more than three) players. Solution functions that encourage the grand coalition in convex games always produce allocations in the core, but the converse is not necessarily true.

1. Cooperative games

We begin by recalling the main concepts and their basic properties. The notation mostly follows [3] and/or [2].

Let $N = \{1, \dots, n\}$. The elements of N are called *players*, its subsets are called *coalitions*, and the set N is called the *grand coalition*. A *cooperative transferable utility game* with n -players is a function $v : 2^N \rightarrow \mathbb{R}$ such that $v(\emptyset) = 0$, where 2^N is the set of all subsets of N .

For a given game v , we often denote $v(\{i\})$ by $v(i)$ or v_i . More generally, for any function $x : N \rightarrow \mathbb{R}$ and $i \in N$, we denote $x(i) = x_i$. Thus we (sometimes) think of functions $x : N \rightarrow \mathbb{R}$ as vectors in \mathbb{R}^n . For a function $x : N \rightarrow \mathbb{R}$ and a coalition $A \subseteq N$, we write $x(A) = \sum_{j \in A} x_j$.

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A game $v : 2^N \rightarrow \mathbb{R}$ is called *super-additive* if, for all disjoint coalitions $A, B \subseteq N$,

$$v(A) + v(B) \leq v(A \cup B)$$

and is called *convex* if, for all coalitions $A, B \subseteq N$,

$$v(A) + v(B) \leq v(A \cup B) + v(A \cap B).$$

Example 1. Define a 4-player game v on $N = \{1, 2, 3, 4\}$ by the diagram in Figure 1 (the value of each coalition is provided at the vertex representing the coalition). The

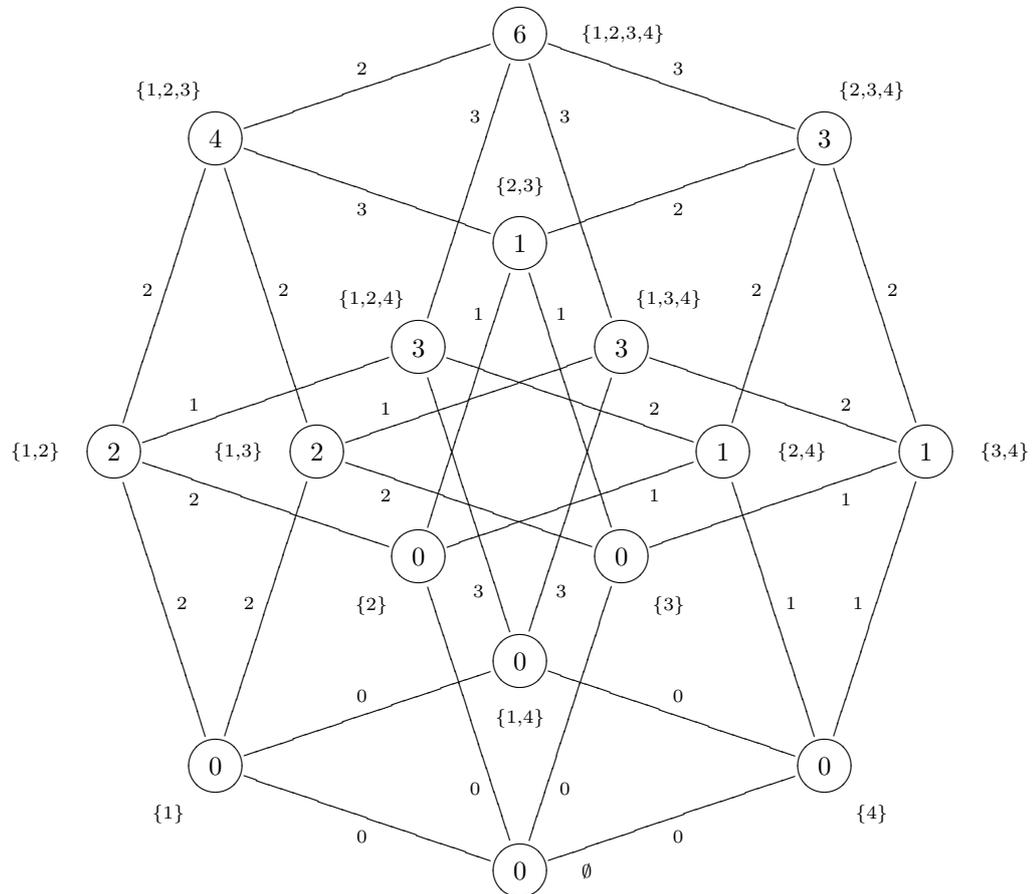


FIGURE 1. A convex 4-player game

same game is given in a tabular form in Table 1.

A	$v(A)$	A	$v(A)$	A	$v(A)$	A	$v(A)$
$\{1\}$	0	$\{2\}$	0	$\{3\}$	0	$\{4\}$	0
$\{1, 2\}$	2	$\{1, 3\}$	2	$\{1, 4\}$	0	$\{2, 3\}$	1
$\{2, 4\}$	1	$\{3, 4\}$	1	$\{1, 2, 3\}$	4	$\{1, 2, 4\}$	3
$\{1, 3, 4\}$	3	$\{2, 3, 4\}$	3	N	6	\emptyset	0

TABLE 1. A convex 4-player game

It is straightforward to check that the game v is convex.

A way to interpret cooperative games is as follows. Assume that the players in the set N can form various coalitions each of which has value prescribed by v (say $v(A)$ represents the amount the coalition A can earn by cooperating). The super-additivity condition implies that “the whole is larger than the sum of its parts”, i.e., forming larger coalitions positively affects the value. The convexity condition is just a stronger form of the super-additivity condition. It says that it is more (or at least equally) beneficial to add a coalition to a larger coalition than to a smaller one.

Assume that i is not a member of some coalition A . The *marginal contribution* $m_i(A)$ of i to the coalition A is the quantity

$$m_i(A) = v(A \cup i) - v(A),$$

where $A \cup i$ denotes the coalition $A \cup \{i\}$. Therefore, the marginal contribution of i to A measures the added value obtained by bringing player i into the coalition A .

A game is convex if and only if, for every player i , and all coalitions $A \subseteq B$ that do not contain i ,

$$m_i(A) \leq m_i(B),$$

i.e., it is more beneficial to add a player to a larger coalition than to a smaller one (this is a well known fact; see for instance [3, Theorem 1.4.2] or [2, Theorem 4.9]).

Example 2. The marginal contributions in the game from Example 1 are written on the edges of the lattice of coalitions. For instance, the fact that $m_2(\{1, 3\}) =$

$v(\{1, 2, 3\}) - v(\{1, 3\}) = 4 - 2 = 2$ is indicated by the label 2 on the edge between $\{1, 3\}$ and $\{1, 3\} \cup \{2\} = \{1, 2, 3\}$.

The top marginal contributions $m_1(N - \{1\}), \dots, m_n(N - \{n\})$ are often denoted by m_1, \dots, m_n . Further, we denote

$$M = \sum_{i \in N} m_i, \quad T = v(N), \quad V = \sum_{i \in N} v_i.$$

Note that, in a convex game, $M \geq T \geq V$.

Example 3. We provide a diagram for a general example of a game on three players. The marginal contributions are indicated on the edges, Note that, for $i, j \in N = \{1, 2, 3\}$, $m_i(\emptyset) = v_i$, and whenever $i \neq j$, we denote $m_i(\{j\}) = m_{ij}$.

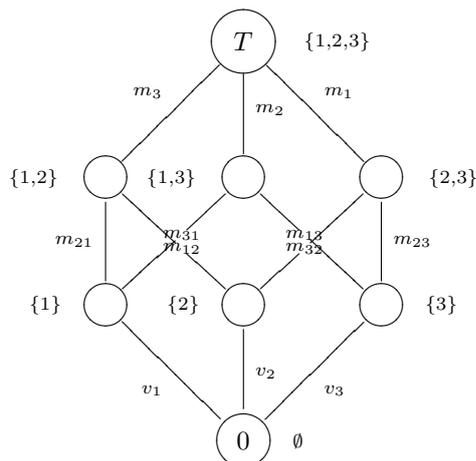


FIGURE 2. A game on 3 players

Note that, if the triple (i, j, k) is a permutation of N then

$$v_i + m_{ji} + m_k = T.$$

The convexity of the game is equivalent to the system of inequalities

$$v_i + v_j + m_k \leq T \leq m_i + m_j + v_k, \tag{1}$$

where (i, j, k) ranges over the permutations of N (see the appendix for details).

An *efficient allocation* is a function $x : N \rightarrow \mathbb{R}$ such that $x(N) = v(N)$. If in addition $x_i \geq v_i$, for $i \in N$, the allocation is called *individually rational*.

An efficient allocation assigns revenue to each player in the game in such a way that the total revenue shared among the players is exactly the value of the grand coalition N . The individual rationality of an allocation then just means that each player should be assigned revenue that is not below the individual value of that player (otherwise that player would choose not to cooperate).

A convex game is *essential* if $T > V$. In an inessential game, $m_i(A) = v_i$, for all coalitions A not containing the player i , and there exists a unique efficient and individually rational allocation, namely $x_i = v_i$, for all $i \in N$.

For a permutation π of N , and a player i in N , denote by $P_i(\pi)$ the set of *predecessors* of i in π . This is the set of players that appear before i (to the left of i) in the one-line representation of the permutation π . For instance, if $n = 6$ and $\pi = 142536$, then $P_3(\pi) = \{1, 4, 2, 5\}$.

The set of permutations of N , denoted Π_n , represents all possible orders in which the grand coalition can be formed by adding the players one by one to the coalition. For each such order, the players have different marginal contributions depending on the set of players that has already joined. The *marginal contribution* of the player i to the permutation π , denoted $m_i(\pi)$, is the marginal contribution of the player i to the coalition $P_i(\pi)$ consisting of the predecessors of i in π . In a convex game, for any permutation $\pi \in \Pi_n$, we have $m_i(\pi) \geq m_i(\emptyset) = v_i$ and $\sum_{i \in N} m_i(\pi) = T$. Thus the marginal contribution vector along π represents an efficient and individually rational allocation for v .

An *efficient solution function* f is a function that assigns an efficient allocation f^v to every convex game (we emphasize that we are not concerned with non-convex games).

Recall the definition of a well known efficient solution function introduced by Shapley [4].

Definition 1. The *Shapley value* of a convex game $v : 2^N \rightarrow \mathbb{R}$ is the allocation s given by

$$s_i = \frac{1}{n!} \sum_{\pi \in \Pi_n} m_i(\pi).$$

Thus the Shapley value is the average of all marginal contribution vectors along all permutations of N .

We also recall the definition of τ -value, introduced by Tijs [6].

Definition 2. The τ -value of an essential convex game $v : 2^N \rightarrow \mathbb{R}$ is the allocation given by

$$\tau_i = \frac{M - T}{M - V} v_i + \frac{T - V}{M - V} m_i.$$

In the case of an inessential game, the τ value is the unique efficient and individually rational allocation.

Note that, for an essential game, $\tau_i = \lambda v_i + (1 - \lambda)m_i$, where $\lambda = \frac{M - T}{M - V}$ is the unique real number in $[0, 1]$ making the allocation efficient. For an inessential game, $v_i = m_i$, for all i , and therefore the formula $\tau_i = \lambda v_i + (1 - \lambda)m_i$ gives the correct τ -value for all λ in the interval $[0, 1]$, i.e., the normalizing coefficient λ is not unique.

Both the Shapley value and the τ -value are efficient solution functions that assign an individually rational allocation to every convex game.

Example 4. For any convex 2-player game,

$$s_1 = \tau_1 = \frac{1}{2}(T + v_1 - v_2), \quad s_2 = \tau_2 = \frac{1}{2}(T + v_2 - v_1).$$

2. Encouraging the grand coalition

We come to our main definition.

Definition 3. An efficient solution function f *encourages the grand coalition* if for every convex game $v : 2^N \rightarrow \mathbb{R}$ and every coalition $A \subseteq N$,

$$f_i^v \geq f_i^{v_A},$$

where $v_A : 2^A \rightarrow \mathbb{R}$ is the convex sub-game of v obtained by restriction on the coalition A .

Thus an efficient solution functions encourages the grand coalitions if no player in any convex game would prefer any coalition (and its associated allocation) over the grand coalition. If f is an efficient solution that encourages the grand coalition and if all players were to vote for all coalitions they like (based on maximizing the revenue they would obtain by applying the proposed solution function f) the grand coalition would be chosen by each player (even though some players may like some additional choices).

Note that the property of encouraging the grand coalition is a global property of solution functions and not of individual allocations (the property requires that we compare allocations in different games).

Theorem 1. *The Shapley value encourages the grand coalition in convex games.*

Proof. Without loss of generality, it is sufficient to show that player 1 does not prefer any coalition $M = \{1, \dots, m\}$ to the grand coalition, i.e., it is sufficient to show that

$$\frac{1}{n!} \sum_{\pi \in \Pi_n} m_1(\pi) \geq \frac{1}{m!} \sum_{\sigma \in \Pi_m} m_1(\sigma),$$

for $1 \leq m \leq n$.

Define a map $\bar{\cdot} : \Pi_n \rightarrow \Pi_m$ by flattening the permutations of N to permutations of M . Namely, for a permutation $\pi \in \Pi_n$ define the permutation $\bar{\pi} \in \Pi_m$ by deleting the symbols $m+1, \dots, n$ from π and keeping the relative order of the symbols $1, \dots, m$ the same as in π (for instance, if $n = 6$, $m = 4$ and $\pi = 153462$, then $\bar{\pi} = 1342$). Every permutation in Π_m is the image of exactly $n!/m!$ permutations in Π_n under the flattening map.

Note that the set of predecessors $P_1(\pi)$ of 1 in the permutation π contains the set of predecessors $P_1(\bar{\pi})$ of 1 in the flattened permutation $\bar{\pi}$. Therefore, by the convexity of the game, $m_1(\pi) = m_1(P_1(\pi)) \geq m_1(P_1(\bar{\pi})) = m_1(\bar{\pi})$.

It follows that

$$\frac{1}{n!} \sum_{\pi \in \Pi_n} m_1(\pi) \geq \frac{1}{n!} \sum_{\pi \in \Pi_n} m_1(\bar{\pi}) = \frac{1}{n!} \cdot \frac{n!}{m!} \sum_{\sigma \in \Pi_m} m_1(\sigma) = \frac{1}{m!} \sum_{\sigma \in \Pi_m} m_1(\sigma),$$

which is what we needed to prove. \square

Theorem 2. *The τ -value encourages the grand coalition in all convex games of up to three players.*

Proof. Since the τ -value coincides with the Shapley value for 2-player convex games and Shapley value encourages the grand coalition, it suffices to consider only 3-player games.

Further, the τ -value always produces individually rational allocations. Thus, it suffices to consider only 3-player games and their 2-player sub-games.

By symmetry, it suffices to show that the convexity of a 3-player game v on $N = \{1, 2, 3\}$ implies the inequality

$$\tau_1^v \geq \tau_1^{v'}$$

where v' is the sub-game corresponding to the coalition $A = \{1, 2\}$.

If the game v is inessential then so is its sub-game v' and the τ -values for v and v' agree on A .

Thus we may assume that v is essential and we need to show that the convexity of v implies

$$\frac{M-T}{M-V} v_1 + \frac{T-V}{M-V} m_1 \geq \frac{1}{2}(T' + v_1 - v_2), \quad (2)$$

where $T' = v(A) = T - m_3$ is the value of the coalition $A = \{1, 2\}$.

Denote $m_{12} = T' - v_2 = T - m_3 - v_2$ (Figure 2 may be useful for visualization; the marginal contribution vector along the permutation $\pi = 213$ is important in our considerations). Taking into account that $M > V$ (from the fact that v is convex and essential) the inequality (2) takes the form

$$(M-T)v_1 + (T-V)m_1 \geq \frac{1}{2}(T' + v_1 - v_2)(M-V),$$

which is equivalent to

$$v_1(m_1 + m_3 + v_2 - v_1 - v_3 - m_2) + m_{12}(m_2 + m_3 + v_1 - v_2 - v_3 - m_1) \leq 2m_1(m_3 - v_3),$$

after substituting $V = v_1 + v_2 + v_3$, $M = m_1 + m_2 + m_3$, $T = v_2 + m_{12} + m_3$, and $T' = v_2 + m_{12}$, and performing simple algebraic manipulations. The convexity

implies that $v_1 \leq m_{12} \leq m_1$, as well as that $m_1 + m_3 + v_2 - v_1 - v_3 - m_2 \geq 0$ and $m_2 + m_3 + v_1 - v_2 - v_3 - m_1 \geq 0$ (see the inequalities in (1)). Thus

$$\begin{aligned} & v_1(m_1 + m_3 + v_2 - v_1 - v_3 - m_2) + m_{12}(m_2 + m_3 + v_1 - v_2 - v_3 - m_1) \leq \\ & \leq m_1(m_1 + m_3 + v_2 - v_1 - v_3 - m_2 + m_2 + m_3 + v_1 - v_2 - v_3 - m_1) = 2m_1(m_3 - v_3), \end{aligned}$$

which is what we needed to prove. \square

Example 5. Consider again the convex game in Example 1. This example shows that the τ -value does not necessarily encourage the grand coalition for convex 4-player games.

Indeed, we have $T = 6$, $V = 0$, $M = 11$, which shows that the normalizing coefficient λ in the formula for the τ -value is $\lambda = (M - T)/(M - V) = 5/11$. Direct calculation then gives the τ -values for v

$$\tau_1^v = \frac{18}{11} \approx 1.64, \quad \tau_2^v = \frac{18}{11} \approx 1.64, \quad \tau_3^v = \frac{18}{11} \approx 1.64, \quad \tau_4^v = \frac{12}{11} \approx 1.09 .$$

On the other hand, for the 3-player sub-game v' determined by the coalition $A = \{1, 2, 3\}$, we have $T' = 4$, $V' = 0$, $M' = 7$, $\lambda' = 3/7$ and the τ -values for v' are

$$\tau_1^{v'} = \frac{12}{7} \approx 1.71, \quad \tau_2^{v'} = \frac{8}{7} \approx 1.14, \quad \tau_3^{v'} = \frac{8}{7} \approx 1.14 .$$

Thus player 1 would prefer the coalition A to the grand coalition, showing that the τ -value does not necessarily encourage the grand coalition.

3. Relation to the core

We consider the relation between efficient solution functions that encourage the grand coalition and the core of a convex game.

Definition 4. The *core* of a convex game $v : 2^N \rightarrow \mathbb{R}$ is the set of all efficient allocations $x : N \rightarrow \mathbb{R}$ such that, for every coalition $A \subseteq N$,

$$x(A) \geq v(A).$$

Note that every allocation in the core is individually rational and we may say that the core allocations are rational with respect to any coalition.

Proposition 1. *Let f be an efficient solution function that encourages the grand coalition in convex games. Then, for every convex game v , the allocation f^v is in the core of v .*

Proof. Let f be an efficient solution function that encourages the grand coalition in convex games and let v be a convex game. Then, for any coalition A ,

$$f^v(A) = \sum_{i \in A} f_i^v \geq \sum_{i \in A} f_i^{v_A} = v(A).$$

Thus f^v is in the core of v . □

Since the τ -value does not always produce allocations in the core of a convex function, we could immediately see that it cannot encourage the grand coalition in general. However, in Example 5 the τ -value of the game v on N is in the core, as is the τ -value of all of its sub-games, but this was still not sufficient to encourage the grand coalition.

The proof of Proposition 1 indicates that, for essential solution functions, the property of encouraging the grand coalition is a refinement of the property of producing solutions in the core. Indeed, the core condition requires that, for each coalition $A \subseteq N$, the sum $f^v(A) = \sum_{i \in A} f_i^v$ is at least as large as the sum $\sum_{i \in A} f_i^{v_A} = v(A)$. On the other hand, for solution functions that encourage the grand coalition, each term in the former sum must be at least as large as the corresponding term in the latter sum. In order to see that this refinement is proper, we provide an example of an efficient solution function that always produces allocations in the core of convex games, but nevertheless fails to encourage the grand coalition.

Example 6. For any convex game v and any permutation π of N , the vector of marginal contributions along π is an efficient solution in the core of v . By convexity of the core, any convex linear combination of marginal contributions along several permutations is also in the core. Therefore, we may define an efficient solution function f as follows. Among all permutations of N select those that give the largest vectors (in the usual sense in \mathbb{R}^n) of marginal contributions and calculate their average. Thus,

if

$$L = \left\{ \pi \in \Pi_n \mid \sum_{i \in N} m_i^v(\pi)^2 \geq \sum_{i \in N} m_i^v(\sigma)^2, \text{ for all } \sigma \in \Pi_n \right\},$$

define

$$f_i^v = \frac{1}{|L|} \sum_{\pi \in L} m_i^v(\pi).$$

To see that f does not encourage the grand coalition in convex games, even though it always produces allocations in the core, consider the game in Example 1 restricted to $N = \{1, 2, 3\}$ (completely ignore player 4).

In this game, the largest marginal vectors are the two vectors along $\pi_1 = 231$ and $\pi_2 = 321$ giving

$$f_1^v = 3, \quad f_2^v = \frac{1}{2}, \quad f_3^v = \frac{1}{2}.$$

On the other hand, if we restrict to the sub-game v' defined by the coalition $A = \{1, 2\}$ we obtain

$$f_1^{v'} = 1, \quad f_2^{v'} = 1.$$

Thus player 2 would prefer the coalition A to the grand coalition.

4. Relation to population monotone allocation schemes

The notion of a population monotone allocation scheme was introduced in [5].

Given a game v , a monotone allocation scheme is a set of efficient allocations $\{x^{v_A} \mid A \subseteq N\}$ associated to the sub-games of v , in such a way that, for every player i and all coalitions A and B with $i \in A \subseteq B \subseteq N$,

$$x_i^{v_A} \leq x_i^{v_B}.$$

This definition is close in spirit to our definition of solution functions that encourage the grand coalition. However, the emphasis goes in different direction. We study efficient solution functions that behave well on convex games, while Sprumont studies games for which well behaved allocation schemes exist. More precisely, the main thrust of Sprumont's work is a characterization of games for which population monotone allocation schemes exist (this includes all convex games, but not all games with non-empty core). For us, on the other hand, the important question is which

solution functions always produce (or fail to produce) population monotone allocation schemes in all convex games.

Sprumont shows that every 3-player game that is totally balanced (see the appendix for a definition) always has a population monotone allocation scheme. Nevertheless, Theorem 2 does not follow directly from this observation (we still need to prove that the specific scheme induced by the τ -value solution function provides such an allocation scheme).

Further, Sprumont shows that the glove game on 4 players fails to have a population monotone allocation scheme. Again, this example is not helpful in our considerations, since the glove game is not convex (the main point of Example 5 is that the τ -value fails to provide a population monotone allocation scheme on a convex game; on the other hand this game certainly has a population monotone allocation scheme, namely the one induced by the Shapley value).

5. On necessity versus desirability

Observe that even if a solution function that does not encourage the grand coalition is used and, for a concrete game v , there exists a player that prefers some smaller coalition over the grand one, this does not mean that the grand coalition will not be formed. For instance, in Example 5 player 1 prefers $A = \{1, 2, 3\}$ to N , but will have difficulties convincing player 2 and player 3 to form this coalition, since they certainly prefer the payout provided to them by the grand coalition. Therefore player 1 would perhaps choose to join the grand coalition (however grudgingly), since it is still offering a better payoff than going-it-alone (which would bring a payoff of 0 to player 1). However, even if player 1 joins the grand coalition, it would be unsatisfied with the situation and may show its discontent by actively and visibly (or covertly and by using inappropriate means) working to undermine the grand coalition and exclude player 4.

Thus encouraging the grand coalition is not necessary to coalesce all players into the grand coalition, but may be desirable in practice.

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Appendix A. Remarks on 3-player games

In Example 3 we provided a quick remark on a condition on 3-player games that is equivalent to convexity and we used this condition in the course of the proof of Theorem 2. We provide a brief justification.

Proposition 2. *A 3-player game v is convex if and only if, for every permutation (i, j, k) of N ,*

$$v_i + v_j + m_k \leq T \leq m_i + m_j + v_k. \quad (3)$$

Proof. As we already remarked, a game is convex if and only if, for every player i , and all coalitions $A \subseteq B$ that do not contain i , $m_i(A) \leq m_i(B)$.

Therefore, in the context of a 3-player game the convexity is equivalent to the system of inequalities

$$v_i \leq m_{ij} \leq m_i, \quad (4)$$

for $i, j \in N$, $i \neq j$.

The inequality (4) is equivalent to

$$v_i + v_j + m_k \leq v_j + m_{ij} + m_k \leq m_i + v_j + m_k,$$

where k is the third player (different from i and j). Since $T = v_j + m_{ij} + m_k$ we obtain

$$v_i + v_j + m_k \leq T \leq m_i + v_j + m_k.$$

Thus, when looked as systems of inequalities, (3) and (4) are equivalent. \square

The games with non-empty core were characterized by Bondareva [1]. Namely, a game has a non-empty core if and only if it is balanced. A game v

balanced if, for every sequence of non-empty subsets A_1, \dots, A_s of N and every sequence of positive real numbers $\lambda_1, \dots, \lambda_s$ such that

$$\sum_{\ell=1}^s \lambda_\ell \chi_{A_\ell} = \chi_N, \quad (5)$$

where χ_{A_ℓ} and χ_N denote the characteristic function of the sets A_ℓ and N , we have

$$\sum_{\ell=1}^s \lambda_\ell v(A_\ell) \leq v(N).$$

A game is totally balanced if all of its sub-games are balanced.

The following modification of the balancing condition is also valid.

Proposition 3. *A game v has non-empty core if and only if, for every sequence of non-empty subsets A_1, \dots, A_s of N and every sequence of positive real numbers $\lambda_1, \dots, \lambda_s$ such that*

$$\sum_{\ell=1}^s \lambda_\ell \chi_{A_\ell} \leq \chi_N, \quad (6)$$

where the inequality is considered pointwise, we have

$$\sum_{\ell=1}^s \lambda_\ell v(A_\ell) \leq v(N). \quad (7)$$

Proof. Let x be an efficient allocation in the core of v , and let $\sum_{\ell=1}^s \lambda_\ell \chi_{A_\ell} \leq \chi_N$, for some positive real numbers $\lambda_1, \dots, \lambda_s$ and a sequence of non-empty subsets A_1, \dots, A_s of N . We have

$$\begin{aligned} \sum_{\ell=1}^s \lambda_\ell v(A_\ell) &\leq \sum_{\ell=1}^s \lambda_\ell x(A_\ell) = \sum_{\ell=1}^s \lambda_\ell \sum_{i \in A_\ell} x_i = \sum_{\ell=1}^s \lambda_\ell \sum_{i=1}^n \chi_{A_\ell}(i) x_i = \\ &= \sum_{i=1}^n \left(\sum_{\ell=1}^s \lambda_\ell \chi_{A_\ell}(i) \right) x_i \leq \sum_{i=1}^n x_i = v(N). \end{aligned}$$

The other direction follows from the result of Bondareva. Namely, if (7) holds whenever (6) does, then (7) also holds whenever (5) does. Therefore the core of v is non-empty. \square

It is easy to see that for a 2-player game, convexity, super-additivity, and the existence of core allocations are equivalent properties and it is well known that these properties are not equivalent for more than 2 players.

Proposition 4. *Let v be a 3-player super-additive game. Define $M_{12} = v_{12} - v_1 - v_2$, $M_{13} = v_{13} - v_1 - v_3$, $M_{23} = v_{23} - v_2 - v_3$, and $S = v(N) - v_1 - v_2 - v_3$, where v_{ij} is the value of the coalition $\{i, j\}$.*

(a) *The game v has a non-empty core if and only if*

$$S \geq \frac{1}{2}(M_{12} + M_{13} + M_{23}). \quad (8)$$

(b) *The game v is convex if and only if*

$$S \geq \max\{M_{12} + M_{13}, M_{12} + M_{23}, M_{13} + M_{23}\}. \quad (9)$$

Proof. (a) Assume v has a non-empty core. By Proposition 3 (or directly by the argument used in the proof), since $\chi_{12} + \chi_{23} + \chi_{31} = 2\chi_N$, we obtain that $v_{12} + v_{13} + v_{23} \leq 2v(N)$. Therefore, $M_{12} + M_{13} + M_{23} = v_{12} + v_{13} + v_{23} - 2(v_1 + v_2 + v_3) \leq 2v(N) - 2(v_1 + v_2 + v_3) = 2S$.

Conversely, assume that (8) holds. Instead of trying to use Proposition 3, we construct explicitly an element in the core.

Assume that the sum of every pair of numbers from $\{M_{12}, M_{13}, M_{23}\}$ is no smaller than the third one (triangle-like inequalities hold). Set $a_1 = \frac{M_{12} + M_{13} - M_{23}}{2}$, $a_2 = \frac{M_{12} + M_{23} - M_{13}}{2}$, $a_3 = \frac{M_{13} + M_{23} - M_{12}}{2}$, and $t = \frac{1}{3}(S - \frac{1}{2}(M_{12} + M_{13} + M_{23}))$. Then a_1, a_2, a_3 , and t are non-negative. Set $x_1 = v_1 + a_1 + t$, $x_2 = v_2 + a_2 + t$, and $x_3 = v_3 + a_3 + t$. Since $x_1 + x_2 + x_3 = v(N)$, the allocation x is efficient. The allocation x is individually rational (by the non-negativity of a_1, a_2, a_3 , and t). We also have

$$x_1 + x_2 = v_1 + v_2 + M_{12} + 2t = v_{12} + 2t \geq v_{12}.$$

Thus the allocation x is rational for the coalition $\{1, 2\}$. By symmetry, x is rational for the other two 2-element coalitions as well. Note that we have not used yet the super-additivity property.

Assume that the sum of two of the numbers M_{12}, M_{13}, M_{23} is smaller than the third, say $M_{12} > M_{13} + M_{23}$ and set $t = \frac{1}{2}(S - M_{13} - M_{23})$. The super-additivity implies that $v(N) \geq v_{12} + v_3$. Therefore $S = v(N) - v_1 - v_2 - v_3 \geq v_{12} + v_3 - v_1 - v_2 - v_3 = M_{12}$. Since $S \geq M_{12} > M_{13} + M_{23}$, we have that $t > 0$. Set $x_1 = v_1 + M_{13} + t$,

$x_2 = v_2 + M_{23} + t$, and $x_3 = v_3$. Since $x_1 + x_2 + x_3 = v(N)$, the allocation x is efficient. The allocation x is individually rational by the non-negativity of M_{12} , M_{13} , M_{23} , and t (for $i \neq j$, M_{ij} is non-negative by the super-additivity property). Further,

$$x_1 + x_3 = v_1 + v_3 + M_{13} + t = v_{13} + t \geq v_{13}$$

and, by symmetry,

$$x_2 + x_3 \geq v_{23}.$$

We also have

$$x_1 + x_2 = v_1 + M_{13} + t + v_2 + M_{23} + t = v_1 + v_2 + S = v_{12} - M_{12} + S \geq v_{12}.$$

Thus the allocation x is rational for all 2-element coalitions.

(b) Note that the convexity needs to be checked only for coalitions that are not comparable (the convexity condition is trivially satisfied when one of the coalitions is included in the other). Therefore, given the super-additivity of the game, v is convex if and only if, for every permutation (i, j, k) of N

$$v_{ij} + v_{ik} \leq v(N) + v_i.$$

The last inequality is equivalent to

$$M_{ij} + M_{ik} \leq S.$$

□

Therefore, we see that the convexity and the existence of the core are not equivalent for 3-player games even in the presence of super-additivity. For instance, if $v_1 = v_2 = v_3 = 0$, $v_{12} = v_{13} = v_{23} = 1$ and $v_N = 3/2$, we have a super-additive, non-convex game with non-empty core.

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