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A MIXED MONTE CARLO AND QUASI-MONTE CARLO METHOD WITH APPLICATIONS TO MATHEMATICAL FINANCE

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Abstract. In this paper, we apply a mixed Monte Carlo and Quasi-Monte Carlo method, which we proposed in a previous paper, to problems from mathematical finance. We estimate the value of an European Call option and of an Asian option using our mixed method, under different horizont times. We assume that the stock price of the underlaying asset $S = S(t)$ is driven by a Lévy process $L(t)$. We compare our estimates with the estimates obtained by using the Monte Carlo and Quasi-Monte Carlo methods. Numerical results show that a considerable improvement can be achieved by using the mixed method.

1. Introduction

The valuation of financial derivatives is one of the most important problems from mathematical finance. The risk-neutral price of such a derivative can be expressed in terms of a risk-neutral expectation of a random payoff. In some cases, the expectation is explicitly computable, such as the Black & Scholes formula for pricing call options on assets modelled by a geometric Brownian motion. However, if we consider an Asian option, there exists no longer closed form expressions for the price, and therefore numerical methods are involved. This is the case, even if we consider call options written on assets with non-normal returns. Among these methods, Monte Carlo (MC) and Quasi-Monte Carlo (QMC) methods play an increasingly important role.

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One of the first applications of the MC method in this field appeared in Boyle [2], who used simulation to estimate the value of a standard European option. Applications of the QMC method to option pricing problems can be found in [15] and [12].

Barndorff-Nielsen [1] proposed to model the log returns of asset prices by using the normal inverse Gaussian (NIG) distribution. This family of non-normal distributions has proven to fit the semi-heavily tails observed in financial time series of various kinds extremely well (see Rydberg [21] or Eberlein and Keller [7]). The time dynamics of the asset prices are modelled by an exponential Lévy process. To price such derivatives, even simple call and put options, we need to consider the numerical evaluation of the expectation. Raible [18] has considered a Fourier method to evaluate call and put options. Alternatives to this method are the MC method or the QMC method. The QMC method has been applied with succes in financial applications by many authors (see [8]), and has strong convergence properties. Majority of the work done on applying these simulation techniques to financial problems was in direction where one needs to simulate from the normal distribution. One exception is Kainhofer (see [13]), who proposes a QMC algorithm for NIG variables, based on a technique proposed by Hlawka and Mück (see [11]) to generate low-discrepancy sequences for general distributions.

In a recent paper [19], we proposed a mixed MC and QMC method for estimating an s-dimensional integral I and we defined a new hybrid sequence that we called the H-mixed sequence. Other authors who combine the ideas from MC and QMC methods in estimating multidimensional integrals are G. Ökten (see [16]) and N. Roşca (see [20]). Using these sequences, we defined a new estimator and proved a central limit theorem for this estimator. In this paper, we apply our mixed method to practical problems from financial mathematics. First, we remember the theoretical background of our method and give some important results. Then, we apply the H-mixed sequence to valuation of an European Call option and compare the effectiveness of it with that of pseudorandom and low-discrepancy sequences. At the end, we apply the mixed method to a more difficult problem from finance, namely the valuation of Asian options. We also compare numerically our method with the MC and QMC methods.

2. H-mixed sequences

Let us consider the problem of estimating integrals of the form

$$I = \int_{[0,1]^s} f(x) dH(x), \tag{1}$$

where $f : [0,1]^s \to \mathbb{R}$ is the function we want to integrate and $H : \mathbb{R}^s \to [0,1]$ is a distribution function on $[0,1]^s$. In the continuous case, the integral I can be rewritten as

$$I = \int_{[0,1]^s} f(x) h(x) dx,$$

where h is the density function corresponding to the distribution function H.

In the MC method (see [22]), the integral I is estimated by sums of the form

$$\hat{I}_N = \frac{1}{N} \sum_{k=1}^{N} f(x_k),$$

where $x_k = (x_k^{(1)}, \ldots, x_k^{(s)})$, $k \geq 1$, are independent identically distributed random points on $[0,1]^s$, with the common density function h.

In the QMC method (see [22]), the integral I is approximated by sums of the form $\frac{1}{N}\sum_{k=1}^{N} f(x_k)$, where $(x_k)_{k\geq 1}$ is a H-distributed low-discrepancy sequence on $[0,1]^s$.

Next, the notions of discrepancy and marginal distributions are introduced.

Definition 1 (H-discrepancy). *Consider an s-dimensional continuous distribution on $[0,1]^s$, with distribution function H. Let λ_H be the probability measure induced by H. Let $P = (x_1, \ldots, x_N)$ be a sequence of points in $[0,1]^s$. The H-discrepancy of sequence P is defined as*

$$D_{N,H}(P) = \sup_{J \subseteq [0,1]^s} \left| \frac{1}{N} A_N(J,P) - \lambda_H(J) \right|,$$

where the supremum is calculated over all subintervals $J = \prod_{i=1}^{s}[a_i, b_i] \subseteq [0,1]^s$*;* $A_N(J,P)$ *counts the number of elements of the set* $(x_1,\ldots,x_N)$*, falling into the interval* J*, i.e.*

$$A_N(J,P) = \sum_{k=1}^{N} 1_J(x_k).$$

1_J *is the characteristic function of* J*.*

The sequence P *is called* H*-distributed if* $D_{N,H}(P) \to 0$ *as* $N \to \infty$*.*

The H*-distributed sequence* P *is said to be a low-discrepancy sequence if* $D_{N,H}(P) = \mathcal{O}\big((\log N)^s/N\big)$*.*

The non-uniform Koksma-Hlawka inequality ([3]) gives an upper bound for the approximation error in QMC integration, when H-distributed low-discrepancy sequences are used.

Theorem 2 (non-uniform Koksma-Hlawka inequality)**.** *Let* $f : [0,1]^s \to \mathbb{R}$ *be a function of bounded variation in the sense of Hardy and Krause and* $(x_1,\ldots,x_N)$ *be a sequence of points in* $[0,1]^s$*. Consider an* s*-dimensional continuous distribution on* $[0,1]^s$*, with distribution function* H*. Then, for any* $N > 0$

$$\left| \int_{[0,1]^s} f(x)dH(x) - \frac{1}{N}\sum_{k=1}^{N} f(x_k) \right| \leq V_{HK}(f) D_{N,H}(x_1,\ldots,x_N), \tag{2}$$

where $V_{HK}(f)$ *is the variation of* f *in the sense of Hardy and Krause.*

Definition 3. *Consider an* s*-dimensional continuous distribution on* $[0,1]^s$*, with density function* h *and distribution function* H*. For a point* $u = \big(u^{(1)},\ldots,u^{(s)}\big) \in [0,1]^s$*, the marginal density functions* h_l*,* $l = 1,\ldots,s$*, are defined by*

$$h_l\big(u^{(l)}\big) = \underbrace{\int \ldots \int}_{[0,1]^{s-1}} h\big(t^{(1)},\ldots,t^{(l-1)},u^{(l)},t^{(l+1)},\ldots t^{(s)}\big)dt^{(1)}\ldots dt^{(l-1)}dt^{(l+1)}\ldots dt^{(s)},$$

and the marginal distribution functions H_l*,* $l = 1,\ldots,s$*, are defined by*

$$H_l\big(u^{(l)}\big) = \int_0^{u^{(l)}} h_l(t)dt.$$

We consider s-dimensional continuous distributions on $[0,1]^s$, with independent marginals, i.e.,

$$H(u)=\prod_{l=1}^{s} H_l(u^{(l)}),\ \forall u=(u^{(1)},\ldots,u^{(s)})\in[0,1]^s.$$

This can be expressed, using the marginal density functions, as follows:

$$h(u)=\prod_{l=1}^{s} h_l(u^{(l)}),\ \forall u=(u^{(1)},\ldots,u^{(s)})\in[0,1]^s.$$

Consider an integer $0<d<s$. Using the marginal density functions, we construct the following density functions on $[0,1]^d$ and $[0,1]^{s-d}$, respectively:

$$h_q(u)=\prod_{l=1}^{d} h_l(u^{(l)}),\ \forall u=(u^{(1)},\ldots,u^{(d)})\in[0,1]^d,$$

and

$$h_X(u)=\prod_{l=d+1}^{s} h_l(u^{(l)}),\ \forall u=(u^{(d+1)},\ldots,u^{(s)})\in[0,1]^{s-d}.$$

The corresponding distribution functions are

$$H_q(u)=\int_0^{u^{(1)}}\cdots\int_0^{u^{(d)}} h_q\big(t^{(1)},\ldots,t^{(d)}\big)dt^{(1)}\ldots dt^{(d)},$$
$$u=(u^{(1)},\ldots,u^{(d)})\in[0,1]^d, \tag{3}$$

and

$$H_X(u)=\int_0^{u^{(d+1)}}\cdots\int_0^{u^{(s)}} h_X\big(t^{(d+1)},\ldots,t^{(s)}\big)dt^{(d+1)}\ldots dt^{(s)},$$
$$u=(u^{(d+1)},\ldots,u^{(s)})\in[0,1]^{s-d}. \tag{4}$$

Next, we introduce the new notion of a H-mixed sequence.

Definition 4 (H-mixed sequence)**.** *([19])*

Consider an s-dimensional continuous distribution on $[0,1]^s$, with distribution function H and independent marginals H_l, $l=1,\ldots,s$. Let H_q and H_X be the distribution functions defined in (3) and (4), respectively.

Let $(q_k)_{k\geq 1}$ be a H_q-distributed low-discrepancy sequence on $[0,1]^d$, with $q_k=(q_k^{(1)},\ldots,q_k^{(d)})$, and X_k, $k\geq 1$, be independent and identically distributed random vectors on $[0,1]^{s-d}$, with distribution function H_X, where $X_k=(X_k^{(d+1)},\ldots,X_k^{(s)})$.

A sequence $(m_k)_{k\geq 1}$*, with the general term given by*

$$m_k = (q_k, X_k), \quad k \geq 1, \tag{5}$$

is called a H-mixed sequence on $[0,1]^s$*.*

Remark 5. *For an interval* $J = \prod_{l=1}^{s}[a_l, b_l] \subseteq [0,1]^s$*, we define the subintervals* $J' = \prod_{l=1}^{d}[a_l, b_l] \subseteq [0,1]^d$ *and* $J'' = \prod_{l=d+1}^{s}[a_l, b_l] \subseteq [0,1]^{s-d}$ *(i.e.* $J = J' \times J''$*).*

Let $(m_k)_{k\geq 1}$ be a H-mixed sequence on $[0,1]^s$, with the general term given by (5).

Based on definitions (1) and (4), the H-discrepancy of the sequence $(m_1, \ldots, m_N)$ can be expressed as

$$D_{N,H}(m_1, \ldots, m_N) = \sup_{J\subseteq[0,1]^s} \left| \frac{1}{N} \sum_{k=1}^{N} 1_J(m_k) - \prod_{l=1}^{s}[H_l(b_l) - H_l(a_l)] \right|,$$

and the H_q-discrepancy of the sequence $(q_1, \ldots, q_N)$ is given by

$$D_{N,H_q}(q_1, \ldots, q_N) = \sup_{J'\subseteq[0,1]^d} \left| \frac{1}{N} \sum_{k=1}^{N} 1_{J'}(q_k) - \prod_{l=1}^{d}[H_l(b_l) - H_l(a_l)] \right|.$$

The following result gives a probabilistic error bound for the H-mixed sequences.

Theorem 6. *(*[19]*) If* $(m_k)_{k\geq 1} = (q_k, X_k)_{k\geq 1}$ *is a* H*-mixed sequence, then* $\forall \varepsilon > 0$ *we have*

$$P\Big(D_{N,H}(m_1, \ldots, m_N) \leq \varepsilon + D_{N,H_q}(q_1, \ldots, q_N)\Big) \geq 1 - \frac{1}{\varepsilon^2}\frac{1}{4N}\Big(D_{N,H_q}(q_1, \ldots, q_N) + 1\Big). \tag{6}$$

In order to estimate general integrals of the form (1), we introduce the following estimator.

Definition 7. *(*[19]*) Let* $(m_k)_{k\geq 1} = (q_k, X_k)_{k\geq 1}$ *be an s-dimensional H-mixed sequence, introduced by us in Definition 4, with* $q_k = (q_k^{(1)}, \ldots, q_k^{(d)})$ *and* $X_k = (X_k^{(d+1)}, \ldots, X_k^{(s)})$*. We define the following estimator for the integral* I*:*

$$\theta_m = \frac{1}{N} \sum_{k=1}^{N} f(m_k). \tag{7}$$

We consider the independent random variables:

$$Y_k = f(m_k) = f(q_k^{(1)}, \ldots, q_k^{(d)}, X_k^{(d+1)}, \ldots, X_k^{(s)}), \quad k \geq 1. \tag{8}$$

We denote the expectation of Y_k by

$$E(Y_k) = \mu_k, \tag{9}$$

and the variance of Y_k by

$$Var(Y_k) = \sigma_k^2. \tag{10}$$

We assume that

$$0 < \sigma_k^2 < \infty, \tag{11}$$

and we denote

$$0 < \sigma_{(N)}^2 = \sigma_1^2 + \ldots + \sigma_N^2 < \infty. \tag{12}$$

In what follows, we give and prove an important result, concerning the estimator (7) introduced previously by us.

Theorem 8. *Let $(m_k)_{k\geq 1} = (q_k, X_k)_{k\geq 1}$ be an s-dimensional H-mixed sequence, defined in (5). We assume that the integrant f is bounded on $[0,1]^s$ and that the function*

$$g(x^{(1)}, \ldots, x^{(d)}) = \int_{[0,1]^{s-d}} f(x^{(1)}, \ldots, x^{(s)}) \Big(\prod_{l=d+1}^{s} h_l(x^{(l)}) \Big) dx^{(d+1)} \cdot \ldots \cdot dx^{(s)},$$

is of bounded variation in the sense of Hardy and Krause. Then, the estimator θ_m, defined in relation (7), is asymptotically unbiased i.e.,

$$E(\theta_m) \to I, \text{ as } N \to \infty.$$

Proof. As $(q_k)_{k\geq 1}$, with $q_k = (q_k^{(1)}, \ldots, q_k^{(d)})$, is a H_q-distributed low-discrepancy sequence on $[0,1]^d$, it follows that

$$D_{N,H_q}(q_1, \ldots, q_N) \to 0, \text{ as } N \to \infty. \tag{13}$$

Using this and the fact that function g is of bounded variation in the sense of Hardy and Krause, it follows from Koksma-Hlawka inequality (2) that

$$\frac{1}{N}\sum_{k=1}^{N} g(q_k^{(1)},\dots,q_k^{(d)}) \longrightarrow \int_{[0,1]^d} g(x^{(1)},\dots,x^{(d)})\Big(\prod_{l=1}^{d} h_l(x^{(l)})\Big)dx^{(1)}\dots dx^{(d)}$$

$$= \int_{[0,1]^d}\Big[\int_{[0,1]^{s-d}} f(x^{(1)},\dots,x^{(s)})\Big(\prod_{l=d+1}^{s} h_l(x^{(l)})\Big)dx^{(d+1)}\cdot\ldots\cdot dx^{(s)}\Big]$$

$$\cdot\Big(\prod_{l=1}^{d} h_l(x^{(l)})\Big)dx^{(1)}\dots dx^{(d)}$$

$$= \int_{[0,1]^s} f(x^{(1)},\dots,x^{(s)})\Big(\prod_{l=1}^{s} h_l(x^{(l)})\Big)dx^{(1)}\cdot\ldots\cdot dx^{(s)} = I,\ \ as\ N\to\infty.$$

The expectation of our estimator is

$$E(\theta_m) = E\Bigg(\frac{1}{N}\sum_{k=1}^{N} f(m_k)\Bigg) = \frac{1}{N}\sum_{k=1}^{N} E(f(q_k^{(1)},\dots,q_k^{(d)},X_k^{(d+1)},\dots,X_k^{(s)}))$$

$$= \frac{1}{N}\sum_{k=1}^{N}\int_{[0,1]^{s-d}} f(q_k^{(1)},\dots,q_k^{(d)},x^{(d+1)},\dots,x^{(s)})\Big(\prod_{l=d+1}^{s} h_l(x^{(l)})\Big)dx^{(d+1)}\cdot\ldots\cdot dx^{(s)}$$

$$= \frac{1}{N}\sum_{k=1}^{N} g(q_k^{(1)},\dots,q_k^{(d)}).$$

Hence, we get in the end that

$$\lim_{N\to\infty} E(\theta_m) = I.$$

□

We call the method of estimating the integral I, based on the estimator θ_m, defined in (7), the mixed method.

Proposition 9. *([19]) Let $(m_k)_{k\geq 1} = (q_k, X_k)_{k\geq 1}$ be an s-dimensional H-mixed sequence. We assume that f is bounded on $[0,1]^s$ and that the functions*

$$f_1(x^{(1)},\dots,x^{(d)}) = \int_{[0,1]^{s-d}} (f(x^{(1)},\dots,x^{(s)}))^2\Big(\prod_{l=d+1}^{s} h_l(x^{(l)})\Big)dx^{(d+1)}\cdot\ldots\cdot dx^{(s)},$$

$$f_2(x^{(1)},\dots,x^{(d)}) = \Big[\int_{[0,1]^{s-d}} f(x^{(1)},\dots,x^{(s)})\Big(\prod_{l=d+1}^{s} h_l(x^{(l)})\Big)dx^{(d+1)}\cdot\ldots\cdot dx^{(s)}\Big]^2$$

are of bounded variation in the sense of Hardy and Krause. Then, we have

$$\frac{\sigma^2_{(N)}}{N} \longrightarrow L, \text{ as } N \longrightarrow \infty,$$

where

$$L = \int_{[0,1]^s} (f(x^{(1)}, \dots, x^{(s)}))^2 \Big(\prod_{l=1}^{s} h_l(x^{(l)}) \Big) dx^{(1)} \dots dx^{(s)} -$$

$$- \int_{[0,1]^d} \Big[\int_{[0,1]^{s-d}} f(x^{(1)}, \dots, x^{(s)}) \Big(\prod_{l=d+1}^{s} h_l(x^{(l)}) \Big) dx^{(d+1)} \cdot \dots \cdot dx^{(s)} \Big]^2$$

$$\cdot \Big(\prod_{l=1}^{d} h_l(x^{(l)}) \Big) dx^{(1)} \dots dx^{(d)}.$$

Another important result regarding the estimator defined before is recalled next.

Theorem 10. *(*[19]*) In the same hypothesis as in Proposition 9 and, in addition, assuming that $L \neq 0$, we have*

a)

$$Y_{(N)} = \frac{\sum_{k=1}^{N} Y_k - \sum_{k=1}^{N} \mu_k}{\sigma_{(N)}} \longrightarrow Y, \quad \text{as } N \to \infty, \tag{14}$$

where the random variable Y has the standard normal distribution.

b) *If we denote the crude Monte Carlo estimator for the integral (1) by θ_{MC}, then*

$$Var(\theta_m) \leq Var(\theta_{MC}), \tag{15}$$

meaning that, by using our estimator, we obtain asymptotically a smaller variance than by using the classical Monte Carlo method.

3. **Application to finance: European options**

In this section, we apply our mixed method to a problem from mathematical finance. The general setting of the problem is presented next. We consider the situation where the stock price of the underlaying asset $S = S(t)$ is driven by a Lévy process $L(t)$,

$$S(t) = S(0)e^{L(t)}. \tag{16}$$

Lévy processes can be characterized by the distribution of the random variable $L(1)$. This distribution can be hyperbolic (see [7]), normal inverse gaussian (NIG), variance-gamma (see [14]), or Meixner distribution.

According to the fundamental theory of asset pricing (see [5]), the risk-neutral price of an option, $C(0)$, is given by

$$C(0) = e^{-rT} E^Q(C_T(S)), \tag{17}$$

where $C_T(S)$ is the so-called payoff of the derivative, which coincides with its value at expiration or exercise time T, and Q is an equivalent martingale measure. In this paper, we are mostly concerned with exponential NIG-Lévy processes, meaning that $L(t)$ has independent increments, distributed according to a NIG distribution. For a detailed discussion of the NIG distribution and the corresponding Lévy process, we refer to Barndorff-Nielsen [1] and Rydberg [21]. In the situation of exponential NIG-Lévy models, we have an incomplete market, leading to a continuum of equivalent martingale measures Q, which can be used for risk-neutral pricing. Here, we choose the approach of Raible [18] and consider the measure obtained by Esscher transform method. This approach is so-called structure preserving, in the sense that the distribution of $L(1)$ remains in the class of NIG distributions.

In the following, we consider the evaluation of so-called European Call options, which have to be valued by simulation. The risk-neutral price of such an option is

$$C(0) = e^{-rT} E^Q(\max\{S(T) - K, 0\}) = e^{-rT} E^Q((S(T) - K)_+), \tag{18}$$

where the constant K is called the strike price. If we replace the stock price by (16), we obtain

$$C(0) = e^{-rT} E^Q((S(0)e^{L(T)} - K)_+). \tag{19}$$

From practice, we know that the evaluation of the stock price $S(t)$ is made at discrete times $0 = t_0 < t_1 < t_2 < \ldots < t_s = T$. For simplicity, we focus on regular time intervals, $\Delta t = t_i - t_{i-1}$. We note that

$$X_i = L(t_i) - L(t_{i-1}) = L(t_{i-1} + \Delta t) - L(t_{i-1}) \sim L(\Delta t), \quad i = 1, \ldots, s,$$

are independent and identically distributed NIG random variables with the same distribution as $L(t_1)$. Dropping the discounted factor from the risk-neutral option price, we get the expected payoff under the Esscher transform measure of the European Call option

$$E^Q((S(0)e^{L(T)} - K)_+) = E((S(0)e^{\sum_{i=1}^s X_i} - K)_+), \tag{20}$$

Our purpose is to evaluate the expected payoff (20). For this, we rewrite the expectation (20) as a multidimensional integral on $\mathbb{R}^s$

$$I = \int_{\mathbb{R}^s} \underbrace{\left(S(0)e^{\sum_{i=1}^s x^{(i)}} - K\right)_+}_{E(x)} dG(x) = \int_{\mathbb{R}^s} E(x)dG(x), \tag{21}$$

where $G(x) = \prod_{i=1}^s G_i(x^{(i)}),\ \forall x = (x^{(1)}, \ldots, x^{(s)}) \in \mathbb{R}^s$, and $G_i(x^{(i)})$ denotes the distribution function of the so-called log returns induced by $L(t_1)$, with the corresponding density function $g_i(x^{(i)})$. These log increments are independent and NIG distributed, having a common probability density

$$f_{NIG}(x; \mu, \beta, \alpha, \delta) = \frac{\alpha}{\pi} \exp\left(\delta\sqrt{\alpha^2 - \beta^2} + \beta(x-\mu)\right) \frac{\delta K_1(\alpha\sqrt{\delta^2 + (x-\mu)^2})}{\sqrt{\delta^2 + (x-\mu)^2}} \tag{22}$$

where $K_1(x)$ denotes the modified Bessel function of third type of order 1 (see [17]).

In order to approximate the integral (21), we have to transform it to an integral on $[0,1]^s$. We can do this using an integral transformation, as follows.

We first consider the family of independent double-exponential distributions with the same parameter $\lambda > 0$, having the cumulative distribution functions $G_{\lambda,i} : \mathbb{R} \to [0,1],\ i = 1, \ldots, s$,

$$G_{\lambda,i}(x) = \begin{cases} \frac{1}{2}e^{\lambda x} & , x < 0 \\ 1 - \frac{1}{2}e^{-\lambda x} & , x \geq 0, \end{cases} \tag{23}$$

and the inverses $G_{\lambda,i}^{-1} : [0,1] \to \mathbb{R},\ i = 1, \ldots, s$, given by

$$G_{\lambda,i}^{-1}(x) = \begin{cases} \frac{1}{\lambda}\log(2x) & , x \leq \frac{1}{2} \\ -\frac{1}{\lambda}\log(2-2x) & , x > \frac{1}{2}. \end{cases} \tag{24}$$

Next, we consider the substitutions $x^{(i)} = G_{\lambda,i}^{-1}(1 - y^{(i)}),\ i = 1, \ldots, s$, and then take $y^{(i)} = 1 - z^{(i)},\ i = 1, \ldots, s$.

The integral (21) becomes

$$I = \int_{[0,1]^s} \underbrace{\left(S(0)e^{\sum_{i=1}^{s} G_{\lambda,i}^{-1}(z^{(i)})} - K\right)_+}_{f(z)} dH(z) = \int_{[0,1]^s} f(z)dH(z), \tag{25}$$

where $H : [0,1]^s \to [0,1]$, defined by

$$H(z) = \prod_{i=1}^{s} (G_i \circ G_{\lambda,i}^{-1})(z^{(i)}),\ \forall z = (z^{(1)}, \ldots, z^{(s)}) \in [0,1]^s, \tag{26}$$

is a distribution function on $[0,1]^s$, with independent marginals $H_i = G_i \circ G_{\lambda,i}^{-1}$, $i = 1, \ldots, s$.

In the following, we compare numerically our mixed method with the MC and QMC methods. As a measure of comparison, we will use the absolute errors produced by these three methods in the approximation of the integral (25).

The MC estimate is defined as follows:

$$\theta_{MC} = \frac{1}{N} \sum_{k=1}^{N} f(x_k^{(1)}, \ldots, x_k^{(s)}), \tag{27}$$

where $x_k = (x_k^{(1)}, \ldots, x_k^{(s)})$, $k \geq 1$, are independent identically distributed random points on $[0,1]^s$, with the common distribution function H defined in (26).

In order to generate such a point x_k, we proceed as follows. We first generate a random point $\omega_k = (\omega_k^{(1)}, \ldots, \omega_k^{(s)})$, where $\omega_k^{(i)}$ is a point uniformly distributed on $[0,1]$, for each $i = 1, \ldots, s$. Then, for each component $\omega_k^{(i)}$, $i = 1, \ldots, s$, we apply the inversion method (see [4] and [6]), and obtain that $H_i^{-1}(\omega_k^{(i)}) = (G_{\lambda,i} \circ G_i^{-1})(\omega_k^{(i)})$ is a point with the distribution function H_i. As the s-dimensional distribution with the distribution function H has independent marginals, it follows that $x_k = ((G_{\lambda,1} \circ G_1^{-1})(\omega_k^{(1)}), \ldots, (G_{\lambda,s} \circ G_s^{-1})(\omega_k^{(s)}))$ is a point on $[0,1]^s$, with the distribution function H. As we can see, in order to generate non-uniform random points on $[0,1]^s$, with distribution function H, we need to know the inverse of the distribution function of a NIG distributed random variable or, at least an approximation of it. As the inverse function is not explicitly known, an approximation of it is needed in our simulations. In order to obtain an approximation of the inverse, we use the Matlab function "niginv" as implemented by R. Werner, based on a method proposed by K. Prause in his Ph.D. dissertation [17].

The QMC estimate is defined as follows:

$$\theta_{QMC} = \frac{1}{N}\sum_{k=1}^{N} f(x_k^{(1)}, \ldots, x_k^{(s)}), \tag{28}$$

where $x = (x_k)_{k\geq 1}$ is a H-distributed low-discrepancy sequence on $[0,1]^s$, with $x_k = (x_k^{(1)}, \ldots, x_k^{(s)})$, $k \geq 1$.

In order to generate such a sequence, we apply a method proposed by Hlawka and Mück in [11]. In their method, they create directly H-distributed low-discrepancy sequences, where H can be any distribution function on $[0,1]^s$, with density function h, which can be factored into a product of independent, one-dimensional densities. The method is based on the following theoretical result.

Theorem 11. *([10]) Consider an s-dimensional continuous distribution on $[0,1]^s$, with distribution function H and density function $h(u) = \prod_{j=1}^{s} h_j(u^{(j)})$, $\forall u = \big(u^{(1)}, \ldots, u^{(s)}\big) \in [0,1]^s$. Assume that $h_j(t) \neq 0$, for almost every $t \in [0,1]$ and for all $j = 1, \ldots, s$. Furthermore, assume that h_j, $j = 1, \ldots, s$, are continuous on $[0,1]$. Denote by $M_f = \sup_{u\in[0,1]^s} f(u)$. Let $\omega = (\omega_1, \ldots, \omega_N)$ be a sequence in $[0,1]^s$. Generate the sequence $x = (x_1, \ldots, x_N)$, with*

$$x_k^{(j)} = \frac{1}{N}\sum_{r=1}^{N}\big[1 + \omega_k^{(j)} - H_j\big(\omega_r^{(j)}\big)\big] = \frac{1}{N}\sum_{r=1}^{N} 1_{[0,\omega_k^{(j)}]}\big(H_j\big(\omega_r^{(j)}\big)\big), \tag{29}$$

for all $k = 1, \ldots, N$ and all $j = 1, \ldots, s$, where $[a]$ denotes the integer part of a. Then the generated sequence x has a H-discrepancy of

$$D_{N,H}(x_1, \ldots, x_N) \leq (2 + 6sM_f)D_N(\omega_1, \ldots, \omega_N).$$

As our distribution function H can be factored into independent marginals, and has the support on $[0,1]^s$, we can apply directly the above theorem, to generate H-distributed low-discrepancy sequences. During our experiments, we employed as low-discrepancy sequences $\omega = (\omega_k)_{k\geq 1}$ on $[0,1]^s$, the Halton sequences (see [9]).

All points constructed by the Hlawka-Mück method are of the form i/N, $i = 0, \ldots, N$, in particular some elements of the sequence $x = (x_1, \ldots, x_N)$ might assume a value of 0 or 1. A value of 1 is a singularity of the function $f(x)$, due to the logarithm from the definition of $G^{-1}_{\lambda,i}(x)$, which becomes unbounded if $x = 1$. Hence, the sequence constructed with Hlawka-Mück method is not directly suited for unbounded problems. To overcome this problem, Kainhofer (see [13]) suggests to define a new sequence, in which the value 1 is replaced by $1/N$, where N is the number of points in the set. This slight modification of the sequence is shown to have a minor influence, as the transformed set does not loose its low-discrepancy and can be used for QMC integration.

The estimate proposed by us earlier is:

$$\theta_m = \frac{1}{N}\sum_{k=1}^{N} f(q_k^{(1)}, \ldots, q_k^{(d)}, X_k^{(d+1)}, \ldots, X_k^{(s)}), \tag{30}$$

where $(q_k, X_k)_{k\geq 1}$ is an s-dimensional H*-mixed sequence* on $[0,1]^s$.

In order to obtain such a H-mixed sequence, we first construct the H_q-distributed low-discrepancy sequence $(q_k)_{k\geq 1}$ on $[0,1]^d$, using the Hlawka-Mück method (the distribution function H_q was defined in (3)). Next, we generate the independent and identically distributed random points x_k, $k \geq 1$ on $[0,1]^{s-d}$, with the common distribution function H_X, using the inversion method (the distribution function H_X was defined in (4)). Finally, we concatenate q_k and x_k for each $k \geq 1$, and get our H-mixed sequence on $[0,1]^s$.

In our experiments, we used as low-discrepancy sequences on $[0,1]^d$, for the generation of H-mixed sequences, the Halton sequences (see [9]).

We suppose that the parameters of the NIG-distributed log-returns under the equivalent martingale measure given by the Esscher transform are given by

$$\mu = 0.00079 * 5,\ \beta = -15.1977,\ \alpha = 136.29,\ \delta = 0.0059 * 5, \tag{31}$$

and they are the same as in Kainhofer (see [13]). We observe that these parameters are relevant for daily observed stock price log-returns (see [21]). As the class of NIG distributions is closed under convolution, we can derive weekly stock prices by using a factor of 5 for the parameters μ and δ. We suppose further that the initial stock price is $S(0) = 100$ and the risk-free annual interest rate is $r = 3.75\%$.

The option is sampled at weekly time intervals. We also let the option to have maturities of 12 and 20 weeks. Hence, our problem is a 12 and 20-dimensional integral, respectively, over the payoff function.

We are going to compare the three estimates in terms of their absolute error, where the "exact" option price is obtained as the average of 10 MC simulations, with $N = 100000$ for the initial integral (21).

In our tests we have considered the following dimensions of the transformed integral (25) on $[0,1]^s$: $s = 12,\ 20$. The MC and H-mixed estimates are the mean values of 10 independent runs, while the QMC estimate is the result of a single run. The results are presented in two tables, each table containing the number of samples N, which varies from 5000 to 8500 with a step of 500, and the absolute error of the three estimates.

N	Absolute error MC	Absolute error QMC	Absolute error Mixed Method
5000	0.014731	0.012385	0.007676
5500	0.004485	0.016085	0.003780
6000	0.009268	0.011866	0.001892
6500	0.020887	0.014721	0.002547
7000	0.027395	0.014732	0.008411
7500	0.006316	0.012404	0.017385
8000	0.015027	0.010519	0.012538
8500	0.009207	0.010140	0.007248

Table 1: European Call Option. Case $d = 4$ and $s = 12$.

The numerical results for $s = 12$ and $d = 4$ are presented in Table 1. The results produced by our H-mixed sequence are much better than the ones obtained by using pseudorandom or low-discrepancy sequences, in almost all situations.

N	Absolute error MC	Absolute error QMC	Absolute error Mixed Method
5000	0.006381	0.049304	0.004311
5500	0.030035	0.039222	0.008018
6000	0.017018	0.042674	0.019373
6500	0.004735	0.041674	0.012044
7000	0.023534	0.038581	0.013131
7500	0.020561	0.030509	0.001833
8000	0.028440	0.027873	0.008792
8500	0.012737	0.032972	0.007264

Table 2: European Call Option. Case $d = 7$ and $s = 20$.

To increase the difficulty of the problem, we increase the dimension of the integral to $s = 20$. Table 2 displays the results we get for $s = 20$ and $d = 7$. From this simulations, we see again that the H-mixed sequence outperforms both the pseudorandom and low-discrepancy sequences, for almost all sample sizes N. The absolute error produced by our H-mixed sequence is smaller than the one produced by the low-discrepancy sequence, in all situations.

As a general conclusion for this option pricing problem, we can say that by the use of H-mixed sequences, we obtain increasing advantages over the classical pseudorandom and low-discrepancy sequences, for relatively high dimensions and moderate sample sizes.

4. **Application to finance: Asian options**

In this section, we consider an Asian option pricing problem. We compare numerically our mixed method with the MC and QMC methods, when they are applied to so-called (discrete sampled) Asian options driven by the asset dynamics $S(t)$, as defined in (16). The general setting remains the same as in the previous section, but the payoff function is changed. The payoff of an Asian call option is defined as

$$C_T(S) = \Big(\frac{1}{s}\sum_{i=1}^{s} S(t_i) - K\Big)_+ = \max\Big\{\frac{1}{s}\sum_{i=1}^{s} S(t_i) - K, 0\Big\}, \tag{32}$$

with $0 = t_0 < t_1 < t_2 < \ldots < t_s = T$. The constant $K \geq 0$ is called the strike price. Hence, we get the following integration problem:

$$I = \int_{\mathbb{R}^s} \underbrace{\Big(\frac{S(0)}{s} \sum_{i=1}^{s} e^{\sum_{j=1}^{i} x^{(j)}} - K \Big)_+}_{A(x)} dG(x) = \int_{\mathbb{R}^s} A(x) dG(x), \tag{33}$$

where $G(x) = \prod_{i=1}^{s} G_i(x^{(i)}),\ \forall x = (x^{(1)}, \ldots, x^{(s)}) \in \mathbb{R}^s$, and $G_i(x^{(i)})$ denotes the distribution function of the so-called log returns induced by $L(t_1)$, with the corresponding density function $g_i(x^{(i)})$. These log increments are independent and NIG distributed, having the common density function defined in (22).

In order to approximate the integral (33), we have to transform it to an integral on $[0,1]^s$. We can do this in a similar way as we did for European Call options, in the previous section. In the end, we get the following integration problem on $[0,1]^s$:

$$I = \int_{[0,1]^s} \underbrace{\Big(\frac{S(0)}{s} \sum_{i=1}^{s} e^{\sum_{j=1}^{i} G_{\lambda,i}^{-1}(z^{(j)})} - K \Big)_+}_{f(z)} dH(z) = \int_{[0,1]^s} f(z) dH(z), \tag{34}$$

where $H : [0,1]^s \to [0,1]$, defined by

$$H(z) = \prod_{i=1}^{s} (G_i \circ G_{\lambda,i}^{-1})(z^{(i)}),\ \forall z = (z^{(1)}, \ldots, z^{(s)}) \in [0,1]^s, \tag{35}$$

is a distribution function on $[0,1]^s$, with independent marginals $H_i = G_i \circ G_{\lambda,i}^{-1}$, $i = 1, \ldots, s$.

Next, we compare numerically our estimator θ_m, with the estimators obtained using the MC and QMC methods. All three estimators θ_m, θ_{MC} and θ_{QMC}, and the corresponding sequences are defined in the previous section. The function $f(z)$ is defined in relation (34). As a measure of comparison, we will use the absolute errors produced by these three methods, in the approximation of the integral (34).

We suppose that the parameters of the NIG-distributed log-returns under the equivalent martingale measure given by the Esscher transform are the same as in (31). We assume that the initial stock price is $S(0) = 100$, and the risk-free annual interest rate is $r = 3.75\%$.

For our mixed method and QMC estimate, we use a Halton sequence as low-discrepancy sequence on $[0,1]^s$. The Asian call option is sampled weekly. We also let the option to have maturities of 12 and 20 weeks. Hence, our problem is a 12 and 20-dimensional integral, respectively, over the payoff function.

We are going to compare the three estimates in terms of their absolute error, where the "true" price is obtained as the average of 10 MC simulations, with $N = 100000$. The MC and H-mixed estimates are the mean values of 10 independent runs, while the QMC estimate is the result of a single run. The results are presented in two tables, each table containing the number of samples N, which varies from 4000 to 7000 with a step size of 500, and the absolute error of the three estimates.

N	Absolute error MC	Absolute error QMC	Absolute error Mixed Method
4000	0.004833	0.000723	0.000690
4500	0.003060	0.001083	0.000977
5000	0.001095	0.000380	0.001653
5500	0.000293	0.000618	0.000599
6000	0.011389	0.001482	0.000898
6500	0.001733	0.003187	0.000218
7000	0.008720	0.001582	0.000047

Table 3: Asian Option. Case $d = 4$ and $s = 12$.

In Table 3 we present the results obtained for $s = 12$ and $d = 4$. The H-mixed sequence gives excellent estimates for almost all N, clearly dominating both the pseudorandom and low-discrepancy sequences.

N	Absolute error MC	Absolute error QMC	Absolute error Mixed Method
4000	0.036868	0.014541	0.004318
4500	0.007101	0.011003	0.016654
5000	0.002396	0.009723	0.012004
5500	0.016070	0.008897	0.000818
6000	0.004666	0.008920	0.000003
6500	0.017100	0.009541	0.003007
7000	0.010705	0.009437	0.002776

Table 4: Asian Option. Case $d = 7$ and $s = 20$.

The estimates presented in Table 4 are the results of the simulations for a higher dimensional problem, with $s = 20$ and $d = 7$. Again, the H-mixed method outperforms the conventional MC and QMC methods, in almost all situations.

We can conclude that our mixed method can give considerable improvements over the MC and QMC methods, in estimating high dimensional integrals, which we encounter in problems from financial mathematics, such as valuation of Asian options and European options.

References

[1] Barndorff-Nielsen, O.E., *Processes of normal invers Gaussian type*, Finance and Stochastic, **2**(1998), 41-68.

[2] Boyle, P.P., *Options: A Monte Carlo Approach*, J. Financial Economics, **4**(1977), 323-338.

[3] Chelson, P., *Quasi-random techniques for Monte Carlo methods*, Ph.D. Dissertation, The Claremont Graduate School, 1976.

[4] Deak, I., *Random number generators and simulation*, Akademiai Kiado, Budapest, 1990.

[5] Delbaen, F., Schachermayer, W., *A general version of the fundamental theorem of assest pricing*, Math. Ann., **300**(3)(1994), 463-520.

[6] Devroye, L., *Non-uniform random variate generation*, Springer-Verlag, New-York, 1986.

[7] Eberlein, E., Keller, U., *Hyperbolic distribution in finance*, Bernoulli, **1**(1995), 281-299.

[8] Glasserman, P., *Monte Carlo methods in financial engineering*, Springer-Verlag, New-York, 2003.

[9] Halton, J.H., *On the efficiency of certain quasi-random sequences of points in evaluating multidimensional integrals*, Numer. Math., **2**(1960), 84-90.

[10] Hlawka, E., *Gleichverteilung und Simulation*, Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II, **206**(1997), 183-216.

[11] Hlawka, E., Mück, R., *A transformation of equidistributed sequences*, Applications of number theory to numerical analysis, Academic Press, New-York, 1972, 371-388.

[12] Joy, C., Boyle, P.P., Tan, K.S., *Quasi-Monte Carlo methods in numerical finance*, Management Science, Vol. 42, No. **6**(1996), 926-938.

[13] Kainhofer, R.F., *Quasi-Monte Carlo algorithms with applications in numerical analysis and finance*, Ph.D. Dissertation, Graz University of Technology, Austria, 2003.

[14] Madan, D.B., and Seneta, E., *The variance gamma (V.G.) model for share market returns*, Journal of Business, **63**(1990), 511-524.

[15] Ninomya, S., Tezuka, S., *Toward real time pricing of complex finacial derivatives*, Applied Mathematical Finance, Vol. **3**(1996), 1-20.

[16] Ökten, G., Tuffin, B., Burago, V., *A central limit theorem and improved error bounds for a hybrid-Monte Carlo sequence with applications in computational finance*, Journal of Complexity, Vol. 22, No. **4**(2006), 435-458.

[17] Prause, K., *The generalized hyperbolic model: estimation, financial derivatives and risk measures*, Ph.D. Dissertation, Albert-Ludwigs-Universitat, Freiburg, 1999.

[18] Raible, S., *Lévy processes in finance: theory, numerics and empirical facts*, Ph.D. Dissertation, Albert-Ludwigs-Universitat, Freiburg, 2000.

[19] Roşca, A.V., *A mixed Monte Carlo and quasi-Monte Carlo sequence for multidimensional integral estimation*, Acta Universitatis Apulensis, Mathematics-Informatics, No. **14**(2007), 141-160.

[20] Roşca, N., *A combined Monte Carlo and quasi-Monte Carlo method for estimating multidimensional integrals*, Studia Universitatis Babes-Bolyai, Mathematica, Vol. LII, No. **1**(2007), 125-140.

[21] Rydberg, T.H., *The normal inverse Gaussian Lévy process: simulation and approximation*, Comm. Statist. Stochastic Models, Vol. 13, No. **4**(1997), 887-910.

[22] Stancu, D.D., Coman, Gh., Blaga, P., *Numerical analysis and approximation theory*, (in Romanian) Vol. II, Presa Universitară Clujeană, Cluj-Napoca, 2002.

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