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ON THE TIME-DEPENDENT MOTION OF A VISCOUS INCOMPRESSIBLE FLUID THROUGH A TUBE WITH COMPLIANT WALLS

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Abstract. In this note we study the flow of a viscous, incompressible fluid through an elastic cylinder which is very long when compared to its diameter. The fluid flows due to a given small time-dependent pressure drop between the inflow and the outflow boundary. This creeping flow is modeled by the Stokes equations for a viscous, incompressible flow, while Navier's equations for an elastic membrane describe the behavior of the flexible tube. We show existence and uniqueness of the solution for the system consisting of these equations and the corresponding boundary conditions.

1. **Introduction**

Fluid-structure interaction problems arise in many practical applications, like in aerospace, naval engineering, biomechanics and biomedical engineering (see e.g., [8], [4], [5]). A main issue in this context is haemodynamics.

The cardiovascular system is a very complex system, having a great variety of blood vessels, from large arteries through medium caliber vessels to capillaries. The blood flow is thus a very complicated phenomenon and blood itself is a fluid not easy to describe mathematically. Unless for the very tiny capillaries, it may be regarded as a continuum [2]) and (although Nonnewtonian) as Newtonian and incompressible, excepting some pathological situations [10], [12], [15].

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The problem we study is the following: a viscous incompressible fluid flows through an elastic tube which is very long when compared to its radius. The flow is driven by the difference of the pressures at both ends of the tube. The stress on the fluid depends on the displacement of the flexible wall; this in turn depends on the stresses exerted by the fluid on the interface between the two media. The only stress acting on the structure is supposed to come from the fluid. Fluid and solid mechanics are coupled through the wall position and the traction exerted by the fluid on the tube wall. This scenario can be seen as describing e.g., the blood flow through a segment of a smaller artery.

Our aim is to prove the existence of a unique solution of the above coupled problem. In [13] this has been done in the stationary case for the three-dimensional problem of a fluid flow through an elastic cylindrical tube with thickness and periodic conditions at the ends of the cylinder and in [14] for the full Navier-Stokes equations for the fluid and the nonlinear Navier-Lamé equations for the elastic structure with more general boundary conditions. For the two-dimensional case, when the equations of the fluid were coupled with the ones of an elastic beam we refer to [6]. Another model for a steady-state slow flow in a collapsible tube is studied numerically in [7], where geometrically nonlinear shell theory is used to accurately model the behavior of the tube wall, however by further simplifying the equations for the fluid. Instationary fluid-structure interaction problems (when the fluid domain has moving boundaries depending on time) are considered for instance in [11] and [9] for the two-dimensional case, where the flexible wall is modeled by the equations of an elastic beam or in [3] for the three-dimensional case of a fluid interacting with a structure having a finite number of elastic modes.

2. **The mathematical model**

The fluid is considered viscous, incompressible, unsteady and axisymmetric. We suppose that the pressure drop between the inflow and the outflow ends of the tube is small and that the viscous effects of blood are strongly predominant when compared to the inertial ones. The flexible structure is a thin, long cylinder with very small thickness (an elastic cylindrical membrane). We thus model the fluid by the Stokes equations without time derivatives (a creeping flow) and the flexible tube by Navier's equations for a cylindrical elastic membrane. This seems to be a good model for blood flow in small arteries [5].

We denote by Ω the following domain:

$$\Omega := \{\mathbf{x} \in \mathbf{R}^3 \ : \ \mathbf{x} = (r\cos\theta, r\sin\theta, z),\ 0 \le r \le R,\ 0 \le z \le L\}, \tag{1}$$

where R and L are the radius, respectively the length of the cylinder.

We denote by S the lateral surface (elastic wall) of the cylinder and suppose its evolution is described by Navier's equations [12]:

$$\rho_w h \frac{\partial^2 u_r}{\partial t^2} = kGh\frac{\partial^2 u_r}{\partial z^2} - \frac{Eh}{1-\zeta^2}\left(\frac{\zeta}{R}\frac{\partial u_z}{\partial z} + \frac{u_r}{R^2}\right) + \Phi_r \text{ in } S \times (0,T) \tag{2}$$

$$\rho_w h \frac{\partial^2 u_z}{\partial t^2} = \frac{Eh}{1-\zeta^2}\left(\frac{\zeta}{R}\frac{\partial u_r}{\partial z} + \frac{\partial^2 u_z}{\partial z^2}\right) + \Phi_z \text{ in } S \times (0,T). \tag{3}$$

The unknown variables u_r and u_z represent the radial, respectively longitudinal displacement in the local frame of reference (cylindrical coordinates) (r, θ, z), h is the wall thickness, R is the arterial reference radius at rest, k is the Timoshenko shear correction factor, G is the shear modulus, E the Young modulus of elasticity, ζ the Poisson ratio ($\zeta = \frac{1}{2}$ for an incompressible material), ρ_w is the arterial wall volumetric mass. $\mathbf{\Phi} = (\Phi_r, \Phi_z)^t$ is the forcing term due to the external forces, included the stress coming from the fluid ($\mathbf{\Phi}$ depends on the velocity $\mathbf{v}$ and the pressure p of the fluid, that is of the blood).

Note that this model is based on a Lagrangian description of the motion of the elastic wall. It is referred to a material domain $\Omega(0)$, corresponding to the rest position where $u_r = u_z = 0$.

We also need initial and boundary conditions for the system (2), (3). For the former we consider the rest position and assume that initially there is no deformation of the elastic membrane. Since (2) and (3) are of second order in time, we also need a condition at rest for the time derivatives of the displacements:

$$u_r(0) = u_z(0) = 0, \quad \frac{\partial u_r}{\partial t}(0) = \frac{\partial u_z}{\partial t}(0) = 0 \text{ on } S := \{r = R\} \times (0, L). \tag{4}$$

We further consider the ends of the elastic membrane fixed and take as corresponding boundary conditions the following:

$$u_r = u_z = 0 \text{ for } z = 0 \text{ and } u_r = u_z = 0 \text{ for } z = L, \ \forall t \in \mathbf{R}_+. \tag{5}$$

We come now to the equations modeling the fluid flow. Initially, the elastic tube is filled with fluid and the whole system is in equilibrium. This is the reference state. The pressure drop between the inflow and the outflow gives rise to a deviation from the reference state. If we assume that the acceleration of the fluid is small relatively to the predominant viscous effects (creeping flow), we can write for the fluid the following Stokes equations in cylindrical coordinates (we assume rotational symmetry and thus neglect the circumferential component of the velocity):

$$-\nu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{1}{r^2}v_r + \frac{\partial^2 v_r}{\partial z^2}\right) + \frac{\partial p}{\partial r} = 0 \text{ in } \Omega \times (0, T) \tag{6}$$

$$-\nu\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2}\right) + \frac{\partial p}{\partial z} = 0 \text{ in } \Omega \times (0, T) \tag{7}$$

$$\frac{v_r}{r} + \frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} = 0 \text{ in } \Omega \times (0, T). \tag{8}$$

Here v_r, v_z are the radial, respectively the longitudinal component of the fluid velocity, ν is the viscosity of the fluid and p is the pressure. Equation (8) represents the incompressibility condition div $\mathbf{v} = 0$, written in cylindrical coordinates.

We also need initial and boundary conditions for this system. We assume the initial velocity zero:

$$\mathbf{v} = 0 \text{ in } \Omega \times \{0\} \tag{9}$$

and take the following boundary conditions at the inflow and outflow:

$$v_r = 0, \ p = 0 \text{ on } (\partial\Omega \cap \{z = 0\}) \times (0, T) \tag{10}$$

$$v_r = 0, \ p = P(t) \text{ on } (\partial\Omega \cap \{z = L\}) \times (0, T), \tag{11}$$

where $P(t)$ is the pressure drop driving the fluid.

The following condition, imposed on the rest of the fluid boundary, is a coupling condition and it ensures the continuity of the velocity field:

$$\mathbf{v} = \frac{\partial \mathbf{u}}{\partial t} \text{ on } S \times (0, T). \tag{12}$$

There is one more coupling condition to be satisfied, namely the continuity of stresses. This means that the forcing term on the elastic structure is due to the stresses exerted by the fluid (and possibly by external terms due, for instance, to surrounding organs or muscle tissue, which, however, we neglect here). Thus, the forcing term $\mathbf{\Phi}$ in (2), (3) takes the form:

$$\mathbf{\Phi} = -(p\mathbf{I} - 2\nu\mathbf{e}(\mathbf{v})) \cdot \mathbf{e}_r \text{ on } S \times (0, T). \tag{13}$$

Here $\mathbf{e}(\mathbf{v}) := \frac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^t)$ is the strain tensor (the symmetrized gradient of the velocity).

Remark 2.1.

- The boundary conditions for the structure considered in (5) have been taken homogeneous just for the sake of simplicity. More natural boundary conditions at the outflow should not be zero, since the ends of the elastic structure ($z = 0$ and $z = L$) are tipically "artificial boundaries" (just like the inflow and outflow ends of the fluid) and one should choose them in order not to perturb the numerics.
- We take the pressure drop $P(t)$ in (11) as being as regular as we need in all our further considerations.
- Here we consider the case of a fixed fluid-structure interface. It is known in general that the movement of a solid body implies rigid body motions and displacements caused by the stresses and strains induced in the solid body by the loads coming from the fluid which interacts with the structure. If these displacements are small enough, then one may assume that the interface is stationary, i.e. it does not move in time (unlikely for large displacements). However, even if the displacements are small, the velocity

of deformation is not, therefore we have to take condition (12) instead of a homogeneous Dirichlet type condition for the velocity at the interface.

Thus, the problem can be stated as follows:

Problem 2.2. *Determine a solution* $(\mathbf{u}, \mathbf{v})$ *of the system (2), (3), (6)-(8) in* $S \times \Omega$, *with the initial conditions (4) and (9) and the boundary conditions (5), (10), (11) and (12), where the force* $\mathbf{\Phi}$ *in the equations for the elastic structure is given by the fluid stresses as in (13).*

Our aim is to prove the existence of a unique solution to the coupled problem. This will be done with the aid of Galerkin approximations.

3. **Weak formulation and main result**

In this section we give the weak formulation of the coupled problem and state the main result.

Let us define the space of test functions by:

$$\mathbf{\Psi} := \{\psi \in \mathbf{H}^1(\Omega) \ : \ \psi_r, \psi_z \in H^1(0,L), \ \mathrm{div}\ \psi = 0 \text{ in } \Omega \text{ and}$$

$$\psi_r(r,0) = \psi_r(r,L) = \psi_z(R,L) = \psi_z(R,0) = 0 \text{ for } r \in [0,R]\} \tag{14}$$

Definition 3.1. $(\mathbf{v}, \mathbf{u}) \in H^1(0,T;\mathbf{H}^1(\Omega)) \times L^2(0,T;\mathbf{H}^1(0,L))$ *with* $\mathbf{u}' \in L^2(0,T;$ $\mathbf{L}^2(0,L))$ *and* $\mathbf{u}'' \in L^2(0,T;\mathbf{H}^{-1}(0,L))$ *is called a weak solution of Problem 2.2 if for all* $\psi \in \mathbf{\Psi}$ *the following variational formulation is satisfied in the sense of distributions (in* $\mathcal{D}'(0,T)$*):*

$$R\rho_w h \frac{d^2}{dt^2}\int_0^L (u_r\psi_r + u_z\psi_z)dz + R\int_0^L \left[kGh\frac{\partial u_r}{\partial z}\frac{\partial \psi_r}{\partial z}\right.$$

$$\left. + \frac{Eh}{1-\zeta^2}\left(\frac{\zeta}{R}\frac{\partial u_z}{\partial z}\psi_r + \frac{u_r}{R^2}\psi_r + \frac{\partial u_z}{\partial z}\frac{\partial \psi_z}{\partial z} - \frac{\zeta}{R}\frac{\partial u_r}{\partial z}\psi_z\right)\right]dz \tag{15}$$

$$+2\nu\int_\Omega \mathbf{e}(\mathbf{v}) : \mathbf{e}(\psi) r dr dz = -\int_0^R P(t)\psi_z r dr$$

and

$$\mathbf{u} = \frac{\partial \mathbf{u}}{\partial t} = 0 \text{ on } S \times \{0\} \text{ and } \mathbf{v} = 0 \text{ in } \Omega \times \{0\}. \tag{16}$$

(15) has been obtained by testing in (2) and (3) with ψ_r, respectively ψ_z and in (6), (7) with ψ, integrating by parts on the corresponding domains, using equation (8) and conditions (5), (10), (11), (13) and summing up the equations resulted after the testing.

We now can state the main result.

Theorem 3.1. *There exists a unique weak solution of Problem 2.2.*

4. **Proof of the existence**

4.1. **Galerkin Approximations.** The proof is based on the method of Galerkin, that is we build a weak solution of the problem by first constructing solutions of certain finite dimensional approximations and then passing to limits. We therefore take the functions $\mathbf{w}_k = \mathbf{w}_k(r, z)$ $(k = 1, 2, ...)$ such that

$$\{\mathbf{w}_k\}_{k=1,...,\infty} \text{ is a basis of } \mathbf{\Psi}. \tag{17}$$

In particular, we take $\{\mathbf{w}_k\}_k$ to be the complete set of eigenfunctions of the eigenvalue problem

$$\mathbf{w} \in \mathbf{\Psi},\ (\nabla \mathbf{w}, \nabla \boldsymbol{\psi})_{(0,L)} + (\mathbf{e}(\mathbf{w}), \mathbf{e}(\boldsymbol{\psi}))_\Omega = \lambda[(\mathbf{w}, \boldsymbol{\psi})_{(0,L)} + (\mathbf{w}, \boldsymbol{\psi})_\Omega],\ \boldsymbol{\psi} \in \mathbf{\Psi};$$

we also assume that $\{\mathbf{w}_k\}_k$ is orthonormalized w.r.t. the $\mathbf{H}^1_{0,ends}(\Omega \cup S)$-inner product[1] $(\nabla \cdot, \nabla \cdot)_{(0,L)} + (\mathbf{e}(\cdot), \mathbf{e}(\cdot))_\Omega$. Also observe that $\{\mathbf{w}_k\}_k$ is orthogonal w.r.t. the $\mathbf{L}^2$-inner product in the right hand side of the equation above.

Fix a positive integer m and write

$$\mathbf{v}_m(t) := \sum_{k=1}^{m} c_{km}(t)\mathbf{w}_k, \tag{18}$$

where the coefficients $c_{km}(t),\ k = 1, ..., m,\ 0 \le t \le T$ are intended to satisfy

$$c_{km}(0) = 0,\ k = 1, ..., m. \tag{19}$$

[1] We have denoted by $\mathbf{H}^1_{0,ends}$ the functions which are in $\mathbf{H}^1$ and vanish at the ends of the cylinder.

This would be the approximation of the fluid's velocity. By

$$u_{m,r}(t) := \sum_{k=1}^{m} \alpha_{km}(t) w_{k,r} \text{ and } u_{m,z}(t) := \sum_{k=1}^{m} \alpha_{km}(t) w_{k,z} \tag{20}$$

we construct (with the same basis $\{\mathbf{w}_k\}_{k=1,...}$) an approximation of the displacement of the elastic membrane. The coefficients of these approximations should satisfy (by the continuity of velocities on $S \times (0,T)$) the equation

$$\alpha_{km}(t) = \int_0^t c_{km}(s) ds. \tag{21}$$

Observe that in virtue of (18), (20) and (21) we may write

$$\frac{\partial u_{m,r}}{\partial t} = \sum_{k=1}^{m} c_{km}(t) w_{k,r} \text{ in } S \times (0,T) \tag{22}$$

$$\frac{\partial u_{m,z}}{\partial t} = \sum_{k=1}^{m} c_{km}(t) w_{k,z} \text{ in } S \times (0,T). \tag{23}$$

By (19), the coefficients $\alpha_{km}(t),\ k = 1, ..., m,\ 0 \le t \le T$ satisfy

$$\alpha_{km}(0) = 0 \text{ and } \alpha'_{km}(0) = 0,\ k = 1, ..., m. \tag{24}$$

The Galerkin approximation corresponding to (15) writes ($0 \le t \le T,\ k = 1, ..., m$):

$$\begin{aligned} R\rho_w h(\mathbf{u}''_m(t), \mathbf{w}_k)_{(0,L)} + C[u_{m,r}, \mathbf{w}_k; t] + D[u_{m,z}, \mathbf{w}_k; t] \\ + B[\mathbf{v}_m, \mathbf{w}_k; t] = -\int_0^R P(t) w_{k,z} r dr, \end{aligned} \tag{25}$$

where

$$B[\mathbf{v}, \mathbf{w}; t] := 2\nu \int_\Omega \mathbf{e}(\mathbf{v}) : \mathbf{e}(\mathbf{w}) r dr dz,$$

$$C[u_r, \mathbf{w}; t] := R \int_0^L \left[kGh \frac{\partial u_r}{\partial z} \frac{\partial w_r}{\partial z} + \frac{Eh}{1-\zeta^2} \left(\frac{u_r}{R^2} w_r - \frac{\zeta}{R} \frac{\partial u_r}{\partial z} w_z \right) \right] dz$$

and

$$D[u_z, \mathbf{w}; t] := R \int_0^L \frac{Eh}{1-\zeta^2} \left(\frac{\zeta}{R} \frac{\partial u_z}{\partial z} w_r + \frac{\partial u_z}{\partial z} \frac{\partial w_z}{\partial z} \right) dz.$$

Further (use (17)),

$$B[\mathbf{v}_m, \mathbf{w}_k; t] + C[u_{m,r}, \mathbf{w}_k; t] + D[u_{m,z}, \mathbf{w}_k; t] = \sum_{l=1}^{m} \beta^{kl} \alpha_{lm}(t),$$

where

$$\beta^{kl} := B[\mathbf{w}_l, \mathbf{w}_k] + C[w_{l,r}, \mathbf{w}_k] + D[w_{l,z}, \mathbf{w}_k], \ \ k, l = 1, ..., m.$$

Consequently, (25) becomes the following linear system of ODEs:

$$\alpha''_{km}(t) + \sum_{l=1}^{m} \beta^{kl} \alpha_{lm}(t) = P^k(t) \ (0 \le t \le T, \ \ k = 1, ..., m), \tag{26}$$

where $P^k(t) := -\int\limits_0^R P(t) w_{k,z} r dr$.

The system is subject to the initial conditions (24). By the standard theory for ordinary differential equations (remember that $P(t)$ is regular enough, see Remark 2.1), there exists a unique function $\alpha_m(t) = (\alpha_{1m}(t), ..., \alpha_{mm}(t))$ in C^2, satisfying (24) and solving (26) for $0 \le t \le T$.

4.2. **Energy Estimates.** We intend to pass to the limit with $m \to \infty$ and for this we need some estimates that should be uniform in m.

Theorem 4.1. *There exists a constant $C > 0$ such that*

$$\sup_{0 \le t \le T} \left(||\mathbf{u}'_m(t)||^2_{\mathbf{L}^2(0,L)} + ||\mathbf{u}_m||^2_{\mathbf{H}^1(0,L)} \right) + ||\mathbf{u}''_m(t)||^2_{L^2(0,T;\mathbf{H}^{-1}(0,L))}$$

$$+ ||\mathbf{v}_m||^2_{L^2(0,T;\mathbf{H}^1(\Omega))} \le C(1 + ||P||^2_{L^2(0,T)}) \tag{27}$$

The constant C depends only on Ω, T, R, G, h, k, ζ, ν and ρ_w.

Proof. Multiply (25) by $c_{km}(t)$. By summing up after $k = 1, ..., m$ and taking into account (18), (22), (23), we get:

$$R\rho_w h \int\limits_0^L \frac{\partial^2 \mathbf{u}_m(t)}{\partial t^2} \frac{\partial \mathbf{u}_m(t)}{\partial t} dz + R \int\limits_0^L kGh \frac{\partial u_{m,r}(t)}{\partial z} \frac{\partial^2 u_{m,r}(t)}{\partial t \partial z} dz$$

$$+R \int\limits_0^L \frac{Eh}{1-\zeta^2} \Bigg((\frac{\zeta}{R} \frac{\partial u_{m,z}(t)}{\partial z} + \frac{u_{m,r}(t)}{R^2}) \frac{\partial u_{m,r}(t)}{\partial t} + \frac{\partial u_{m,z}(t)}{\partial z} \frac{\partial^2 u_{m,z}(t)}{\partial t \partial z}$$

$$-\frac{\zeta}{R}\frac{\partial u_{m,r}(t)}{\partial z}\frac{\partial u_{m,z}(t)}{\partial t}\Bigg)dz + 2\nu\int_{\Omega}\mathbf{e}(\mathbf{v}_m(t)) : \mathbf{e}(\mathbf{v}_m(t))rdrdz$$

$$= -\int_0^R P(t)v_{m,z}(t,r,L)rdr.$$

Let us have a closer look at the term whose coefficient is $R\frac{Eh}{1-\zeta^2}$. If we perform a partial integration on the last term of it, use (5) to get rid of the boundary terms and rearrange what we get, it takes the form:

$$R\frac{Eh}{1-\zeta^2}\cdot\frac{1}{2}\frac{d}{dt}\int_0^L\left[(1-\zeta)\left(\Big(\frac{\partial u_{m,z}}{\partial z}\Big)^2+\Big(\frac{u_{m,r}}{R}\Big)^2\right)+\zeta\Big(\frac{u_{m,r}}{R}+\frac{\partial u_{m,z}}{\partial z}\Big)^2\right]dz.$$

Thus, the above identity becomes:

$$\frac{R}{2}\frac{d}{dt}\Bigg[\rho_w h||\mathbf{u}'_m(t)||^2_{L^2(0,L)} + kGh||u_{m,r}(t)||^2_{H^1(0,L)}$$

$$+\frac{Eh}{1-\zeta^2}\Big(\zeta||\frac{u_{m,r}(t)}{R}+\frac{\partial u_{m,z}(t)}{\partial z}||^2_{L^2(0,L)} + (1-\zeta)\Big(||\frac{u_{m,r}(t)}{R}||^2_{L^2(0,L)}$$

$$+||u_{m,z}(t)||^2_{H^1(0,L)}\Big)\Big)\Bigg] + 2\nu|\mathbf{e}(\mathbf{v}_m)|^2_{L^2(\Omega)} = -\int_0^R P(t)v_{m,z}(t,r,L)rdr.$$

The right hand side above may be majorized as follows:

$$-\int_0^R P(t)v_{m,z}(t,r,L)rdr \le |\int_0^R P(t)v_{m,z}(t,r)rdr| \le \frac{1}{L}|P(t)|\int_{\Omega}|v_{m,z}(t)|$$

$$\le \frac{\delta}{L^2}|P(t)|^2 + \frac{1}{\delta}|v_m(t)|^2_{L^2(\Omega)} \le \frac{\delta}{L^2}|P(t)|^2 + \frac{1}{\delta}||v_m(t)||^2_{H^1(\Omega)}.$$

We use this estimation and Korn's inequality (see, for instance, [1]) in the identity above to obtain:

$$\frac{R}{2}\frac{d}{dt}\Bigg[\rho_w h||\mathbf{u}'_m(t)||^2_{L^2(0,L)} + kGh||u_{m,r}(t)||^2_{H^1(0,L)}$$

$$+\frac{Eh}{1-\zeta^2}\Big(\zeta||\frac{u_{m,r}(t)}{R}+\frac{\partial u_{m,z}(t)}{\partial z}||^2_{L^2(0,L)} + (1-\zeta)\Big(||\frac{u_{m,r}(t)}{R}||^2_{L^2(0,L)}$$

$$+||u_{m,z}(t)||^2_{H^1(0,L)}\Big)\Big)\Bigg] + C_1||\mathbf{v}_m||^2_{H^1(\Omega)} \le \frac{\delta}{L^2}|P(t)|^2. \tag{28}$$

Here $C_1 > 0$ is a constant depending on ν, δ and the constant in Korn's inequality.

We now integrate (28) from 0 to t ($t > 0$) and use the initial conditions (4), in order to get the following:

$$\frac{R}{2}\Bigg[\rho_w h||\mathbf{u}'_m(t)||^2_{L^2(0,L)} + kGh||u_{m,r}(t)||^2_{H^1(0,L)}$$

$$+\frac{Eh}{1-\zeta^2}\Big(\zeta||\frac{u_{m,r}(t)}{R} + \frac{\partial u_{m,z}(t)}{\partial z}||^2_{L^2(0,L)} + (1-\zeta)\Big(||\frac{u_{m,r}(t)}{R}||^2_{L^2(0,L)}$$

$$+||u_{m,z}(t)||^2_{H^1(0,L)}\Big)\Big)\Bigg] + C_1\int_0^t ||\mathbf{v}_m(s)||^2_{H^1(\Omega)}ds \leq \frac{\delta}{L^2}||P||^2_{L^2(0,T)}. \tag{29}$$

From (29) it follows that:

$$\sup{}_{0\leq t\leq T}\left(||\mathbf{u}'_m(t)||^2_{L^2(0,L)} + ||\mathbf{u}_m||^2_{H^1(0,L)}\right) \leq C_2||P||^2_{L^2(0,T)}, \tag{30}$$

where $0 < C2 := \frac{\delta}{L^2}\frac{2}{R}\cdot(\min\{\rho_w h,\ kGh,\ \frac{Eh}{1+\zeta}\})^{-1}$.

Now integrate (28) from 0 to T, use again (4) and obtain:

$$\int_0^T ||\mathbf{v}_m(t)||^2_{H^1(\Omega)}dt \leq C_3||P||^2_{L^2(0,T)}, \tag{31}$$

where $0 < C_3 := \frac{\delta}{L^2 C_1}$.

In order to obtain (27), we still need some estimate for the second derivative in time of $\mathbf{u}_m$. In order to do that, let us fix any $\xi \in \mathbf{\Psi}$ with $||\xi||_{\mathbf{H}^1} \leq 1$ and write $\xi = \xi_1 + \xi_2$, where $\xi_1 \in$ span $\{\mathbf{w}_k\}_{k=1,...,m}$ and $(\xi_2, \mathbf{w}_k) = 0$ ($k = 1, ..., m$).

Notice that

$$||\xi_1||_{\mathbf{H}^1} = ||\xi - \xi_2||_{\mathbf{H}^1} \leq ||\xi||_{\mathbf{H}^1} + ||\xi_2||_{\mathbf{H}^1} \leq 1.$$

We also consider that the only nonzero component of ξ is the radial one: $\xi = \xi_1 + \xi_2 = (\xi_{1,r} + \xi_{2,r})\mathbf{e}_r + (\xi_{1,z} + \xi_{2,z})\mathbf{e}_z = (\xi_{1,r} + \xi_{2,r})\mathbf{e}_r$.

Now we test in (25) with ξ subject to the above conditions and obtain (remember (20):

$$R\rho_w h(u''_{m,r}(t), \xi_{1,r})_{(0,L)} + C[u_{m,r}, \xi_1; t] + D[u_{m,z}, \xi_1; t] + B[\mathbf{v}_m, \xi_1; t] = 0. \tag{32}$$

Here

$$B[\mathbf{v}_m, \xi_1; t] = 2\nu \int_\Omega \mathbf{e}(\mathbf{v}_m) : \mathbf{e}(\xi_1) r dr dz,$$

thus

$$|B[\mathbf{v}_m, \xi_1; t]| \leq \text{const } ||\mathbf{v}_m||_{H^1(\Omega)} ||\xi_1||_{H^1(\Omega)} \leq \text{const } ||\mathbf{v}_m||_{H^1(\Omega)}.$$

Further,

$$|C[u_{m,r}, \xi_1; t]| \leq \text{const } (1 + ||u_{m,r}||_{L^2(0,L)})$$

and

$$|D[u_{m,z}, \xi_1; t]| \leq \text{const } ||\mathbf{u}_m||_{H^1(0,L)}.$$

By using these estimates, (32), (30) and (31), it follows:

$$|(u''_{m,r}, \xi_{1,r})| \leq \text{const } (||\mathbf{v}_m||_{H^1(\Omega)} + 1 + ||u_{m,r}||_{L^2(0,L)}).$$

Thus,

$$\int_0^T ||u''_{m,r}(t)||^2_{H^{-1}(0,L)} dt \leq C_4, \tag{33}$$

where the constant $C_4 > 0$ depends on C_2 and C_3.

Now we consider that the only nonzero component of ξ is the longitudinal one. Testing under the above conditions in (25) with ξ leads to:

$$R\rho_w h(u''_{m,z}(t), \xi_{1,z})_{(0,L)} + C[u_{m,r}, \xi_1; t] + D[u_{m,z}, \xi_1; t] + B[\mathbf{v}_m, \xi_1; t] = 0. \tag{34}$$

Analogously as above, it follows that

$$\int_0^T ||u''_{m,z}(t)||^2_{H^{-1}(0,L)} dt \leq C_5(1 + ||P||^2_{L^\infty(0,T)}), \tag{35}$$

where, again, the constant $C_5 > 0$ depends on C_2 and C_3.

(27) follows now from (30), (31), (33) and (35), where the constant C may be taken as $\sum_{i=2}^{5} C_i$. □

4.3. **Existence of a Weak Solution.** We now pass to limits (for $m \to \infty$) in our Galerkin approximations.

The estimate (27) implies that:

$$(\mathbf{u}_m)_m \text{ is bounded in } L^2(0,T;\mathbf{H}^1(0,L)) \tag{36}$$

$$(\mathbf{u}'_m)_m \text{ is bounded in } L^2(0,T;\mathbf{L}^2(0,L)) \tag{37}$$

$$(\mathbf{u}''_m)_m \text{ is bounded in } L^2(0,T;\mathbf{H}^{-1}(0,L)) \tag{38}$$

and

$$(\mathbf{v}_m)_m \text{ is bounded in } L^2(0,T;\mathbf{H}^1(\Omega)). \tag{39}$$

Consequently, there exist some sequences $(\mathbf{u}_{m_k})_k \subset (\mathbf{u}_m)_m$ and $(\mathbf{v}_{m_k})_k \subset (\mathbf{v}_m)_m$ and the functions $\mathbf{u} \in L^2(0,T;\mathbf{H}^1(0,L))$ with $\mathbf{u}' \in L^2(0,T;\mathbf{L}^2(0,L))$, $\mathbf{u}'' \in L^2(0,T;$
$H^{-1}(0,L))$, $\mathbf{v} \in L^2(0,T;\mathbf{H}^1(\Omega))$ such that

$$\mathbf{u}_{m_k} \overset{k\to\infty}{\rightharpoonup} \mathbf{u} \text{ in } L^2(0,T;\mathbf{H}^1(0,L)) \tag{40}$$

$$\mathbf{u}'_{m_k} \overset{k\to\infty}{\rightharpoonup} \mathbf{u}' \text{ in } L^2(0,T;\mathbf{L}^2(0,L)) \tag{41}$$

$$\mathbf{u}''_{m_k} \overset{k\to\infty}{\rightharpoonup} \mathbf{u}'' \text{ in } L^2(0,T;\mathbf{H}^{-1}(0,L)) \tag{42}$$

and

$$\mathbf{v}_{m_k} \overset{k\to\infty}{\rightharpoonup} \mathbf{v} \text{ in } L^2(0,T;\mathbf{H}^1(\Omega)). \tag{43}$$

We now fix an integer N and choose a function $\varphi \in C^1(0,T;\mathbf{\Psi})$ of the form

$$\varphi(t) := \sum_{k=1}^{N} \alpha_k(t)\mathbf{w}_k, \tag{44}$$

where $\{\alpha_k\}_{k=\overline{1,N}}$ are smooth functions. We choose N such that $N \le m$, multiply (25) by $\alpha_k(t)$, sum after $k = 1, ..., N$ and integrate by parts to obtain:

$$R\rho_w h \int_0^T (\mathbf{u}''_m(t), \varphi(t))_{(0,L)} dt + \int_0^T \{C[u_{m,r}, \varphi; t] + D[u_{m,z}, \varphi; t]$$
$$+B[\mathbf{v}_m, \varphi; t]\} dt = -\int_0^T \int_0^R P(t)\varphi_z r dr dt. \tag{45}$$

We may now pass to the limit in the above identity, in virtue of (40), (41), (42) and (43) (set $m = m_k$) and obtain:

$$R\rho_w h \int_0^T (\mathbf{u}''(t), \varphi(t))_{(0,L)} dt + \int_0^T \{C[u_r, \varphi; t] + D[u_z, \varphi; t]$$

$$+B[\mathbf{v}, \varphi; t]\} dt = -\int_0^T \int_0^R P(t)\varphi_z r dr dt. \tag{46}$$

Note that (46) holds for all functions $\varphi \in L^2(0, T; \mathbf{\Psi})$, since functions of the form (44) are dense in this space. It also follows from (46) that

$$R\rho_w h(\mathbf{u}'', \varphi)_{(0,L)} + C[u_r, \varphi; t] + D[u_z, \varphi; t] + B[\mathbf{v}, \varphi; t] = -\int_0^R P(t)\varphi_z r dr$$

for all $\varphi \in \mathbf{\Psi}$ and a.e. $0 \le t \le T$.

Also notice that $\mathbf{u} \in C([0, T]; \mathbf{L}^2(0, L))$ and $\mathbf{u}' \in C([0, T]; \mathbf{H}^{-1}(0, L))$.

We still have to verify that

$$\mathbf{u}(0) = 0, \ \mathbf{u}'(0) = 0 \text{ in } S \tag{47}$$

and

$$\mathbf{v}(0) = 0 \text{ in } \Omega. \tag{48}$$

We therefore choose any function $\varphi \in C^2([0, T]; \mathbf{\Psi})$, with $\varphi(T) = \varphi'(T) = 0$. We then integrate by parts twice in time in (46) to obtain

$$R\rho_w h \int_0^T (\mathbf{u}(t), \varphi''(t))_{(0,L)} dt + \int_0^T \{C[u_r, \varphi; t] + D[u_z, \varphi; t] + B[\mathbf{v}, \varphi; t]\} dt$$

$$= -\int_0^T \int_0^R P(t)\varphi_z r dr dt - (\mathbf{u}(0), \varphi'(0))_{(0,L)} + (\mathbf{u}'(0), \varphi(0))_{(0,L)}. \tag{49}$$

Analogously, we deduce from (45) that

$$R\rho_w h \int_0^T (\mathbf{u}_m, \varphi'')_{(0,L)} dt + \int_0^T \{C[u_{m,r}, \varphi; t] + D[u_{m,z}, \varphi; t] + B[\mathbf{v}_m, \varphi; t]\} dt$$

$$= -\int_0^T \int_0^R P(t)\varphi_z r dr dt - (\mathbf{u}_m(0), \varphi'(0))_{(0,L)} + (\mathbf{u}'_m(0), \varphi(0))_{(0,L)}.$$

We set again $m = m_k$ and deduce from (24), (40), (41), (42) and (43) that

$$R\rho_w h \int_0^T (\mathbf{u}_m(t), \varphi''(t))_{(0,L)} dt + \int_0^T \{C[u_{m,r}, \varphi; t] + D[u_{m,z}, \varphi; t]$$

$$+B[\mathbf{v}_m, \varphi; t]\} dt = -\int_0^T \int_0^R P(t)\varphi_z r dr dt. \tag{50}$$

Compare now the identities (49) and (50) to deduce (47), since $\varphi(0)$, $\varphi'(0)$ are arbitrary.

We now intend to verify (48). For this we need some estimate on $\mathbf{v}'_m$. We therefore differentiate in (25) with respect to time and get:

$$R\rho_w h(\mathbf{v}''_m(t), \mathbf{w}_k)_{(0,L)} + C[v_{m,r}(t), \mathbf{w}_k; t] + D[v_{m,z}(t), \mathbf{w}_k; t]$$

$$+B[\mathbf{v}'_m(t), \mathbf{w}_k; t] = -\int_0^R P'(t) w_{k,z} r dr. \tag{51}$$

Multiply (51) with $c'_{km}(t)$ and sum after $k = 1, ..., m$. It follows:

$$R\rho_w h \frac{1}{2}\frac{d}{dt}|\mathbf{v}_m(t)|^2_{L^2(0,L)} + C[v_{m,r}, \mathbf{v}'_m; t] + D[v_{m,z}, \mathbf{v}'_m; t]$$

$$+2\nu|\mathbf{e}(\mathbf{v}'_m(t))|^2_{L^2(\Omega)} = -\int_0^R P'(t) v'_{m,z} r dr. \tag{52}$$

The right hand side in (52) can be majorized in the following way:

$$-\int_0^R P'(t) v'_{m,z} r dr \leq \gamma |P'(t)|^2 + \frac{1}{\gamma}|\mathbf{v}'_m|^2_{L^2(\Omega)},$$

where γ is a positive constant.

Applying again Korn's inequality and using Gronwall's inequality it follows from (52) that

$$\mathbf{v}'_m \text{ is bounded in } L^2(0,T;\mathbf{L}^2(\Omega)).$$

Integrate in (52) from 0 to T and using the boundedness of $\mathbf{v}'_m$ in $L^2(0,T;\mathbf{L}^2(\Omega))$, we obtain that

$$\mathbf{v}'_m \text{ is bounded in } L^2(0,T;\mathbf{H}^1(\Omega)).$$

Consequently, there exists a subsequence $(\mathbf{v}_{m_k})_k$ of $(\mathbf{v}_m)_m$ and $\mathbf{v}' \in L^2(0,T;\mathbf{H}^1(\Omega))$ with

$$\mathbf{v}'_{m_k} \overset{k\to\infty}{\rightharpoonup} \mathbf{v}' \text{ in } L^2(0,T;\mathbf{H}^1(\Omega)). \tag{53}$$

Multiply (25) by $\alpha'_k(t)$, sum after $k = 1, ..., m$, use the assumptions on φ and integrate by parts with respect to time the term with $B[.,.;t]$ to get:

$$R\rho_w h \int_0^T (\mathbf{u}''_m(t), \varphi'(t))_{(0,L)} dt + \int_0^T [C[u_{m,r}, \varphi'; t] + D[u_{m,z}, \varphi'; t]] dt$$
$$- \int_0^T B[\mathbf{v}'_m, \varphi; t] dt = - \int_0^T \int_0^R P(t) \varphi'_z r dr dt. \tag{54}$$

We may now pass to the limit (take $m_k = m$) in (54), in virtue of the weak convergences obtained so far. It follows that:

$$R\rho_w h \int_0^T (\mathbf{u}''(t), \varphi'(t))_{(0,L)} dt + \int_0^T [C[u_r, \varphi'; t] + D[u_z, \varphi'; t]] dt$$
$$- \int_0^T B[\mathbf{v}', \varphi; t] dt = - \int_0^T \int_0^R P(t) \varphi'_z r dr dt. \tag{55}$$

If we pass to the limit in (54) before integrating by parts the term with $B[.,.;t]$, we obtain (doing the integration by parts *afterwards*):

$$R\rho_w h \int_0^T (\mathbf{u}''(t), \varphi'(t))_{(0,L)} dt + \int_0^T [C[u_r, \varphi'; t] + D[u_z, \varphi'; t]] dt$$
$$- \int_0^T B[\mathbf{v}', \varphi; t] dt - B[\mathbf{v}(0), \varphi(0); 0] = - \int_0^T \int_0^R P(t) \varphi'_z r dr dt. \tag{56}$$

Now, by comparing (55) and (56), since $\varphi(0)$ is arbitrary, we obtain (48). Consequently, $(\mathbf{u}, \mathbf{v})$ is a weak solution of Problem 2.2, corresponding to the weak formulation (15), (16). $\square$

4.4. **Proof of the uniqueness.** In this section we prove the uniqueness of the weak solution found in the previous section. For this it suffices to show that the only weak solution of Problem 2.2 with $P(t) \equiv 0$ is

$$(\mathbf{u}, \mathbf{v}) \equiv \mathbf{0}. \tag{57}$$

Fix $0 \leq s \leq T$ and take

$$\boldsymbol{\zeta}(t) := \begin{cases} \int_t^s \mathbf{v}(\tau) d\tau & \text{if} \quad 0 \leq t \leq s \\ 0 & \text{if} \quad s \leq t \leq T \end{cases} . \tag{58}$$

Observe that

$$\boldsymbol{\zeta}'(t) = -\mathbf{v}(t),$$

thus on $S \times (0, T)$ also $\boldsymbol{\zeta}(t) = -\mathbf{u}(t)$.

Then from the regularity properties of $\mathbf{v}$ and $\mathbf{u}$ it follows that $\boldsymbol{\zeta}(t) \in \mathbf{H}^1(\Omega)$, $\forall\ 0 \leq t \leq T$ with $\zeta_r(t),\ \zeta_z(t) \in H^1(0, L),\ \zeta_r(R, L) = \zeta_r(R, 0) = 0,\ \zeta_z(R, L) = \zeta_z(R, 0) = 0$ and thus we can write (see (25)):

$$R\rho_w h \int_0^s (\mathbf{u}''(t), \boldsymbol{\zeta}(t))_{(0,L)} dt + \int_0^s \{C[u_r, \boldsymbol{\zeta}; t] + D[u_z, \boldsymbol{\zeta}; t] + B[\mathbf{v}, \boldsymbol{\zeta}; t]\} dt = 0.$$

Integrate by parts with respect to time and use (4) and (58) to write:

$$-R\rho_w h \int_0^s (\mathbf{u}'(t), \boldsymbol{\zeta}'(t))_{(0,L)} dt + \int_0^s B[\mathbf{v}, \boldsymbol{\zeta}; t] dt$$
$$= - \int_0^s \{C[u_r, \boldsymbol{\zeta}; t] + D[u_z, \boldsymbol{\zeta}; t]\} dt.$$

We have

$$R\rho_w h \int_0^s |\mathbf{u}'(t)|^2_{(0,L)} dt - \nu \frac{d}{dt} \int_0^s |\mathbf{e}(\boldsymbol{\zeta}(t))|^2_{L^2(\Omega)} \leq C \int_0^s ||\boldsymbol{\zeta}(t)||^2_{\mathbf{H}^1(\Omega)} dt,$$

thus (use again Korn's inequality)

$$\nu |\boldsymbol{\zeta}(0)|^2_{\mathbf{H}^1(\Omega)} \leq C \int_0^s ||\boldsymbol{\zeta}(t)||^2_{\mathbf{H}^1(\Omega)} dt. \tag{59}$$

We now define $\mathbf{g}(t) := \int\limits_0^t \mathbf{v}(\tau)d\tau$, $0 \le t \le T$. Then note that $\boldsymbol{\zeta}(0) = \mathbf{g}(s)$ and $\boldsymbol{\zeta}(t) = \mathbf{g}(s) - \mathbf{g}(t)$. Consequently, we deduce from (59) that

$$(1 - C(\nu)s)||\mathbf{g}(s)||^2_{\mathbf{H}^1(\Omega)} \le C(\nu)\int\limits_0^s ||\mathbf{g}(t)||^2_{\mathbf{H}^1(\Omega)}dt$$

and we choose $0 < T_1$ small enough $(0 \le T_1 \le \frac{1}{2C(\nu)})$.

Then for $0 \le s \le T_1$ we have

$$||\mathbf{g}(s)||^2_{\mathbf{H}^1(\Omega)} \le C(\nu, T_1)\int\limits_0^s ||\mathbf{g}(t)||^2_{\mathbf{H}^1(\Omega)}dt.$$

Applying the integral form of Gronwall's inequality, it follows that $\mathbf{g} \equiv \mathbf{0}$, thus $\boldsymbol{\zeta} \equiv \mathbf{0}$ and so $\mathbf{v} \equiv \mathbf{0}$ on $\Omega \times [0, T_1]$ and $\mathbf{u} \equiv \mathbf{0}$ on $S \times [0, T_1]$.

We apply the same argument on the intervals $[T_1, 2T_1]$, $[2T_1, 3T_1]$, etc. to eventually obtain that $(\mathbf{u}, \mathbf{v}) \equiv \mathbf{0}$. $\square$

References

[1] Ciarlet, P., *Mathematical elasticity. Volume I: Three-dimensional elasticity*, North-Holland, 1988.

[2] Cokelet, G., *The rheology and tube flow of blood*, in R. Skalak and S. Chen (eds.), *Handbook of Bioengineering*, McGraw-Hill, 1987.

[3] Desjardins, B., Esteban, M., Grandmont, C., Le Tallec, P., *Weak solutions for a fluid-elastic structure interaction model*, Rev. Matematica Complutense, **XIV**(2000), 523-538.

[4] Fung, Y.C., *Biomechanics. Motion, flow, stress, and growth*, Springer, 1990.

[5] Fung, Y.C., *Biomechanics. Circulation*, second edition, Springer, 1996.

[6] Grandmont, C., *Éxistence et unicité des solutions d'un problème de couplage fluide-structure bidimensionnel stationnaire*, Note aux CRAS, Paris, T.326, Série I, 1998.

[7] Heil, M., *Large deformations of cylindrical shells conveying viscous flow*, Ph.D. Thesis, Univ. of Leeds, 1995.

[8] Lighthill, J., *Biofluiddynamics: A survey*, in *Fluid Dynamics in Biology*, ed. by A.Y.Cheer and C.P.van Dam, Contemporary Mathematics, **141**(1993), 1-32.

[9] Litvinov, W.G., *A Problem of nonsteady flow of a nonlinear viscous fluid in a deformable pipe*, Methods of Functional Analysis and Topology, vol. **2**(1996), 85-113.

[10] Nobile, F., *Problemi di interazione fluido-struttura in emodinamica*, diploma thesis, Milano 1998.

[11] Prouse, G., *On the motion of a viscous incompressible fluid in a tube with permeable and deformable wall*, Rend. Sc. Fis. Mat. e Nat. Lincei, **L**(1971), 663-675.

[12] Quarteroni, A., Tuveri, M., Veneziani, A., *Computational vascular fluid dynamics: Problems, models and methods*, Comput. Vis. Sci., **2**(2000), 163-197.

[13] Surulescu, C., *On the stationary motion of a Stokes fluid in a thick elastic tube: a 3D/3D interaction problem*, AMUC, **75**(2006), 95-106.

[14] Surulescu, C., *On the stationary interaction of a Navier-Stokes fluid with an elastic tube wall*, Applicable Analysis, **86**(2007), 149-165.

[15] Veneziani, A., *Mathematical and numerical modelling of blood flow problems*, Ph.D. Thesis, Politecnico di Milano 1998.

[16] Zeidler, E., *Nonlinear functional analysis and its applications*, Springer, 1985.

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