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MULTIPLE SOLUTIONS FOR A DOUBLE EIGENVALUE ELLIPTIC PROBLEM IN DOUBLE WEIGHTED SOBOLEV SPACES

ILDIKÓ ILONA MEZEI

Abstract. In this paper we study a semilinear double eigenvalue problem with nonlinear boundary conditions in an unbounded domain $\Omega \in \mathbb{R}^N$. To obtain existence and multiplicity results for this problem we use the Mountain Pass Theorem applied to double weighted Sobolev spaces and a recent result proved by G. Bonanno (Nonlinear Analysis, **54**(2003), 651-665) concerning critical points. This result completes some recent results obtained in this direction.

1. **Main result**

Let $\Omega \subset \mathbb{R}^N,\ (N \geq 3)$ be an unbounded domain with smooth boundary Γ. For a positive measurable function u and a positive measurable function w defined in Ω, we define the weighted p-norm $(1 \leq p < \infty)$

$$||u||_{p,\Omega,w} = \left(\int_\Omega |u(x)|^p w(x) dx \right)^{\frac{1}{p}}$$

and denote by $L^q(\Omega; w)$ the space of all measurable functions u such that $||u||_{q,\Omega,w}$ is finite. The double weighted Sobolev space

$$W^{1,p}(\Omega; v_0, v_1)$$

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is defined as the space of all functions $u \in L^p(\Omega; v_0)$ such that all derivatives $\frac{\partial u}{\partial x_i}$ belong to $L^p(\Omega; v_1)$. The corresponding norm is defined by

$$||u||_{p,\Omega,v_0,v_1} = \left(\int_\Omega |\nabla u(x)|^p v_1(x) + |u(x)|^p v_0 dx \right)^{\frac{1}{p}}.$$

The Muckenhoupt class A_p is defined as the set of all positive functions v in $\mathbb{R}^N$, which satisfy

$$\frac{1}{|Q|} \left(\int_\Omega v \ dx \right)^{\frac{1}{p}} \left(\int_\Omega v^{-\frac{1}{p-1}} \ dx \right)^{\frac{p-1}{p}} \le \bar{C}, \text{ if } 1 < p < \infty$$

$$\frac{1}{|Q|} \int_\Omega v \ dx \le \bar{C} \text{ ess} \inf_{x \in Q} v(x), \text{ if } p = 1,$$

for all cubes $Q \in \mathbb{R}^N$ and some $\bar{C} > 0$.

In this paper we always assume that the weight functions $v_0, \ v_1, \ w$ are defined in Ω, belong to A_p and are choosen such that the embeddings

$$W^{1,2}(\Omega; v_0, v_1) \hookrightarrow L^p(\Omega; w) \tag{1}$$

and the trace

$$W^{1,2}(\Omega; v_0, v_1) \hookrightarrow L^q(\Gamma; w) \tag{2}$$

are compact for $2 < p < 2N/(N-2)$, $2 < q < 2(N-1)/(N-2)$ and continuous for $2 \le p \le 2N/(N-2)$, $2 \le q \le 2(N-1)/(N-2)$ respectively. Such weight functions there exist, see for example [4], [5]. The best embedding constants are denoted by $C_{p,\Omega}$ and $C_{q,\Gamma}$, i.e. we have the inequalities

$$||u||_{p,\Omega,w} \le C_{p,\Omega} ||u||_{v_0,v_1}, \quad \text{for all } u \in W^{1,2}(\Omega; v_0, v_1) \tag{3}$$

$$||u||_{q,\Gamma,w} \le C_{q,\Gamma} ||u||_{v_0,v_1}, \quad \text{for all } u \in W^{1,2}(\Omega; v_0, v_1) \tag{4}$$

where we used the abbreviation $||u||_{v_0,v_1} = ||u||_{2,\Omega,v_0,v_1}$.

For $\lambda > 0$ and $\mu \in \mathbb{R}$ we consider the following semilinear elliptic double eigenvalue problem

$$(P_{\lambda,\mu}) \qquad \begin{cases} Au \equiv -\Delta u + b(x)u = \lambda f(x,u) \text{ in } \Omega \\ \\ \partial_n u = \lambda \mu g(x,u) \text{ on } \Gamma \end{cases},$$

where b is a positive measurable function, n denotes the unit outward normal on Γ and ∂_n is the outer normal derivative on Γ.

We define a bilinear form associated with A by

$$\langle u, v\rangle_A = \int_\Omega (\nabla u \nabla v + b(x)uv)dx.$$

A weak solution of the problem $(P_{\lambda,\mu})$ is a function $u \in W^{1,2}(\Omega; v_0, v_1)$, such that for every $v \in W^{1,2}(\Omega; v_0, v_1)$ we have

$$\langle u, v\rangle_A - \lambda \int_\Omega f(x, u(x))v(x)dx - \lambda\mu \int_\Gamma g(x, u(x))v(x)d\Gamma = 0.$$

Furthermore we consider the following assumptions:

(A) we assume that A defines a continuous bilinear form $\langle \cdot, \cdot\rangle_A$ on $W^{1,2}(\Omega; v_0, v_1)$ and satisfies the ellipticity condition

$$\langle u, u\rangle_A \geq 2K||u||^2_{v_0,v_1} \text{ for every } u \in W^{1,2}(\Omega; v_0, v_1), \tag{5}$$

with some positive constant $K > 0$;

(F1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with $f(\cdot, 0) = 0$ and

$$|f(x, s)| \leq f_0(x) + f_1(x)|s|^{p-1} \text{ for } x \in \Omega,\ s \in \mathbb{R},$$

where $2 < p < \dfrac{2N}{N-2}$, and f_0, f_1 are positive measurable functions satisfying $f_0 \in L^{\frac{p}{p-1}}(\Omega; w^{\frac{1}{1-p}})$, $f_0(x) \leq C_f w(x)$ and $f_1(x) \leq C_f w(x)$ for a.e. $x \in \Omega$, with an appropiate constant C_f;

(F2) $\lim\limits_{s\to 0} \dfrac{f(x, s)}{f_0(x)|s|} = 0$, uniformly in $x \in \Omega$;

(F3) $\lim\limits_{s\to\infty} \dfrac{F(x, s)}{f_0(x)|s|^2} = 0$, uniformly in $x \in \Omega$,
$\max\limits_{|s|\leq M} F(\cdot, s) \in L^1(\Omega)$, for all $M > 0$, where

$$F(x, u) = \int_0^u f(x, s)ds;$$

(F4) there exist $x_0 \in \Omega$, $s_0 \in \mathbb{R}$ and $R_0 > 0$ such that $\min\limits_{|x-x_0|<R} F(x, s_0) > 0$.

(G1) Let $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ be a a Carathéodory function with $g(\cdot, 0) = 0$ and

$$|g(x, s)| \leq g_0(x) + g_1(x)|s|^{q-1}, \text{ for } x \in \Gamma,\ s \in \mathbb{R}$$

where $2 < q < \dfrac{2(N-1)}{N-2}$, and g_0, g_1 are positive measurable functions satisfying $g_0 \in L^{\frac{q}{q-1}}(\Gamma; w^{\frac{1}{1-q}})$, $g_0(x) \leq C_g w(x)$ and $g_1(x) \leq C_g w(x)$, a.e. $x \in \Gamma$, with an appropiate constant C_g;

(G2) $\lim\limits_{s\to 0} \dfrac{g(x,s)}{g_0(x)|s|} = 0$, uniformly in $x \in \Gamma$;

(G3) $\lim\limits_{s\to +\infty} \dfrac{G(x,s)}{g_0(x)|s|^2} = 0$, uniformly for $x \in \Gamma$, $\max\limits_{|s|\leq M} G(\cdot, s) \in L^1(\Gamma)$, for every $M > 0$, where $G(x,s) = \displaystyle\int_0^u g(x,s)ds$.

Next, we introduce the functionals $J_F, J_G, J_\mu : W^{1,2}(\Omega; v_0, v_1) \to \mathbb{R}$, defined by

$$J_F(u) = \int_\Omega F(x, u(x))dx, \qquad J_G(u) = \int_\Gamma G(x, u(x))d\Gamma,$$

$$J_\mu(u) = J_F(u) + \mu J_G(u)$$

and the energy functional $\mathcal{E}_{\lambda,\mu}(u) : W^{1,2}(\Omega; v_0, v_1) \to \mathbb{R}$ associated to $(P_{\lambda,\mu})$, defined by

$$\mathcal{E}_{\lambda,\mu}(u) = \frac{1}{2}\langle u, u\rangle_A - \lambda J_\mu(u).$$

The main result of this paper is the following

Theorem 1.1. *We suppose that the assumption (A) is satisfied and the functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ satisfy the conditions $(F1) - (F4)$ and $(G1) - (G3)$ respectively.*

(a) *Then there exists $\lambda_0 > 0$ such that to every $\lambda \in]\lambda_0, +\infty[$ it corresponds a nonempty open interval $I_\lambda \subset \mathbb{R}$ such that for every $\mu \in I_\lambda$ the problem $(P_{\lambda,\mu})$ has at least two distinct, nontrivial weak solutions $u_{\lambda,\mu}$ and $v_{\lambda,\mu}$, with the property*

$$\mathcal{E}_{\lambda,\mu}(u_{\lambda,\mu}) < 0 < \mathcal{E}_{\lambda,\mu}(v_{\lambda,\mu}).$$

(b) *Then there exists $\mu_0 > 0$ such that to every $\mu \in [-\mu_0, \mu_0]$ it corresponds a nonempty open interval $\Gamma_\mu \in]0, +\infty[$ and a number $\sigma_\mu > 0$ for which*

$(P_{\lambda,\mu})$ has at least two distinct, nontrivial weak solutions: $u^1_{\lambda,\mu}$ and $u^2_{\lambda,\mu}$, with the property

$$max\{||u^1_{\lambda,\mu}||_{v_0,v_1}, ||u^2_{\lambda,\mu}||_{v_0,v_1}\} \leq \sigma_\mu,$$

whenever $\lambda \in \Gamma_\mu$.

2. **Preliminaries**

In this section we denote by p' and q' the conjugates of p respective q, i.e. $p' = \frac{p}{p-1}$ and $q' = \frac{q}{q-1}$.

The following result deals with the Nemytskii operator of a Carathéodory function $h : \Omega \times \mathbb{R} \to \mathbb{R}$, which is the function defined by $N_h(u) = h(x, u(x))$. Then we have the following result.

Lemma 2.1. *Assume that the conditions* (F1)*,* (G1) *are satisfied. Then the Nemytskii operators $N_f : L^p(\Omega; w) \to L^{\frac{p}{p-1}}(\Omega; w^{\frac{1}{1-p}})$, $N_F : L^p(\Omega; w) \to L^1(\Omega)$, $N_g : L^q(\Gamma; w) \to L^{\frac{q}{q-1}}(\Gamma; w^{\frac{1}{1-q}})$ and $N_F : L^q(\Gamma; w) \to L^1(\Gamma)$ are bounded and continuous.*

Proof. We will use the following result: for all $s \in (0, \infty)$ there is a constant $C_s > 0$ such that

$$(x + y)^s \leq C_s(x^s + y^s), \quad \text{for any } x, y \in (0, \infty). \tag{6}$$

To prove that N_f is bounded, we choose an arbitrary set $A \subseteq L^p(\Omega; w)$ and prove that $N_f(A)$ is bounded. For this, let $u \in A$ be an arbitrary element and we claim that $N_f(u)$ is bounded in $L^{\frac{p}{p-1}}(\Omega; w^{\frac{1}{1-p}})$. Using the (F1) condition, the (6), the Hölder's inequalities, we have

$$||N_f(u)||^{\frac{1}{p'}}_{\frac{p}{p-1},\Omega,w^{\frac{1}{1-p}}} = \int_\Omega |f(x, u(x))|^{p'} w(x)^{\frac{1}{1-p}} dx \leq$$

$$\leq \int_\Omega \left(f_0(x) + f_1(x)|u(x)|^{p-1}\right)^{p'} w(x)^{\frac{1}{1-p}} dx \leq$$

$$\leq C_{p'} \left(\int_\Omega f_0(x)^{p'} w(x)^{\frac{1}{1-p}} dx + \int_\Omega f_1(x)^{p'} |u(x)|^{(p-1)p'} w(x)^{\frac{1}{1-p}} dx \right) \leq$$

$$\leq C_{p'} \left(C + \int_\Omega C_f^{p'} w(x)^{p'} w(x)^{\frac{1}{1-p}} |u(x)|^p dx \right) =$$

$$= C_{p'}C + C_{p'}C_f^{p'}\int_\Omega |u(x)|^p w(x)dx = C_{p'}C + C_{p'}C_f^{p'}||u||^p_{p,\Omega,w},$$

where in the last inequality we used that $f_0 \in L^{\frac{p}{p-1}}(\Omega; w^{\frac{1}{1-p}})$, so there exists $C > 0$ such that $\int_\Omega f_0(x)^{\frac{p}{p-1}} w(x)^{\frac{1}{1-p}} dx \le C$. Since $u \in A \subseteq L^p(\Omega; w)$, we have that $||u||^p_{p,\Omega,w}$ is finite, therefore N_f is bounded. Then the continuity follows from standard properties of the Nemytskii operators.

In the same way we obtain for $u \in L^p(\Omega; w)$

$$\int_\Omega |F(x, u(x))|dx \le \int_\Omega (f_0(x)|u(x)| + f_1(x)|u(x)|^p)\, dx =$$

$$= \int_\Omega f_0(x)w(x)^{-\frac{1}{p}}|u(x)|w(x)^{\frac{1}{p}}dx + \int_\Omega f_1(x)|u(x)|^p dx \le$$

$$\le \left(\int_\Omega f_0(x)^{p'} w(x)^{\frac{1}{1-p}}dx\right)^{\frac{1}{p'}} \left(\int_\Omega |u(x)|^p w(x)dx\right)^{\frac{1}{p}} + C_f \int_\Omega |u(x)|^p w(x)dx \le$$

$$\le C^{\frac{1}{p'}}||u||_{p,\Omega,w} + C_f||u||^p_{p,\Omega,w},$$

therefore N_F is bounded. For the operators N_g and N_G the arguments are identical, therefore we omit the details here. □

Lemma 2.2. [5] *The energy functional $\mathcal{E}_{\lambda,\mu}$ is Fréchet differentiable in $W^{1,2}(\Omega; v_0, v_1)$ and its derivative is given by*

$$\langle \mathcal{E}'_{\lambda,\mu}(u), v\rangle = \langle u, v\rangle_A - \lambda \int_\Omega f(x, u(x))v(x)dx - \lambda\mu \int_\Gamma g(x, u(x))v(x)d\Gamma. \tag{7}$$

for every $v \in W^{1,2}(\Omega; v_0, v_1)$.

Remark 2.1. Due to this result, one can see, that the critical points of $\mathcal{E}_{\lambda,\mu}$ are exactly the weak solutions of $(P_{\lambda,\mu})$.

Lemma 2.3. *Suppose that the conditions* (F2), (F3), (G2) *and* (G3) *are satisfied. Then, for every $\lambda > 0$ and $\mu \in \mathbb{R}$ the functional $\mathcal{E}_{\lambda,\mu}$ is coercive and bounded from below on $W^{1,2}(\Omega; v_0, v_1)$.*

Proof. Let us fix $\lambda > 0$ and $\mu \in \mathbb{R}$ arbitrarily and $a,\ b > 0$ such that

$$\lambda a C_f C^2_{2,\Omega} + \lambda|\mu|bC_g C^2_{2,\Gamma} < K.$$

By the conditions (F2),(F3) and (G2),(G3) there exist the positive functions $k_a \in L^1(\Omega; w)$ and $k_b \in L^1(\Gamma; w)$ such that

$$|F(x,s)| \leq af_0(x)|s|^2 + k_a(x)w(x), \quad \forall (x,s) \in \Omega \times \mathbb{R}$$

$$|G(x,s)| \leq bg_0(x)|s|^2 + k_b(x)w(x), \quad \forall (x,s) \in \Omega \times \mathbb{R}.$$

Thus, for every $u \in W^{1,2}(\Omega; v_0, v_1)$ we obtain

$$\begin{aligned}
\mathcal{E}_{\lambda,\mu}(u) &= \frac{1}{2}\langle u,u\rangle_A - \lambda \int_\Omega F(x,u(x))dx - \lambda\mu \int_\Gamma G(x,u(x)dx) \geq \\
&\geq K||u||^2_{v_0,v_1} - \lambda \int_\Omega af_0(x)|u(x)|^2 dx - \lambda \int_\Omega k_a(x)w(x)dx - \\
&\quad -\lambda|\mu| \int_\Gamma bg_0(x)|u(x)|^2 d\Gamma - \lambda|\mu| \int_\Gamma k_b(x)w(x)d\Gamma \geq \\
&\geq K||u||^2_{v_0,v_1} - \lambda aC_f||u||^2_{2,\Omega,w} - \lambda||k_a||_{1,\Omega,w} - \\
&\quad -\lambda|\mu|bC_g||u||^2_{2,\Gamma,w} - \lambda|\mu|||k_b||_{1,\Gamma,w} \geq \\
&\geq \left(K - \lambda aC_f C^2_{2,\Omega} - \lambda|\mu|bC_g C^2_{2,\Gamma}\right)||u||^2_{v_0,v_1} - \\
&\quad -\lambda||k_a||_{1,\Omega,w} - \lambda|\mu|||k_b||_{1,\Gamma,w}.
\end{aligned}$$

Since $k_a \in L^1(\Omega; w)$, $k_b \in L^1(\Gamma; w)$, we have that $||k_a||_{1,\Omega,w}$, $||k_b||_{1,\Gamma,w}$ are finite. Therefore $\mathcal{E}_{\lambda,\mu}$ is bounded from below on $W^{1,2}(\Omega; v_0, v_1)$ and $\mathcal{E}_{\lambda,\mu}(u) \to \infty$, whenever $||u||_{v_0,v_1} \to \infty$. Hence $\mathcal{E}_{\lambda,\mu}$ is coercive. □

Lemma 2.4. $\mathcal{E}_{\lambda,\mu} : W^{1,2}(\Omega; v_0, v_1) \to \mathbb{R}$ *satisfies the Palais-Smale condition on* $W^{1,2}(\Omega; v_0, v_1)$*, for every* $\lambda > 0$ *and* $\mu \in \mathbb{R}$.

Proof. Let $\{u_n\} \subset W^{1,2}(\Omega; v_0, v_1)$ be an arbitrary Palais-Smale sequence for $\mathcal{E}_{\lambda,\mu}$, i.e.

(a) $\{\mathcal{E}_{\lambda,\mu}(u_n)\}$ is bounded;

(b) $\mathcal{E}'_{\lambda,\mu}(u_n) \to 0$.

We have to prove that $\{u_n\}$ contains a strongly convergent subsequence. Since $\mathcal{E}_{\lambda,\mu}$ is coercive, we have that $\{u_n\}$ is bounded. $W^{1,2}(\Omega; v_0, v_1)$ is a reflexive Banach space, so taking a subsequence if necessary (denoted in the same way), we get an element $u \in W^{1,2}(\Omega; v_0, v_1)$ such that $u_n \to u$ weakly in $W^{1,2}(\Omega; v_0, v_1)$. Because the embeddings (1) and (2) are compact for $2 < p < 2N/(N-2)$, $2 < q < 2(N-1)/(N-2)$, we have that $u_n \to u$ strongly in $L^p(\Omega; w)$ and $L^q(\Gamma; w)$.

From the condition (b) we have that $\left|\langle \mathcal{E}'_{\lambda,\mu}(u_n), \frac{u_n}{||u_n||_{v_0,v_1}}\rangle\right| \leq \varepsilon$, for every $\varepsilon > 0$ and large $n \in \mathbb{N}$. Then

$$-\langle u_n, u_n\rangle_A + \lambda \int_\Omega f(x, u_n(x))u_n(x)dx + \lambda\mu \int_\Gamma g(x, u_n(x))u_n(x)d\Gamma \leq \varepsilon ||u_n||_{v_0,v_1}.$$

Then we have

$$2K||u_n - u||^2_{v_0,v_1} \leq \langle u_n - u, u_n - u\rangle_A \leq |\langle u_n, u_n - u\rangle_A| + |\langle u, u_n - u\rangle_A| \leq$$

$$\leq 2\varepsilon||u_n - u||_{v_0,v_1}+$$

$$+\lambda \left|\int_\Omega f(x, u_n(x))(u_n(x) - u(x))dx\right| + \lambda \left|\int_\Omega f(x, u(x))(u_n(x) - u(x))dx\right| +$$

$$+\lambda|\mu| \left|\int_\Gamma g(x, u_n(x))(u_n(x) - u(x))d\Gamma\right| + \lambda|\mu| \left|\int_\Gamma g(x, u(x))(u_n(x) - u(x))d\Gamma\right|.$$

Using the Hölder's inequality we get

$$\left|\int_\Omega f(x, u_n(x))(u_n(x) - u(x))dx\right| \leq$$

$$\leq \int_\Omega \left|f(x, u_n(x))w(x)^{-\frac{1}{p}}\right| \left|(u_n(x) - u(x))w(x)^{\frac{1}{p}}\right| dx \leq$$

$$\leq \left(\int_\Omega |f(x, u_n(x))|^{p'} w(x)^{-\frac{p'}{p}} dx\right)^{\frac{1}{p'}} \left(\int_\Omega |u_n(x) - u(x)|^p w(x)dx\right)^{\frac{1}{p}} =$$

$$= \left(\int_\Omega |f(x, u_n(x))|^{p'} w(x)^{\frac{1}{1-p}} dx\right)^{\frac{1}{p'}} ||u_n - u||_{p,\Omega,w}$$

and arguing in the same way for g, we obtain

$$\left|\int_\Gamma g(x, u_n(x))(u_n(x) - u(x))dx\right| \leq \left(\int_\Gamma |g(x, u_n(x))|^{q'} w(x)^{\frac{1}{1-q}} d\Gamma\right)^{\frac{1}{q'}} ||u_n - u||_{q,\Gamma,w}.$$

Since $\varepsilon > 0$ is arbitrary, $||u_n - u||_{p,\Omega,w}$ and $||u_n - u||_{q,\Gamma,w}$ tend to zero and $\int_\Omega |f(x, u_n(x))|^{p'} w(x)^{\frac{1}{1-p}} dx$, $\int_\Gamma |g(x, u_n(x))|^{q'} w(x)^{\frac{1}{1-q}} d\Gamma$ are bounded (by Lemma 2.1, using that $\{u_n\}$ is bounded), it follows that $||u_n - u||_{v_0,v_1}$ tends to zero. □

Lemma 2.5. [3, Lemma 3.2] *Assume that (F4) is satisfied. Then there exist an* $u_0 \in W^{1,2}(\Omega; v_0, v_1)$ *such that* $J_F(u_0) > 0$.

Let us define $m = \int_\Gamma |G(x, u_0(x))|d\Gamma$, $\lambda_0 = \dfrac{\frac{1}{2}\langle u_0, u_0\rangle_A}{J_F(u_0)} > 0$ and $\mu^*_\lambda = \frac{1}{\lambda(1+m)}$. $(\lambda - \lambda_0)J_F(u_0) > 0$.

Lemma 2.6. *For $\lambda > \lambda_0$ and $|\mu| \in]0, \mu^*_\lambda]$ we have*

$$\inf_{u \in W^{1,2}(\Omega; v_0, v_1)} \mathcal{E}_{\lambda,\mu}(u) < 0.$$

Proof. It is sufficient to prove, that for $\lambda > \lambda_0$ and $|\mu| \in]0, \mu^*_\lambda]$ we have $\mathcal{E}_{\lambda,\mu}(u_0) < 0$. Indeed,

$$\begin{aligned}
\mathcal{E}_{\lambda,\mu}(u_0) &= \frac{1}{2}\langle u_0, u_0\rangle_A - \lambda J_F(u_0) - \lambda\mu J_G(u_0) \leq \\
&\leq \lambda_0 J_F(u_0) - \lambda J_F(u_0) + \lambda|\mu|m = \\
&= (\lambda_0 - \lambda) J_F(u_0) + \lambda|\mu|m = \\
&= (\lambda_0 - \lambda)\frac{\lambda(1+m)\mu^*_\lambda}{\lambda - \lambda_0} + \lambda|\mu|m = \\
&= -(1+m)\lambda\mu^*_\lambda + \lambda|\mu|m = \\
&= -\lambda\mu^*_\lambda - m\lambda(\mu^*_\lambda - |\mu|) < 0.
\end{aligned}$$

for all $\lambda > \lambda_0$ and $|\mu| \in]0, \mu^*_\lambda]$. □

Lemma 2.7. *For every $\lambda > \lambda_0$ and $\mu \in]0, \mu^*_\lambda]$, the functional $\mathcal{E}_{\lambda,\mu}$ satisfies the Mountain Pass geometry.*

Proof. From the assumptions (F1), (F2), (G1) and (G2) results the existence of $\hat{c}_1(\varepsilon)$, $\hat{c}_2(\varepsilon) > 0$ such that, for every $\hat{\varepsilon} > 0$ we have

$$|f(x,s)| \leq \hat{\varepsilon} f_0(x)|s| + \hat{c}_1(\varepsilon) f_1(x)|s|^{p-1}, \text{ for } 2 < p < \frac{2N}{N-2}, \tag{8}$$

$$|g(x,s)| \leq \hat{\varepsilon} g_0(x)|s| + \hat{c}_2(\varepsilon) g_1(x)|s|^{q-1}, \text{ for } 2 < q < \frac{2(N-1)}{N-2}. \tag{9}$$

Then integrating with respect to the second variable, from 0 to $u(x)$, we get the existence of $c_1(\varepsilon)$, $c_2(\varepsilon) > 0$ such that, for every $\varepsilon > 0$ we have

$$|F(x,u(x))| \leq \varepsilon f_0(x)|u(x)|^2 + c_1(\varepsilon) f_1(x)|u(x)|^p, \text{ for } 2 < p < \frac{2N}{N-2}, \tag{10}$$

$$|G(x,u(x))| \leq \varepsilon g_0(x)|u(x)|^2 + c_2(\varepsilon) g_1(x)|u(x)|^q, \text{ for } 2 < q < \frac{2(N-1)}{N-2}. \tag{11}$$

Fix $\lambda > \lambda_0$ and $\mu \in]0, \mu^*_\lambda[$, then using the (10)and (11) inequalities for every $u \in W^{1,2}(\Omega; v_0, v_1)$ we have

$$\begin{aligned}
\mathcal{E}_{\lambda,\mu}(u) &= \frac{1}{2}\langle u,u\rangle_A - \lambda J_\mu(u) \geq \\
&\geq K||u||^2_{v_0,v_1} - \lambda \int_\Omega |F(x,u(x))|dx - \lambda|\mu| \int_\Gamma |G(x,u(x))|d\Gamma \geq \\
&= K||u||^2_{v_0,v_1} - \lambda\varepsilon C_f ||u||^2_{2,\Omega,w} - \lambda c_1(\varepsilon) C_f ||u||^p_{p,\Omega,w} - \\
&\quad -\lambda|\mu|\varepsilon C_g ||u||^2_{2,\Gamma,w} - \lambda|\mu| c_2(\varepsilon) C_g ||u||^q_{q,\Omega,w} \geq \\
&\geq \left(K - \lambda\varepsilon C_f C^2_{2,\Omega} - \lambda|\mu|\varepsilon C_g C^2_{2,\Gamma}\right) ||u||^2_{v_0,v_1} - \\
&- \lambda c_1(\varepsilon) C_f C^p_{p,\Omega} ||u||^p_{v_0,v_1} - \lambda|\mu| c_2(\varepsilon) C_g C^q_{q,\Gamma} ||u||^q_{v_0,v_1}.
\end{aligned}$$

Using the notations $A = \left(K - \lambda\varepsilon C_f C^2_{2,\Omega} - \lambda|\mu|\varepsilon C_g C^2_{2,\Gamma}\right)$, $B = \lambda c_1(\varepsilon) C_f C^p_{p,\Omega}$, $C = \lambda|\mu| c_2(\varepsilon) C_g C^q_{q,\Gamma}$, we get

$$\mathcal{E}_{\lambda,\mu}(u) \geq (A - B||u||^{p-2}_{v_0,v_1} - C||u||^{q-2}_{v_0,v_1})||u||^2_{v_0,v_1}.$$

We choose $\varepsilon \in \left]0, \frac{K}{2} \frac{1}{\lambda(C_f C^2_{2,\Omega} + |\mu| C_g C^2_{2,\Gamma})}\right[$, so $A > 0$. Now, let $l : \mathbb{R}_+ \to \mathbb{R}$ be the function defined by $l(t) = A - Bt^{p-2} - Ct^{q-2}$. We can see, that $l(0) = A > 0$, so because l is continuous, there exists an $\varepsilon^* > 0$ such that $l(t) > 0$, for every $t \in]0, \varepsilon^*[$. Then for every $u \in W^{1,2}(\Omega; v_0, v_1)$, with $||u||_{v_0,v_1} = \varepsilon^{**} < \min\{\varepsilon^*, ||u_0||_{v_0,v_1}\}$, we have $\mathcal{E}_{\lambda,\mu}(u) \geq \eta(\lambda, \mu, \varepsilon^*) > 0$. From Lemma 2.6 we have $\mathcal{E}_{\lambda,\mu}(u_0) < 0$.

Therefore the functional $\mathcal{E}_{\lambda,\mu}$ satisfies the Mountain Pass geometry, meaning that $\mathcal{E}_{\lambda,\mu}$ satisfies the conditions of the Mountain Pass Theorem (see Theorem 3.1). □

Lemma 2.8. *For every $\mu \in \mathbb{R}_+$, we have*

$$\lim_{\rho \to 0} \frac{\sup\{J_\mu(u) : \frac{1}{2}\langle u,u\rangle_A < \rho\}}{\rho} = 0.$$

Proof. Fix arbitrarily $\varepsilon > 0$ and $p \in \left]2, \dfrac{2N}{N-p}\right[$, $q \in \left]2, \dfrac{2(N-1)}{N-2}\right[$, then from (10) and (11) and the ellipticity condition (A), it follows that

$$\begin{aligned}
J_\mu(u) &= J_F(u) + \mu J_G(u) \leq \\
&\leq \varepsilon\left(C_f C_{2,\Omega}^2 + |\mu| C_g C_{2,\Gamma}^2\right) ||u||_{v_0,v_1}^2 + c_1(\varepsilon) C_f C_{p,\Omega}^p ||u||_{v_0,v_1}^p + \\
&+ |\mu| c_2(\varepsilon) C_g C_{q,\Gamma}^q ||u||_{v_0,v_1}^q \leq \\
&\leq \varepsilon\left(C_f C_{2,\Omega}^2 + |\mu| C_g C_{2,\Gamma}^2\right) \frac{\langle u,u\rangle_A}{2K} + c_1(\varepsilon) C_f C_{p,\Omega}^p \left(\frac{\langle u,u\rangle_A}{2K}\right)^{\frac{p}{2}} + \\
&+ |\mu| c_2(\varepsilon) C_g C_{q,\Gamma}^q \left(\frac{\langle u,u\rangle_A}{2K}\right)^{\frac{q}{2}}.
\end{aligned}$$

Therefore, we have

$$\sup\{J_\mu(u) : \frac{1}{2}\langle u,u\rangle_A < \rho\} \leq$$

$$\leq \varepsilon \frac{\left(C_f C_{2,\Omega}^2 + |\mu| C_g C_{2,\Gamma}^2\right)}{K}\rho + \frac{c_1(\varepsilon) C_f C_{p,\Omega}^p}{K^{\frac{p}{2}}}\rho^{\frac{p}{2}} + |\mu| \frac{c_2(\varepsilon) C_g C_{q,\Gamma}^q}{K^{\frac{q}{2}}}\rho^{\frac{q}{2}}.$$

Since $p > 2,\ q > 2$, dividing this last inequality with ρ and taking the limit whenever $\rho \to 0$, we have the required equality.

Lemma 2.9. *We assume that the conditions* (F1)-(F3) *and* (G1)-(G3) *are satisfied. Then the functional* $J_\mu = J_F + \mu J_G$ *is sequentially weakly continuous.*

Proof. We argue by contradiction. Let u_n be a sequence from $W^{1,2}(\Omega; v_0, v_1)$ weakly convergent to some $u \in W^{1,2}(\Omega; v_0, v_1)$ and $d > 0$ such that

$$|J_\mu(u_n) - J_\mu(u)| \geq d, \quad \text{for all } n \in \mathbb{N}.$$

At the same time we have

$$\begin{aligned}
|J_\mu(u_n) - J_\mu(u)| &\leq \int_\Omega |F(x, u_n(x)) - F(x, u(x))| dx + \\
&+ |\mu| \int_\Gamma |G(x, u_n(x)) - G(x, u(x))| d\Gamma.
\end{aligned}$$

In the sequel, we will estimate the previous two integrals. For this end, first we use the Mean Value Theorem for the function F on the interval $(u_n(x), u(x))$, then we make use of the (3), (8) and the Hölder inequalities. So, there exists a $\theta \in]0,1[$ such that

$$\begin{aligned}
& \int_\Omega |F(x,u_n(x)) - F(x,u(x))|dx = \\
= \; & \int_\Omega |f(x,(1-\theta)u_n(x)+\theta u(x))||u_n(x)-u(x)|dx \le \\
\le \; & \hat{\varepsilon}\int_\Omega f_0(x)|(1-\theta)u_n(x)+\theta u(x)||u_n(x)-u(x)|dx + \\
+ \; & \hat{c}_1(\varepsilon)\int_\Omega f_1(x)|(1-\theta)u_n(x)+\theta u(x)|^{p-1}|u_n(x)-u(x)|dx \le \\
\le \; & \hat{\varepsilon}\int_\Omega f_0(x)(|u_n(x)|+|u(x)|)|u_n(x)-u(x)|dx + \\
& +\hat{c}_1(\varepsilon)\int_\Omega f_1(x)(|u_n(x)|^{p-1}+|u(x)|^{p-1})|u_n(x)-u(x)|dx \le \\
\le \; & \hat{\varepsilon}C_f\int_\Omega |u_n(x)-u(x)|w(x)^{\frac{1}{2}}w(x)^{\frac{1}{2}}(|u_n(x)|+|u(x)|)dx + \\
& +\hat{c}_1(\varepsilon)C_f\int_\Omega |u_n(x)-u(x)|w(x)^{\frac{1}{p}}w(x)^{\frac{1}{p'}}\left(|u_n(x)|^{p-1}+|u(x)|^{p-1}\right)dx \le \\
\le \; & \hat{\varepsilon}C_f\left(\int_\Omega |u_n(x)-u(x)|^2 w(x)dx\right)^{\frac{1}{2}} \cdot \\
& \cdot\left[\left(\int_\Omega |u_n(x)|^2 w(x)dx\right)^{\frac{1}{2}} + \left(\int_\Omega |u(x)|^2 w(x)dx\right)^{\frac{1}{2}}\right] + \\
& +\hat{c}_1(\varepsilon)C_f\left(\int_\Omega |u_n(x)-u(x)|^p w(x)dx\right)^{\frac{1}{p}} \cdot \\
& \cdot\left[\left(\int_\Omega |u_n(x)|^{(p-1)p'} w(x)dx\right)^{\frac{1}{p'}} + \left(\int_\Omega |u(x)|^{(p-1)p'} w(x)dx\right)^{\frac{1}{p'}}\right] \le \\
\le \; & \hat{\varepsilon}C_f||u_n-u||_{2,\Omega,w}(||u_n||_{2,\Omega,w}+||u_n||_{2,\Omega,w}) + \\
& +\hat{c}_1(\varepsilon)C_f||u_n-u||_{p,\Omega,w}\left(||u_n||_{p,\Omega,w}^{\frac{p}{p'}}+||u||_{p,\Omega,w}^{\frac{p}{p'}}\right) \le \\
\le \; & \hat{\varepsilon}C_f C_{2,\Omega}^2||u_n-u||_{v_0,v_1}(||u_n||_{v_0,v_1}+||u||_{v_0,v_1}) + \\
& +\hat{c}_1(\varepsilon)C_f C_{p,\Omega}^{p-1}||u_n-u||_{p,\Omega,w}\left(||u_n||_{v_0,v_1}^{p-1}+||u||_{v_0,v_1}^{p-1}\right).
\end{aligned}$$

Since u_n is weakly convergent to $u \in W^{1,2}(\Omega; v_0, v_1)$, we can assume without loss of generality that there exist a constant $M > 0$ such that

$$||u_n||_{v_0,v_1} \le M \text{ and } ||u_n - u||_{v_0,v_1} \le M, \text{ for all } n \in \mathbb{N}.$$

Then we have

$$|F(x,u_n(x)) - F(x,u(x))| \le 2\hat{\varepsilon} C_f C_{2,\Omega}^2 M^2 + 2\hat{c}_1(\varepsilon) C_f C_{p,\Omega}^{p-1} M^{p-1} ||u_n - u||_{p,\Omega,w}.$$

Arguing as above for the function G, we obtain

$$|G(x,u_n(x)) - G(x,u(x))| \le 2\hat{\varepsilon} C_g C_{2,\Gamma}^2 M^2 + 2\hat{c}_2(\varepsilon) C_g C_{q,\Gamma}^{q-1} M^{q-1} ||u_n - u||_{q,\Gamma,w}.$$

Therefore

$$d \le |J_\mu(u_n) - J_\mu(u)| \le 2\hat{\varepsilon} M^2 (C_f C_{2,\Omega}^2 + C_g C_{2,\Gamma}^2) +$$
$$+ 2\hat{c}_1(\varepsilon) C_f C_{p,\Omega}^{p-1} M^{p-1} ||u_n - u||_{p,\Omega,w} + 2\hat{c}_2(\varepsilon) C_g C_{q,\Gamma}^{q-1} M^{q-1} ||u_n - u||_{q,\Gamma,w}.$$

Because the embeddings (1) and (2) are compact for $2 < p < 2N/(N-2)$, $2 < q < 2(N-1)/(N-2)$, it follows that $||u_n - u||_{p,\Omega,w} \to 0$ and $||u_n - u||_{q,\Gamma,w} \to 0$. Therefore, if $\hat{\varepsilon} > 0$ is sufficiently small and $n \in \mathbb{N}$ is large enough, we have

$$d \le |J_\mu(u_n) - J_\mu(u)| < d,$$

which is a contradiction.

3. **Proof of Theorem 1.1**

For the reader's convenience we recall here the Mountain Pass Theorem used in the proof of Theorem 1.1 (a).

Theorem 3.1. [6, Theorem 2.2] *Let E be a Banach space and $I \in C^1(E,\mathbb{R})$ a functional, satisfying the Palais-Smale condition. Suppose $I(0) = 0$ and*

(*a*) *there are constants $\alpha > 0$ and $\rho > 0$ such that $I(u) \ge \alpha$, for every $||u|| = \rho$;*

(*b*) *there is an $e \in E$ with $||e|| > \rho$ and $I(e) \le 0$.*

Then the number

$$c = \inf_{g \in \Gamma} \max_{v \in g([0,1])} I(v),$$

where

$$\Gamma = \{g \in C([0,1],E) : g(0) = 0, g(1) = e\},$$

is a critical value of I*, with* $c \geq \alpha$*.*

The main tool in the proof of Theorem 1.1 (b) is the following refinement of a B. Ricceri-type critical point theorem ([7], [8]) established by G. Bonanno in [1].

Theorem 3.2. *Let* X *be a separable and reflexive real Banach space and let* $\Phi, J : X \to \mathbb{R}$ *be two continuously Gâteaux differentiable functionals. Assume that there exists* $x_0 \in X$ *such that* $\Phi(x_0) = J(x_0) = 0$ *and* $\Phi(x) \geq 0$ *for every* $x \in X$*, and there exists* $x_1 \in X$*,* $\rho > 0$ *such that*

(i) $\rho < \Phi(x_1)$ *and* $\sup_{\Phi(x)<\rho} J(x) < \rho \dfrac{J(x_1)}{\Phi(x_1)}$*. Further put*

$$\bar{a} = \frac{\zeta\rho}{\rho\dfrac{J(x_1)}{\Phi(x_1)} - \sup\limits_{\Phi(x)<\rho} J(x)},$$

with $\zeta > 1$*, assume that the functional* $\Phi - \lambda J$ *is sequentially weakly lower semicontinuous, satisfies the Palais-Smale condition and*

(ii) $\lim\limits_{||x||\to+\infty} [\Phi(x) - \lambda J(x)] = +\infty$*, for every* $\lambda \in [0, \bar{a}]$*.*

Then there is an open interval $\Lambda \subset [0, \bar{a}]$ *and a number* $\sigma > 0$ *such that for each* $\lambda \in \Lambda$*, the equation* $\Phi'(x) - \lambda J'(x) = 0$ *admits at least three distinct solutions in X, having norm less than* σ*.*

Proof of Theorem 1.1 (a). Fix $\lambda > \lambda_0$ and $\mu \in]0, \mu^*_\lambda[= I_\lambda$. From the Lemma 2.3 and Lemma 2.4 we have that the functional $\mathcal{E}_{\lambda,\mu}$ is bounded from below and satisfies the (PS)-condition. Then $\mathcal{E}_{\lambda,\mu}$ achieves its infimum, i.e. there exists an element $u_{\lambda,\mu} \in W^{1,2}(\Omega; v_0, v_1)$ such that $\mathcal{E}_{\lambda,\mu}(u_{\lambda,\mu}) = \inf\limits_{v\in W^{1,2}(\Omega;v_0,v_1)} \mathcal{E}_{\lambda,\mu}(v)$ (see[6, Theorem 2.7]). So $\mathcal{E}'_{\lambda,\mu}(u_{\lambda,\mu}) = 0$ and by Lemma 2.6, we have $\mathcal{E}_{\lambda,\mu}(u_{\lambda,\mu}) < 0$.

On the other hand, there exists an element $v_{\lambda,\mu} \in W^{1,2}(\Omega; v_0, v_1)$ such that $\mathcal{E}'_{\lambda,\mu}(v_{\lambda,\mu}) = 0$ and $\mathcal{E}_{\lambda,\mu}(v_{\lambda,\mu}) \geq \eta(\lambda, \mu, \varepsilon^*) > 0$ (by Lemma 2.7 and Theorem 3.1), which completes the proof. □

Proof of Theorem 1.1 (b). Let $u_0 \in W^{1,2}(\Omega; v_0, v_1)$ be the function from Lemma 2.5 and fix

$$\mu_0 = \frac{J_F(u_0)}{1 + |J_G(u_0)|}.$$

Then for every $\mu \in [-\mu_0, \mu_0]$ we have

$$J_\mu(u_0) = J_F(u_0) + \mu J_G(u_0) \geq \frac{J_F(u_0)}{1 + |J_G(u_0)|} > 0.$$

Now, we apply the Theorem 3.2 of Bonanno, by choosing $X = W^{1,2}(\Omega; v_0, v_1)$, $\Phi(u) = \frac{1}{2}\langle u, u\rangle_A$ and $J = J_\mu$, for $\mu \in [-\mu_0, \mu_0]$.

Taking account the lema 2.8 and the inequalities $J_\mu(u_0) > 0$, $\Phi(u_0) > 0$, we can choose for every $\mu \in [-\mu_0, \mu_0]$ a $\rho_\mu > 0$ so small that

$$\rho_\mu < \frac{1}{2}\langle u_0, u_0\rangle_A = \Phi(u_0) \tag{12}$$

$$\frac{sup\{J_\mu(u) : \frac{1}{2}\langle u, u\rangle_A < \rho_\mu\}}{\rho_\mu} < \frac{J_\mu(u_0)}{\Phi(u_0)} \tag{13}$$

Now, choosing $x_1 = u_0$, $x_0 = 0$, $\zeta = 1 + \rho_\mu$ and

$$a = \bar{a}_\mu = \frac{1 + \rho_\mu}{\frac{J_\mu(u_0)}{\Phi(u_0)} - \frac{sup\{J_\mu(u): \frac{1}{2}\langle u,u\rangle_A < \rho_\mu\}}{\rho_\mu}},$$

all the assumptions of the Theorem 3.2 are verified. Then, there is an open interval $\Lambda_\mu \subset [0, \bar{a}_\mu]$ and a number $\sigma_\mu > 0$ such that for any $\lambda \in \Lambda_\mu$, the functional $\mathcal{E}_{\lambda,\mu} = \Phi - \lambda J_\mu$ admits at least three distinct critical points: $u^i_{\lambda,\mu} \in W^{1,2}(\Omega; v_0, v_1)$, $(i \in \{1, 2, 3\})$, having norms less than σ_μ.

We can see, that $u = 0$ is a solution of the problem $(P_{\lambda,\mu})$. So if we are looking for nontrivial solutions, we can affirm that $(P_{\lambda,\mu})$ has at least two distinct, nontrivial solutions in $W^{1,2}(\Omega; v_0, v_1)$, having norms less than σ_μ, concluding the proof of the Theorem 1.1.

Remark. As an example, we consider the weight functions (see [5])

$$v_0(x) = w(x) = \begin{cases} ||x||^{-2}, \text{ if } x \in \mathbb{R}^N \setminus B_1 \\ 1, \text{ if } x \in B_1 \end{cases},$$

$$v_1(x) = 1, \forall x \in \mathbb{R}^N,$$

where $B_1 = \{x \in \mathbb{R}^N : ||x|| \leq 1\}$. For these functions the embeddings $W^{1,2}(\Omega; v_0, 1) \hookrightarrow L^p(\Omega; w)$ and $W^{1,2}(\Omega; v_0, 1) \hookrightarrow L^q(\Gamma; w)$ are compact, if $2 < p < 2N/(N-2)$, $2 < q < 2(N-1)/(N-2)$. Assuming that f and g satisfy the conditions

(F1)-(F4), (G1)-(G3) respectively and A defines a bilinear form with (A), we can apply the Theorem 1.1.

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Faculty of Matematics and Computer Science
University of Babeş Bolyai
Str. M. Kogalniceanu 1, 400084 Cluj Napoca, Romania
E-mail address: mezeiildi@yahoo.com