STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIII**, Number 2, June 2008

THE GENERALIZATION OF VORONOVSKAJA'S THEOREM FOR A CLASS OF BIVARIATE OPERATORS

OVIDIU T. POP

Abstract. In this paper we generalize Voronovakaja's theorem and we give an approximation property for a class of bivariate operators and then, through particular cases, we obtain statements verified by the bivariate operators of Bernstein, Schurer, Durrmeyer, Kantorovich, Stancu and Bleimann, Butzer and Hahn.

1. **Introduction**

In this section, we recall some notions and results which we will use in this article. Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

For $m \in \mathbb{N}$, let $B_m : C([0,1]) \to C([0,1])$ the Bernstein operators, defined for any function $f \in C([0,1])$ by

$$(B_m f)(x) = \sum_{k=0}^{n} p_{m,k}(x) f\left(\frac{k}{m}\right), \tag{1.1}$$

where $p_{m,k}(x)$ are the fundamental polynomials of Bernstein, defined as follows

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}, \tag{1.2}$$

for any $x \in [0,1]$ and any $k \in \{0, 1, \dots, m\}$.

Let $p \in \mathbb{N}_0$. For $m \in \mathbb{N}$, F. Schurer (see [15]) introduced and studied in 1962, the operators $\widetilde{B}_{m,p} : C([0, 1+p]) \to C([0,1])$, named Bernstein-Schurer operators,

Received by the editors: 20.03.2007.

2000 *Mathematics Subject Classification.* 41A10, 41A25, 41A35, 41A36, 41A63.

Key words and phrases. linear positive operators, bivariate operators of Bernstein, Schurer, Durrmeyer, Kantorovich, Stancu and Bleimann, Butzer and Hahn, degree of approximation.

defined for any function $f \in C([0,1+p])$ by

$$\left(\widetilde{B}_{m,p}f\right)(x) = \sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x) f\left(\frac{k}{m}\right), \tag{1.3}$$

where $\widetilde{p}_{m,k}(x)$ denotes the fundamental Bernstein-Schurer polynomials, defined as follows

$$\widetilde{p}_{m,k}(x) = \binom{m+p}{k} x^k (1-x)^{m+p-k} = p_{m+p,k}(x) \tag{1.4}$$

for any $x \in [0,1]$ and any $k \in \{0,1,\dots,m+p\}$.

For $m \in \mathbb{N}$ let the operators $M_n : L_1([0,1]) \to C([0,1])$ defined for any function $f \in L_1([0,1])$ by

$$(M_m f)(x) = (m+1) \sum_{k=0}^{m} p_{m,k}(x) \int_0^1 p_{m,k}(t) f(t) dt, \tag{1.5}$$

for any $x \in [0,1]$.

These operators were introduced in 1967 by J. L. Durrmeyer in [7] and were studied in 1981 by M. M. Derriennic in [5].

For $m \in \mathbb{N}$ let the operator $K_m : L_1([0,1]) \to C([0,1])$ defined for any function $f \in L_1([0,1])$ by

$$(K_m f)(x) = (m+1) \sum_{k=0}^{m} p_{m,k}(x) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t) dt, \tag{1.6}$$

for any $x \in [0,1]$.

The operators K_m, where $m \in \mathbb{N}$, are named Kantorovich operators, introduced and studied in 1930 by L. V. Kantorovich (see [8]).

For the following construction see [11].

Define the natural number m_0 by

$$m_0 = \begin{cases} \max\{1, -[\beta]\}, & \text{if } \beta \in \mathbb{R} \backslash \mathbb{Z} \\ \max\{1, 1-\beta\}, & \text{if } \beta \in \mathbb{Z}. \end{cases} \tag{1.7}$$

For the real number β, we have that

$$m + \beta \geq \gamma_\beta \tag{1.8}$$

for any natural number m, $m \geq m_0$, where

$$\gamma_\beta = m_0 + \beta = \begin{cases} \max\left\{1+\beta, \{\beta\}\right\}, & \text{if} \quad \beta \in \mathbb{R}\backslash\mathbb{Z} \\ \max\{1+\beta, 1\}, & \text{if} \quad \beta \in \mathbb{Z}. \end{cases} \tag{1.9}$$

For the real numbers α, β, $\alpha \geq 0$, we note

$$\mu^{(\alpha,\beta)} = \begin{cases} 1, & \text{if} \quad \alpha \leq \beta \\ 1 + \dfrac{\alpha - \beta}{\gamma_\beta}, & \text{if} \quad \alpha > \beta. \end{cases} \tag{1.10}$$

For the real numbers α and β, $\alpha \geq 0$, we have that $1 \leq \mu^{(\alpha,\beta)}$ and

$$0 \leq \frac{k+\alpha}{m+\beta} \leq \mu^{(\alpha,\beta)} \tag{1.11}$$

for any natural number m, $m \geq m_0$ and for any $k \in \{0, 1, \ldots, m\}$.

For the real numbers α and β, $\alpha \geq 0$, m_0 and $\mu^{(\alpha,\beta)}$ defined by (1.7) - (1.10), let the operators $P_m^{(\alpha,\beta)} : C\big([0, \mu^{(\alpha,\beta)}]\big) \to C\big([0,1]\big)$, defined for any function $f \in C\big([0, \mu^{(\alpha,\beta)}]\big)$ by

$$\big(P_m^{(\alpha,\beta)} f\big)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right), \tag{1.12}$$

for any natural number $m, m \geq m_0$ and for any $x \in [0,1]$.

These operators are named Stancu operators, introduced and studied in 1969 by D. D. Stancu in the paper [16]. In [16], the domain of definition of the Stancu operators is $C([0,1])$ and the numbers α and β verify the condition $0 \leq \alpha \leq \beta$.

In 1980, G. Bleimann, P. L. Butzer and L. Hahn introduced in [4] a sequence of linear positive operators $(L_m)_{m\geq 1}$, $L_m : C_B([0,\infty)) \to C_B([0,\infty))$, defined for any function $f \in C_B([0,\infty))$ by

$$(L_m f)(x) = \frac{1}{(1+x)^m} \sum_{k=0}^{m} \binom{m}{k} x^k f\left(\frac{k}{m+1-k}\right), \tag{1.13}$$

for any $x \in [0,\infty)$ and any $m \in \mathbb{N}$, where $C_B([0,\infty)) = \{f \,|\, f : [0,\infty) \to \mathbb{R},\ f$ bounded and continuous on $[0,\infty)\}$.

Let $I_1, I_2 \subset \mathbb{R}$ be given intervals and $f : I_1 \times I_2 \to \mathbb{R}$ be a bounded function.

The function $\omega_{total}(f;\cdot,*) : [0,\infty)\times[0,\infty)\to\mathbb{R}$, defined for any $(\delta_1,\delta_2)\in[0,\infty)\times[0,\infty)$ by

$$\omega_{total}(f;\delta_1,\delta_2)=\sup\big\{|f(x,y)-f(x',y')| : (x,y),(x',y')\in I_1\times I_2,\quad |x-x'|\le\delta_1,\ |y-y'|\le\delta_2\big\} \tag{1.14}$$

is called the first order modulus of smoothness of function f or total modulus of continuity of function f (see [18]).

The first order modulus of smoothness for bivariate functions has properties similar to the properties of the first order modulus of smoothness for univariate functions. Some of them are contained in Lemma 1.1.

Lemma 1.1. *The first order modulus of smoothness for bounded function $f : I_1\times I_2\to\mathbb{R}$ has the following properties:*

(i) $\omega_{total}(f;\delta_1,\delta_2)\le\omega_{total}(f;\delta_1',\delta_2')$ for any (δ_1,δ_2), $(\delta_1',\delta_2')\in[0,\infty)\times[0,\infty)$ such that $\delta_1\le\delta_1'$ and $\delta_2\le\delta_2'$;

(ii) $\omega_{total}(f;|t-x|,|\tau-y|)\le\left(1+\delta_1^{-2}(t-x)^2\right)\left(1+\delta_2^{-2}(\tau-y)^2\right)\omega_{total}(f;\delta_1,\delta_2)$ for any $(\delta_1,\delta_2)\in(0,\infty)\times(0,\infty)$ and any (t,τ), $(x,y)\in I_1\times I_2$.

For some further informations on this measure of smoothness see for example [18].

2. Preliminaries

Let $I,J\subset\mathbb{R}$ intervals with $I\cap J\ne\emptyset$. For $m\in\mathbb{N}$ we consider the functions $p^*_{m,k} : J\to\mathbb{R}$ with the property that $p^*_{m,k}(x)\ge 0$ for any $x\in J$, $k\in\{0,1,\dots,m\}$ and the linear positive functionals $A_{m,k} : E(I)\to\mathbb{R}$, $k\in\{0,1,\dots,m\}$.

Definition 2.1. Let $m\in\mathbb{N}$. Define the operator $L^*_m : E(I)\to F(J)$ by

$$\left(L^*_m f\right)(x)=\sum_{k=0}^{m}p^*_{m,k}(x)A_{m,k}(f) \tag{2.1}$$

for any function $f\in E(I)$ and any $x\in J$, where $E(I)$ and $F(J)$ are subsets of the set of real functions defined on I, respectively on J.

Proposition 2.1. *The operators $\left(L^*_m\right)_{m\ge1}$ are linear and positive on $E(I\cap J)$.*

Proof. The proof follows immediately. □

Definition 2.2. Let $m \in \mathbb{N}$. For $i \in \mathbb{N}_0$ define $T^*_{m,i}$ by

$$\left(T^*_{m,i}L^*_m\right)(x) = m^i \left(L_m \psi^i_x\right)(x) = m^i \sum_{k=0}^{m} p^*_{m,k}(x) A_{m,k}\left(\psi^i_x\right) \tag{2.2}$$

for any $x \in I \cap J$, where for $x \in I$, $\psi_x : I \to \mathbb{R}$, $\psi_x(t) = t - x$ for any $t \in I$.

In the following, let $s \in \mathbb{N}_0$, s even. We suppose that the operators $(L^*_m)_{m\geq 1}$ verify the conditions: there exists the smallest $\alpha_j \in [0, \infty)$ so that

$$\lim_{m\to\infty} \frac{\left(T^*_{m,j}L^*_m\right)(x)}{m^{\alpha_j}} = B_j(x) \in \mathbb{R} \tag{2.3}$$

for any $x \in I \cap J$, $j \in \{0, 2, 4, \dots, s+2\}$ and

$$\begin{cases} \alpha_{s-2l} + \alpha_{2l} - \alpha_s \leq 0 \\ \alpha_{s-2l+2} + \alpha_{2l} - \alpha_s - 2 < 0 \\ \alpha_{s-2l+2} + \alpha_{2l+2} - \alpha_s - 4 < 0 \end{cases} \tag{2.4}$$

where $l \in \left\{0, 1, 2, \dots, \frac{s}{2}\right\}$.

Remark 2.1. From the first relation from (2.4), for $l = 0$ it results that $\alpha_0 = 0$.

Now, we construct with the $(L^*_m)_{m\geq 1}$ operators the bivariate operators of L^*-type.

For $m, n \in \mathbb{N}$, let the linear positive functionals $A_{m,n,k,j} : E(I \times I) \to \mathbb{R}$ with the property

$$A_{m,n,k,j}\left((\cdot - x)^i (* - y)^l\right) = A_{m,k}\left((\cdot - x)^i\right) A_{n,j}\left((* - y)^l\right) \tag{2.5}$$

for any $k \in \{0, 1, \dots, m\}$, $j \in \{0, 1, \dots, n\}$, $i, l \in \{0, 1, \dots, s\}$, $x, y \in I$, where " $\cdot$ " and " $*$ " stand for the first and second variable.

Definition 2.3. Let $m, n \in \mathbb{N}$. The operator $L^*_{m,n} : E(I \times I) \to F(J \times J)$ defined for any function $f \in E(I \times I)$ and any $(x, y) \in J \times J$ by

$$\left(L^*_{m,n}f\right)(x, y) = \sum_{k=0}^{m} \sum_{j=0}^{n} p^*_{m,k}(x) p^*_{n,j}(y) A_{m,n,k,j}(f) \tag{2.6}$$

is named the bivariate operator of L^*-type.

Proposition 2.2. *The operators* $\left(L^*_{m,n}\right)_{m,n\geq 1}$ *are linear and positive on* $E\left((I\times I)\cap(J\times J)\right)$.

Proof. The proof follows immediately. □

In the following we consider that

$$\left(T^*_{m,0}L^*_m\right)(x)=1 \tag{2.7}$$

for any $x\in I\cap J$, any $m\in\mathbb{N}$.

3. Main results

Theorem 3.1. *Let* $I_1,I_2\subset\mathbb{R}$ *be intervals,* $(a,b)\in I_1\times I_2$, $n\in\mathbb{N}_0$ *and the function* $f:I_1\times I_2\to\mathbb{R}$, f *admits partial derivatives of order* n *continuous in a neighborhood* V *of the point* (a,b). *According to Taylor's expansion theorem for the function* f *around* (a,b), *for* $(x,y)\in V$ *we have*

$$f(x,y)=\sum_{k=0}^{n}\frac{1}{k!}\left(\frac{\partial}{\partial x}(x-a)+\frac{\partial}{\partial y}(y-b)\right)^k f(a,b)+ \tag{3.1}$$
$$+\rho^n(x,y)\mu(x-a,y-b)$$

where

$$\left(\frac{\partial}{\partial x}(x-a)+\frac{\partial}{\partial y}(y-b)\right)^k f(a,b)= \tag{3.2}$$
$$=\sum_{i=0}^{k}\binom{k}{i}\frac{\partial^k f}{\partial x^{k-i}\partial y^i}(a,b)(x-a)^{k-i}(y-b)^i,$$

$k\in\{0,1,\dots,n\}$, μ *is a bounded function with* $\lim\limits_{(x,y)\to(a,b)}\mu(x-a,y-b)=0$ *and*

$$\rho(x,y)=\sqrt{(x-a)^2+(y-b)^2}. \tag{3.3}$$

Then

$$|\mu(x-a,y-b)|\leq\frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}\omega_{total}\left(\frac{\partial^n f}{\partial x^{n-i}\partial y^i};|x-a|,|y-b|\right) \tag{3.4}$$

and for any $\delta_1, \delta_2 > 0$, *we have*

$$|\mu(x-a, y-b)| \le \tag{3.5}$$
$$\le \frac{1}{n!}\left(1+\delta_1^{-2}(x-a)^2\right)\left(1+\delta_2^{-2}(y-b)^2\right)\sum_{i=0}^{n}\binom{n}{i}\omega_{total}\left(\frac{\partial^n f}{\partial x^{n-i}\partial y^i};\delta_1,\delta_2\right).$$

Proof. If $n = 0$, it is verified immediately. Let $n \in \mathbb{N}$. According to Taylor's expansion theorem with the Lagrange's remainder, we have

$$f(x,y) = \sum_{k=0}^{n-1}\frac{1}{k!}\left(\frac{\partial}{\partial x}(x-a) + \frac{\partial}{\partial y}(y-b)\right)^k f(a,b) + \tag{3.6}$$
$$+\frac{1}{n!}\left(\frac{\partial}{\partial x}(x-a) + \frac{\partial}{\partial y}(y-b)\right)^n f(\xi,\eta)$$

where $(\xi,\eta) \in V$, (ξ,η) is on the interval determined by the points (a,b) and (x,y) and

$$|\xi - a| \le |x-a|,\ |\eta - b| \le |y-b|. \tag{3.7}$$

From (3.1) and (3.6) it results that

$$\mu(x-a,y-b) = \frac{1}{n!}\frac{1}{\rho^n(x,y)}\left[\left(\frac{\partial}{\partial x}(x-a)+\frac{\partial}{\partial y}(y-b)\right)^n f(\xi,\eta) - \right.$$
$$\left. -\left(\frac{\partial}{\partial x}(x-a)+\frac{\partial}{\partial y}(y-b)\right)^n f(a,b)\right] =$$
$$= \frac{1}{n!}\frac{1}{\rho^n(x,y)}\sum_{i=0}^{n}\binom{n}{i}\left[\frac{\partial^n f}{\partial x^{n-i}\partial y^i}(\xi,\eta) - \right.$$
$$\left. -\frac{\partial^n f}{\partial x^{n-i}\partial y^i}(a,b)\right](x-a)^{n-i}(y-b)^i.$$

Because $|x-a| \le \rho(x,y)$ and $|y-b| \le \rho(x,y)$, the relation above becomes

$$|\mu(x-a,y-b)| \le \frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}\left|\frac{\partial^n f}{\partial x^{n-i}\partial y^i}(\xi,\eta) - \frac{\partial^n f}{\partial x^{n-i}\partial y^i}(a,b)\right| \cdot$$
$$\cdot \frac{|x-a|^{n-i}}{\rho^{n-i}(x,y)}\frac{|y-b|^i}{\rho^i(x,y)},$$

from where

$$|\mu(x-a,y-b)| \le \frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}\left|\frac{\partial^n f}{\partial x^{n-i}\partial y^i}(\xi,\eta) - \frac{\partial^n f}{\partial x^{n-i}\partial y^i}(a,b)\right|. \tag{3.8}$$

Taking (3.7) into account, from (3.8) we have that

$$|\mu(x-a,y-b)| \leq \frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}\sup\left\{\left|\frac{\partial^n f}{\partial x^{n-i}\partial y^i}(u,v)-\frac{\partial^n f}{\partial x^{n-i}\partial y^i}(u',v')\right| : \right.$$
$$\left. |u-u'|\leq |x-a|,\ |v-v'|\leq |y-b|\right\},$$

from where we obtain the relation (3.4).

From (3.4) taking Lemma 1.2 into account, we obtain the relation (3.5). □

In the following we consider the construction from Preliminaries.

Theorem 3.2. *Let $f : I \times I \to \mathbb{R}$ be a bivariate function.*

If $(x, y) \in (I \times I) \cap (J \times J)$ and f admits partial derivatives of order s continuous in a neighborhood of the point (x, y), then

$$\lim_{m\to\infty} m^{s-\alpha_s}\left[\left(L^*_{m,m}f\right)(x,y)- \right. \tag{3.9}$$
$$\left. -\sum_{i=0}^{s}\frac{1}{m^i i!}\sum_{l=0}^{i}\binom{i}{l}\frac{\partial^i f}{\partial t^{i-l}\partial\tau^l}(x,y)\left(T^*_{m,i-l}L^*_m\right)(x)\left(T^*_{m,l}L^*_m\right)(y)\right]=0.$$

If f admits partial derivatives of order s continuous on $(I \times I) \cap (J \times J)$ and there exists an interval $K \subset I \cap J$ such that there exist $m(s) \in \mathbb{N}$ and $k_{2l} \in \mathbb{R}$ depending on K, so that for any $m \in \mathbb{N}$, $m \geq m(s)$ and for any $x \in K$ we have

$$\frac{\left(T^*_{m,2l}L^*_m\right)(x)}{m^{\alpha_{2l}}} \leq k_{2l} \tag{3.10}$$

where $l \in \left\{0, 1, \ldots, \frac{s}{2}+1\right\}$, *then the convergence given in* (3.9) *is uniform on* $K \times K$ *and*

$$m^{s-\alpha_s}\Bigg| \left(L^*_{m,m} f\right)(x,y) - \tag{3.11}$$

$$- \sum_{i=0}^{s} \frac{1}{m^i i!} \sum_{l=0}^{i} \binom{i}{l} \frac{\partial^i f}{\partial t^{i-l}\partial \tau^l}(x,y) \left(T^*_{m,i-l} L^*_m\right)(x) \left(T^*_{m,l} L^*_m\right)(y)\Bigg| \leq$$

$$\leq \frac{1}{s!} \sum_{l=0}^{\frac{s}{2}} \binom{\frac{s}{2}}{l} (k_{2l} + k_{2l+2})(k_{s-2l} + k_{s-2l+2}) \cdot$$

$$\cdot \sum_{i=0}^{s} \binom{s}{i} \omega_{total}\left(\frac{\partial^s f}{\partial t^{s-i}\partial \tau^i}; \frac{1}{\sqrt{m^{\beta_s}}}, \frac{1}{\sqrt{m^{\beta_s}}}\right)$$

for any $(x,y) \in (K \times K)$, *any natural number* m, $m \geq m(s)$, *where*

$$\beta_s = -\max\left\{\alpha_{s-2l+2} + \alpha_{2l} - \alpha_s - 2, \frac{1}{2}(\alpha_{s-2l+2} + \alpha_{2l+2} - \alpha_s - 4) : l \in \left\{0, 1, ..., \frac{s}{2}\right\}\right\}.$$

Proof. Let $m, n \in \mathbb{N}$. According to Taylor's theorem for the function f around (x,y), we have

$$f(t,\tau) = \sum_{i=0}^{s} \frac{1}{i!}\left(\frac{\partial}{\partial t}(t-x) + \frac{\partial}{\partial \tau}(\tau - y)\right)^i f(x,y) + \rho^s(t,\tau)\mu(t-x, \tau-y),$$

from where

$$f(t,\tau) = \sum_{i=0}^{s} \frac{1}{i!} \sum_{l=0}^{i} \binom{i}{l} \frac{\partial^i f}{\partial t^{i-l}\partial \tau^l}(x,y)(t-x)^{i-l}(\tau-y)^l + \tag{3.12}$$

$$+ \rho^s(t,\tau)\mu(t-x, \tau-y),$$

where μ is a bounded function and $\lim\limits_{(t,\tau)\to(x,y)} \mu(t-x,\tau-y) = 0$.

Because $A_{m,n,k,j}$ is linear positive functional and verifies (2.5), from (3.12) we have

$$A_{m,n,k,j}(f) = \sum_{i=0}^{s} \frac{1}{i!} \sum_{l=0}^{i} \binom{i}{l} \frac{\partial^i f}{\partial t^{i-l}\partial \tau^l}(x,y) A_{m,k}\left((\cdot - x)^{i-l}\right) A_{n,j}\left((* - y)^l\right) +$$

$$+ A_{m,n,k,j}\left(\rho^s(\cdot, *)\mu_{xy}\right),$$

where $\mu_{xy} : (I \times I) \cap (J \times J) \to \mathbb{R}$, $\mu_{xy}(t,\tau) = \mu(t-x, \tau-y)$ for any $(t,\tau) \in (I \times I) \cap (J \times J)$.

Multiplying by $p^*_{m,k}(x)p^*_{n,j}(y)$ and summing after k, j, where $k\in\{0,1,\dots,m\}$, $j\in\{0,1,\dots,n\}$, we obtain

$$\begin{aligned}&\left(L^*_{m,n}f\right)(x,y)=\\&=\sum_{i=0}^{s}\frac{1}{i!}\sum_{l=0}^{i}\binom{i}{l}\frac{\partial^i f}{\partial t^{i-l}\partial\tau^l}(x,y)\frac{1}{m^{i-l}}\,\frac{1}{n^l}\left(T^*_{m,i-l}L^*_m\right)(x)\left(T^*_{n,l}L^*_n\right)(y)+\\&+\sum_{k=0}^{m}\sum_{j=0}^{n}p^*_{m,k}(x)p^*_{n,j}(y)A_{m,n,k,j}\left(\rho^s(\cdot,*)\mu_{xy}\right),\end{aligned}$$

from which

$$\begin{aligned}&m^{s-\alpha_s}\Bigg[\left(L^*_{m,m}f\right)(x,y)-\\&-\sum_{i=0}^{s}\frac{1}{m^i i!}\sum_{l=0}^{i}\binom{i}{l}\frac{\partial^i f}{\partial t^{i-l}\partial\tau^l}(x,y)\left(T^*_{m,i-l}L^*_m\right)(x)\left(T^*_{m,l}L^*_m\right)(y)\Bigg]=\\&=(R_{m,m}f)(x,y),\end{aligned}\tag{3.13}$$

where

$$(R_{m,m}f)(x,y)=m^{s-\alpha_s}\sum_{k=0}^{m}\sum_{j=0}^{m}p^*_{m,k}(x)p^*_{m,j}(y)A_{m,m,k,j}\left(\rho^s(\cdot,*)\mu_{xy}\right).\tag{3.14}$$

Then

$$|(R_{m,m}f)(x,y)|\le m^{s-\alpha_s}\sum_{k=0}^{m}\sum_{j=0}^{m}p^*_{m,k}(x)p^*_{m,j}(y)\left|A_{m,m,k,j}\left(\rho^s(\cdot,*)\mu_{xy}\right)\right|,$$

from where

$$\begin{aligned}&|(R_{m,m}f)(x,y)|\le\\&\le m^{s-\alpha_s}\sum_{k=0}^{m}\sum_{j=0}^{m}p^*_{m,k}(x)p^*_{m,j}(y)A_{m,m,k,j}\left(\rho^s(\cdot,*)|\mu_{xy}|\right).\end{aligned}\tag{3.15}$$

According to the relation (3.5), for any $\delta_1, \delta_2 > 0$ and for any $(t,\tau) \in (I \times I) \cap (J \times j)$, we have that

$$|\mu_{xy}(t,\tau)| = |\mu(t-x,\tau-y)| \leq$$
$$\leq \frac{1}{s!}\left(1+\delta_1^{-2}(t-x)^2+\delta_2^{-2}(\tau-y)^2+\delta_1^{-2}\delta_2^{-2}(t-x)^2(\tau-y)^2\right)\cdot$$
$$\cdot\sum_{i=0}^{s}\binom{s}{i}\omega_{total}\left(\frac{\partial^s f}{\partial t^{s-i}\partial\tau^i};\delta_1,\delta_2\right)$$

and taking $\rho^s(t,\tau)=\sum_{l=0}^{\frac{s}{2}}\binom{\frac{s}{2}}{l}(t-x)^{s-2l}(\tau-y)^{2l}$ into account, (3.16) results

$$A_{m,m,k,j}\left(\rho^s(\cdot\,,*)|\mu_{xy}|\right) \leq \frac{1}{s!}\sum_{l=0}^{\frac{s}{2}}\binom{\frac{s}{2}}{l}\Big[A_{m,k}\left(\psi_x^{s-2l}\right)A_{m,j}\left(\psi_y^{2l}\right)+ \tag{3.16}$$
$$+\delta_1^{-2}A_{m,k}\left(\psi_x^{s-2l+2}\right)A_{m,j}\left(\psi_y^{2l}\right)+\delta_2^{-2}A_{m,k}\left(\psi_x^{s-2l}\right)A_{m,j}\left(\psi_y^{2l+2}\right)+$$
$$+\delta_1^{-2}\delta_2^{-2}A_{m,k}\left(\psi_x^{s-2l+2}\right)A_{m,j}\left(\psi_y^{2l+2}\right)\Big]\cdot$$
$$\cdot\sum_{i=0}^{s}\binom{s}{i}\omega_{total}\left(\frac{\partial^s f}{\partial t^{s-i}\partial\tau^i};\delta_1,\delta_2\right).$$

From (3.15) and (3.16), it results that

$$|(R_{m,m}f)(x,y)| \leq$$
$$\leq \frac{1}{s!}m^{s-\alpha_s}\sum_{l=0}^{\frac{s}{2}}\binom{\frac{s}{2}}{l}\sum_{k=0}^{m}\sum_{j=0}^{m}p_{m,k}^*(x)p_{m,j}^*(y)\Big[A_{m,k}\left(\psi_x^{s-2l}\right)A_{m,j}\left(\psi_y^{2l}\right)+$$
$$+\delta_1^{-2}A_{m,k}\left(\psi_x^{s-2l+2}\right)A_{m,j}\left(\psi_y^{2l}\right)+\delta_2^{-2}A_{m,k}\left(\psi_x^{s-2l}\right)A_{m,j}\left(\psi_y^{2l+2}\right)+$$
$$+\delta_1^{-2}\delta_2^{-2}A_{m,k}\left(\psi_x^{s-2l+2}\right)A_{m,j}\left(\psi_y^{2l+2}\right)\Big]\sum_{i=0}^{s}\binom{s}{i}\omega_{total}\left(\frac{\partial^s f}{\partial t^{s-i}\partial\tau^i};\delta_1,\delta_2\right),$$

or

$$|(R_{m,m}f)(x,y)| \leq \frac{1}{s!}\, m^{s-\alpha_s} \sum_{l=0}^{\frac{s}{2}} \binom{\frac{s}{2}}{l} \left[\frac{\left(T^*_{m,s-2l}L^*_m\right)(x)}{m^{s-2l}} \, \frac{\left(T^*_{m,2l}L^*_m\right)(y)}{m^{2l}} + \right.$$

$$+ \delta_1^{-2} \frac{\left(T^*_{m,s-2l+2}L^*_m\right)(x)}{m^{s-2l+2}} \, \frac{\left(T^*_{m,2l}L^*_m\right)(y)}{m^{2l}} +$$

$$+ \delta_2^{-2} \frac{\left(T^*_{m,s-2l}L^*_m\right)(x)}{m^{s-2l}} \, \frac{\left(T^*_{m,2l+2}L^*_m\right)(y)}{m^{2l+2}} +$$

$$\left. + \delta_1^{-2}\delta_2^{-2} \frac{\left(T^*_{m,s-2l+2}L^*_m\right)(x)}{m^{s-2l+2}} \, \frac{\left(T^*_{m,2l+2}L^*_m\right)(y)}{m^{2l+2}} \right] \cdot$$

$$\cdot \sum_{i=0}^{s} \binom{s}{i} \omega_{total} \left(\frac{\partial^s f}{\partial t^{s-i} \partial \tau^i}; \delta_1, \delta_2 \right),$$

so

$$|(R_{m,m}f)(x,y)| \leq$$

$$\leq \frac{1}{s!} \sum_{l=0}^{\frac{s}{2}} \binom{\frac{s}{2}}{l} \left[\frac{\left(T^*_{m,s-2l}L^*_m\right)(x)}{m^{\alpha_{s-2l}}} \, \frac{\left(T^*_{m,2l}L^*_m\right)(y)}{m^{\alpha_{2l}}} \, m^{\alpha_{s-2l}+\alpha_{2l}-\alpha_s} + \right.$$

$$+ \delta_1^{-2} \frac{\left(T^*_{m,s-2l+2}L^*_m\right)(x)}{m^{\alpha_{s-2l+2}}} \, \frac{\left(T^*_{m,2l}L^*_m\right)(y)}{m^{\alpha_{2l}}} \, m^{\alpha_{s-2l+2}+\alpha_{2l}-\alpha_s-2} +$$

$$+ \delta_2^{-2} \frac{\left(T^*_{m,s-2l}L^*_m\right)(x)}{m^{\alpha_{s-2l}}} \, \frac{\left(T^*_{m,2l+2}L^*_m\right)(y)}{m^{\alpha_{2l+2}}} \, m^{\alpha_{s-2l}+\alpha_{2l+2}-\alpha_s-2} +$$

$$\left. + \delta_1^{-2}\delta_2^{-2} \frac{\left(T^*_{m,s-2l+2}L^*_m\right)(x)}{m^{\alpha_{s-2l+2}}} \, \frac{\left(T^*_{m,2l+2}L^*_m\right)(y)}{m^{\alpha_{2l+2}}} \, m^{\alpha_{s-2l+2}+\alpha_{2l+2}-\alpha_s-4} \right] \cdot$$

$$\cdot \sum_{i=0}^{s} \binom{s}{i} \omega_{total} \left(\frac{\partial^s f}{\partial t^{s-i} \partial \tau^i}; \delta_1, \delta_2 \right).$$

We have

$$\beta_s \leq -(\alpha_{s-2l+2} + \alpha_{2l} - \alpha_s - 2),$$

$$\beta_s \leq -\frac{1}{2}(\alpha_{s-2l+2} + \alpha_{2l+2} - \alpha_s - 4)$$

for any $l \in \left\{0, 1, \ldots, \frac{s}{2}\right\}$, from where $\beta_s + \alpha_{s-2l+2} + \alpha_{2l} - \alpha_s - 2 \leq 0$, $2\beta_s + \alpha_{s-2l+2} + \alpha_{2l+2} - \alpha_s - 4 \leq 0$, for any $l \in \left\{0, 1, \ldots, \frac{s}{2}\right\}$. Replacing l with $l+1$ in the relation $\beta_s + \alpha_{s-2l+2} + \alpha_{2l} - \alpha_s - 2 \leq 0$, we have $\beta_s + \alpha_{s-2l} + \alpha_{2l+2} - \alpha_s - 2 \leq 0$. From the first inequality from (2.4) and from the inequalities above, we have

$$m^{\alpha_{s-2l}+\alpha_{2l}-\alpha_s} \leq 1, \qquad m^{\beta_s+\alpha_{s-2l+2}+\alpha_{2l}-\alpha_s-2} \leq 1,$$
$$m^{\beta_s+\alpha_{s-2l}+\alpha_{2l+2}-\alpha_s-2} \leq 1, \quad m^{2\beta_s+\alpha_{s-2l+2}+\alpha_{2l+2}-\alpha_s-4} \leq 1,$$

where $l \in \left\{0, 1, \ldots, \frac{s}{2}\right\}$.

Considering $\delta_1 = \delta_2 = \dfrac{1}{\sqrt{m^{\beta_s}}}$, we have

$$|(R_{m,m}f)(x,y)| \leq \frac{1}{s!} \sum_{l=0}^{\frac{s}{2}} \binom{\frac{s}{2}}{l} \left[\frac{\left(T^*_{m,s-2l}L^*_m\right)(x)}{m^{\alpha_{s-2l}}} \frac{\left(T^*_{m,2l}L^*_m\right)(y)}{m^{\alpha_{2l}}} + \right. \tag{3.17}$$
$$+ \frac{\left(T^*_{m,s-2l+2}L^*_m\right)(x)}{m^{\alpha_{s-2l+2}}} \frac{\left(T^*_{m,2l}L^*_m\right)(y)}{m^{\alpha_{2l}}} +$$
$$+ \frac{\left(T^*_{m,s-2l}L^*_m\right)(x)}{m^{\alpha_{s-2l}}} \frac{\left(T^*_{m,2l+2}L^*_m\right)(y)}{m^{\alpha_{2l+2}}} +$$
$$\left. + \frac{\left(T^*_{m,s-2l+2}L^*_m\right)(x)}{m^{\alpha_{s-2l+2}}} \frac{\left(T^*_{m,2l+2}L^*_m\right)(y)}{m^{\alpha_{2l+2}}} \right] \cdot$$
$$\cdot \sum_{i=0}^{s} \binom{s}{i} \omega_{total} \left(\frac{\partial^s f}{\partial t^{s-i} \partial \tau^i}; \frac{1}{\sqrt{m^{\beta_s}}}, \frac{1}{\sqrt{m^{\beta_s}}} \right).$$

Taking (2.3) into account and considering the fact that

$$\lim_{m\to\infty} \omega_{total} \left(\frac{\partial^s f}{\partial t^{s-i} \partial \tau^i}; \frac{1}{\sqrt{m^{\beta_s}}}, \frac{1}{\sqrt{m^{\beta_s}}} \right) = \omega_{total} \left(\frac{\partial^s f}{\partial t^{s-i} \partial \tau^i}; 0, 0 \right) = 0,$$

$i \in \{0, 1, \ldots, s\}$, from (3.17) we have that

$$\lim_{m\to\infty} (R_{m,m}f)(x,y) = 0. \tag{3.18}$$

From (3.13) and (3.18), (3.9) follows.

If in addition (3.10) takes place, then (3.17) becomes

$$|(R_{m,m}f)(x,y)| \leq \frac{1}{s!}\sum_{l=0}^{\frac{s}{2}}\binom{\frac{s}{2}}{l}(k_{2l}+k_{2l+2})(k_{s-2l}+k_{s-2l+2})\cdot \tag{3.19}$$

$$\cdot\sum_{i=0}^{s}\binom{s}{i}\omega_{total}\left(\frac{\partial^s f}{\partial t^{s-i}\partial\tau^i};\frac{1}{\sqrt{m^{\beta_s}}},\frac{1}{\sqrt{m^{\beta_s}}}\right)$$

for any $m\in\mathbb{N}$, $m\geq m(s)$ and for any $(x,y)\in K\times K$, from which, the convergence from (3.9) is uniform on $K\times K$. From (3.13) and (3.19), (3.11) follows. □

Corollary 3.1. *Let $f:I\times I\to\mathbb{R}$ be a bivariate function.*

If $(x,y)\in(I\times I)\cap(J\times J)$ and f is continuous in (x,y), then

$$\lim_{m\to\infty}\left(L^*_{m,m}f\right)(x,y)=f(x,y). \tag{3.20}$$

If f is continuous on $(I\times I)\cap(J\times J)$, and there exists an interval $K\subset I\cap J$ such that there exist $m(0)\in\mathbb{N}$ and $k_2\in\mathbb{R}$ depending on K so that for any $m\in\mathbb{N}$, $m\geq m(0)$ and any $x\in K$ we have that

$$\frac{\left(T^*_{m,2}L^*_m\right)(x)}{m^{\alpha_2}}\leq k_2, \tag{3.21}$$

then the convergence given in (3.20) is uniform on $K\times K$ and

$$\left|\left(L^*_{m,m}f\right)(x,y)-f(x,y)\right|\leq(1+k_2)^2\omega_{total}\left(f;\frac{1}{\sqrt{m^{2-\alpha_2}}},\frac{1}{\sqrt{m^{2-\alpha_2}}}\right), \tag{3.22}$$

for any $(x,y)\in K\times K$, any $m\in\mathbb{N}$, $m\geq m(0)$.

Proof. It results from Theorem 3.2 for $s=0$ and one verifies immediately that $\beta_0=2-\alpha_2$, $k_0=1$. □

In the Application 3.1 - 3.4, we consider that $p^*_{m,k}=p_{m,k}$, $m\in\mathbb{N}$ and $k\in\{0,1,\ldots,m\}$. By particularization and applying Theorem 3.2 and Corollary 3.1, we give convergence and approximation theorem for some bivariate operators. In all applications we give the convergence theorem for $s=2$ and the approximation theorem for $s=0$. In every application we have $\alpha_2=1$ and $k_0=1$.

Application 3.1. We consider $I = J = K = [0,1]$ and for any $m \in \mathbb{N}$, let the functionals $A_{m,k} : C([0,1]) \to \mathbb{R}$, $A_{m,k}(f) = f\left(\frac{k}{m}\right)$, for any $f \in C([0,1])$, $k \in \{0,1,\ldots,m\}$. In this application, we obtain the Bernstein operators.

We have that

$$\left(T^*_{m,i}B_m\right)(x) = m^i \sum_{k=0}^{m} p_{m,k}(x)\left(\frac{k}{m} - x\right)^i = T_{m,i}(x), \tag{3.23}$$

$x \in [0,1]$, $m \in \mathbb{N}$, $i \in \mathbb{N}_0$,

$$B_j(x) = [x(1-x)]^{\left[\frac{j}{2}\right]}(a_j x + b_j), \tag{3.24}$$

$$\alpha_j = \left[\frac{j}{2}\right], \tag{3.25}$$

$j \in \mathbb{N}_0$, $x \in [0,1]$,

$$a_j = \begin{cases} 0, & \text{if } j \text{ is even or } j = 1 \\ -(j-1)!! \displaystyle\sum_{k=1}^{\left[\frac{j}{2}\right]} \frac{(2k-1)!!}{(2k-2)!!}, & \text{if } j \text{ is odd, } j \geq 3, \end{cases} \tag{3.26}$$

$$b_j = \begin{cases} 1, & \text{if } j = 0 \\ 0, & \text{if } j = 1 \\ (j-1)!!, & \text{if } j \text{ is even, } j \geq 2 \\ \dfrac{1}{2}(j-1)!! \displaystyle\sum_{k=1}^{\left[\frac{j}{2}\right]} \frac{(2k-1)!!}{(2k-2)!!}, & \text{if } j \text{ is odd, } j \geq 3, \end{cases} \tag{3.27}$$

and

$$k_{2l} = \left(\frac{1}{4}\right)^l b_{2l} + 1, \tag{3.28}$$

$l \in \mathbb{N}_0$ (see [9] and [12]).

Let $m, n \in \mathbb{N}$. The operator $B_{m,n} : C([0,1]\times[0,1]) \to C([0,1]\times[0,1])$ defined for any function $f \in C([0,1]\times[0,1])$ and any $(x,y) \in [0,1]\times[0,1]$ by

$$(B_{m,n}f)(x,y) = \sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,k}(x)p_{n,j}(y)f\left(\frac{k}{m}, \frac{j}{n}\right) \tag{3.29}$$

is named the bivariate operator of Bernstein type.

On verify immediately that the condition (2.4), (3.10) and (3.22) take place and then Theorem 3.2 holds for the bivariate operators of Bernstein type.

We have that $T_{m,0}(x) = 1$, $T_{m,1}(x) = 0$, $T_{m,2}(x) = mx(1-x)$, $m \in \mathbb{N}$, $x \in [0,1]$ and then $k_2 = \dfrac{1}{4}$ and $k_4 = \dfrac{19}{16}$.

Theorem 3.3. *Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be a bivariate function.*

(i) If $(x,y) \in [0,1] \times [0,1]$ and f admits partial derivatives of second order continuous in a neighborhood of the point (x,y), then

$$\lim_{m\to\infty} m\left[(B_{m,m}f)(x,y) - f(x,y)\right] = \tag{3.30}$$
$$= \frac{x(1-x)}{2} f''_{x^2}(x,y) + \frac{y(1-y)}{2} f''_{y^2}(x,y).$$

If f admits partial derivatives of second order continuous on $[0,1] \times [0,1]$, then the convergence given in (3.30) is uniform on $[0,1] \times [0,1]$.

(ii) If f is continuous on $[0,1] \times [0,1]$, then

$$|(B_{m,m}f)(x,y) - f(x,y)| \le \frac{25}{16}\, \omega_{total}\left(f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{m}}\right) \tag{3.31}$$

for any $(x,y) \in [0,1] \times [0,1]$, any $m \in \mathbb{N}$.

Application 3.2. We consider $I = J = K = [0,1]$. For any $m \in \mathbb{N}$, let the functionals $A_{m,k} : L_1([0,1]) \to \mathbb{R}$, $A_{m,k}(f) = (m+1)\displaystyle\int_0^1 p_{m,k}(t)f(t)dt$, for any $f \in L_1([0,1])$, $k \in \{0,1,\dots,m\}$. In this case, we obtain the Durrmeyer operators.

We have that

$$\left(T^*_{m,i}M_m\right)(x) = (-1)^i m^i (m+1) \sum_{k=0}^{m} p_{m,k}(x) \int_0^1 p_{m,k}(t)(x-t)^i dt, \tag{3.32}$$

$x \in [0,1]$, $m \in \mathbb{N}$, $i \in \mathbb{N}_0$,

$$\alpha_j = \left[\frac{j}{2}\right], \tag{3.33}$$

$$B_j(x) = \begin{cases} \dfrac{j!}{\left(\frac{j}{2}\right)!}[x(1-x)]^{\frac{j}{2}}, & \text{if } j \text{ is even} \\[2ex] -\dfrac{(j+1)!}{2\left(\frac{j-1}{2}\right)!}(1-2x)[x(1-x)]^{\frac{j-1}{2}}, & \text{if } j \text{ is odd} \end{cases} \tag{3.34}$$

$j \in \mathbb{N}_0$, $x \in [0,1]$, and in the same way from Application 3.1

$$k_{2l} = \left(\frac{1}{4}\right)^l \frac{(2l)!}{l!} + 1, \tag{3.35}$$

$l \in \mathbb{N}_0$ (see [5] and [12]).

Let $m, n \in \mathbb{N}$. The operator $M_{m,n} : L_1([0,1] \times [0,1]) \to C([0,1] \times [0,1])$ defined for any function $f \in L_1([0,1] \times [0,1])$ and any $(x,y) \in [0,1] \times [0,1]$ by

$$(M_{m,n}f)(x,y) = \tag{3.36}$$

$$= (m+1)(n+1) \sum_{k=0}^{m} \sum_{j=0}^{n} p_{m,k}(x) p_{n,j}(y) \int_0^1 p_{m,k}(t) p_{n,j}(s) f(t,s) dt ds$$

is named the bivariate operator of Durrmeyer type.

The Theorem 3.2 holds for these operators.

We have

$$\left(T^*_{m,0} M_m\right)(x) = 1,$$

$$\left(T^*_{m,1} M_m\right)(x) = \frac{m(1-2x)}{m+2},$$

$$\left(T^*_{m,2} M_m\right)(x) = m^2 \frac{2(m-3)x(1-x)+2}{(m+2)(m+3)}, \quad m \in \mathbb{N},$$

$$B_0(x) = 1, \quad B_1(x) = 1 - 2x, \quad B_2(x) = 2x(1-x), \quad x \in [0,1],$$

$$k_2 = \frac{3}{2} \quad \text{and} \quad k_4 = \frac{7}{4}$$

(see [12]).

Theorem 3.4. *Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be a bivariate function.*

(i) If $(x,y) \in [0,1] \times [0,1]$ and f admits partial derivatives of second order continuous in a neighborhood of the point (x,y), then

$$\lim_{m\to\infty} m\left[(M_{m,m}f)(x,y) - f(x,y)\right] = (1-2x)f'_x(x,y) + \tag{3.37}$$

$$+ (1-2y)f'_y(x,y) + x(1-x)f''_{x^2}(x,y) + y(1-y)f''_{y^2}(x,y).$$

If f admits partial derivatives of second order continuous on $[0,1] \times [0,1]$, then the convergence given in (3.37) is uniform on $[0,1] \times [0,1]$.

(ii) If f is continuous on $[0,1]\times[0,1]$, then

$$|(M_{m,m}f)(x,y)-f(x,y)|\leq\frac{25}{4}\,\omega_{total}\left(f;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{m}}\right),\tag{3.38}$$

for any $(x,y)\in[0,1]\times[0,1]$, any natural number m, $m\geq 3$.

Application 3.3. We consider $I=J=K=[0,1]$. For any $m\in\mathbb{N}$, let the functionals $A_{m,k}:L_1([0,1])\to\mathbb{R}$,

$$A_{m,k}(f)=(m+1)\int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}}f(t)dt,$$

for any $f\in L_1([0,1])$, $k\in\{0,1,\dots,m\}$. In this case, we obtain the Kantorovich operators.

We have

$$\left(T^*_{m,0}K_m\right)(x)=1,$$

$$\left(T^*_{m,1}K_m\right)=\frac{m}{2(m+1)}(1-2x),$$

$$\left(T^*_{m,2}K_m\right)(x)=\left(\frac{m}{m+1}\right)^2\frac{(1-x)^3+x^3+3mx(1-x)}{3},\quad m\in\mathbb{N},\quad x\in[0,1],$$

$$B_0(x)=1,\quad B_1(x)=\frac{1-2x}{2},\quad B_2(x)=x(1-x),\quad x\in[0,1],$$

$k_2=1$ and $k_4=\dfrac{3}{2}$ (see [12]).

Let $m,n\in\mathbb{N}$. The operator $K_{m,n}:L_1([0,1]\times[0,1])\to C([0,1]\times[0,1])$ defined for any function $f\in L_1([0,1]\times[0,1])$ and any $(x,y)\in[0,1]\times[0,1]$ by

$$(K_{m,n}f)(x,y)=\tag{3.39}$$

$$=(m+1)(n+1)\sum_{k=0}^{n}\sum_{j=0}^{n}p_{m,k}(x)p_{n,j}(y)\int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}}\int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}}f(t,s)dtds$$

is named the bivariate operator of Kantorovich type.

Theorem 3.5. *Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be a bivariate function.*

(i) If $(x,y) \in [0,1] \times [0,1]$ and f admits partial derivatives of second order continuous in a neighborhood of the point (x,y), then

$$\lim_{m\to\infty} m\left[(K_{m,m}f)(x,y) - f(x,y)\right] = \frac{1-2x}{2}\, f'_x(x,y) + \tag{3.40}$$
$$+\frac{1-2y}{2}\, f'_y(x,y) + \frac{x(1-x)}{2}\, f''_{x^2}(x,y) + \frac{y(1-y)}{2}\, f''_{y^2}(x,y).$$

If f admits partial derivatives of second order continuous on $[0,1] \times [0,1]$, then the convergence in (3.40) is uniform on $[0,1] \times [0,1]$.

(ii) If f is continuous on $[0,1] \times [0,1]$, then

$$|(K_{m,m}f)(x,y) - f(x,y)| \le 4\omega_{total}\left(f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{m}}\right), \tag{3.41}$$

for any $(x,y) \in [0,1] \times [0,1]$, any natural number m, $m \ge 3$.

Application 3.4. Let $I = [0, \mu^{(\alpha,\beta)}]$, $J = K = [0,1]$ (see (1.7)-(1.10)). For any $m \in \mathbb{N}$, $m \ge m_0$, let the functionals $A_{m,k} : C([0, \mu^{(\alpha,\beta)}]) \to \mathbb{R}$,

$$A_{m,k}(f) = f\left(\frac{k+\alpha}{m+\beta}\right),$$

for any $f \in C([0, \mu^{(\alpha,\beta)}])$, $k \in \{0,1,\ldots,m\}$. In this case, we obtain the Stancu operators.

We have that

$$\left(T^*_{m,0}P_m^{(\alpha,\beta)}\right)(x) = 1,$$

$$\left(T^*_{m,1}P_m^{(\alpha,\beta)}\right)(x) = \frac{m(\alpha-\beta x)}{m+\beta},$$

$$\left(T^*_{m,2}P_m^{(\alpha,\beta)}\right)(x) = \frac{m^2[mx(1-x)+(\alpha-\beta x)^2]}{(m+\beta)^2}, \quad m \in \mathbb{N}, \quad m \ge m_0,$$

$$B_0(x) = 1, \quad B_1(x) = \alpha - \beta x, \quad B_2(x) = x(1-x), \quad x \in [0,1].$$

There exists a natural number $m(0)$ such that $\dfrac{\left(T^*_{m,2}P^{(\alpha,\beta)}\right)(x)}{m} \le \dfrac{5}{4} = k_2$ for any natural number m, $m \ge m(0)$, any $x \in [0,1]$ and $k_4 = 1$ (see [13]).

For the real numbers $\alpha_1, \alpha_2, \beta_1, \beta_2$ with $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$, m_1, m_2, $\mu^{(\alpha_1,\beta_1)}$ and $\mu^{(\alpha_2,\beta_2)}$ are defined through

$$m_i = \begin{cases} \max\{1, -[\beta_i]\}, & \text{if} \quad \beta_i \in \mathbb{R}\backslash\mathbb{Z} \\ \max\{1, 1-\beta_i\}, & \text{if} \quad \beta_i \in \mathbb{Z} \end{cases}, \tag{3.42}$$

$$\gamma_{\beta_i} = m_i + \beta_i = \begin{cases} \max\left\{1+\beta_i, \{\beta_i\}\right\}, & \text{if} \quad \beta_i \in \mathbb{R}\backslash\mathbb{Z} \\ \max\{1+\beta_i, 1\}, & \text{if} \quad \beta_i \in \mathbb{Z} \end{cases}, \tag{3.43}$$

$$\mu^{(\alpha_i,\beta_i)} = \begin{cases} 1, & \text{if} \quad \alpha_i \leq \beta_i \\ 1 + \dfrac{\alpha_i - \beta_i}{\gamma_{\beta_i}}, & \text{if} \quad \alpha_i > \beta_i \end{cases}, \tag{3.44}$$

where $i \in \{1,2\}$.

Let the bivariate operators $P_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)} : C([0,\mu^{(\alpha_1,\beta_1)}] \times [0,\mu^{(\alpha_2,\beta_2)}]) \to C([0,1]\times[0,1])$ defined for any function $f \in C([0,\mu^{(\alpha_1,\beta_1)}] \times [0,\mu^{(\alpha_2,\beta_2)}])$ by

$$\left(P_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)} f\right)(x,y) = \sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,k}(x) p_{n,j}(y) f\left(\frac{k+\alpha_1}{m+\beta_1}, \frac{j+\alpha_2}{n+\beta_2}\right), \tag{3.45}$$

for any $(x,y) \in [0,1]\times[0,1]$ and any natural numbers m,n, $m \geq m_1$ and $n \geq m_2$.

These operators are named the bivariate operators of Stancu type.

Theorem 3.6. *Let $f : [0,\mu^{(\alpha_1,\beta_1)}] \times [0,\mu^{(\alpha_2,\beta_2)}] \to \mathbb{R}$ be a bivariate function.*

(i) If $(x,y) \in [0,1]\times[0,1]$ and f admits partial derivatives of second order continuous in a neighborhood of the point (x,y), then

$$\lim_{m\to\infty} m\left[\left(P_{m,m}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)} f\right)(x,y) - f(x,y)\right] = \tag{3.46}$$
$$= (\alpha_1 - \beta_1 x) f'_x(x,y) + (\alpha_2 - \beta_2 y) f'_y(x,y) +$$
$$+ \frac{x(1-x)}{2} f''_{x^2}(x,y) + \frac{y(1-y)}{2} f''_{y^2}(x,y).$$

If f admits partial derivatives of second order continuous on $[0,1]\times[0,1]$, then the convergence given in (3.46) is uniform on $[0,1]\times[0,1]$.

(ii) If f is continuous on $[0,1]\times[0,1]$, then

$$\left|\left(P_{m,m}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)} f\right)(x,y) - f(x,y)\right| \leq \frac{81}{16}\,\omega_{total}\left(f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{m}}\right), \tag{3.47}$$

for any $(x,y) \in [0,1] \times [0,1]$, *any natural number* m, $m \geq m(0)$.

For the particular case from this applications (see [13]), we obtain the Voronovskaja's type theorem and approximation theorem for the bivariate operator of Bernstein, Schurer and Schurer-Stancu.

Application 3.5. In this application, $I = J = [0,\infty)$ and for any $m \in \mathbb{N}$, we consider $p^*_{m,k}(x) = \binom{m}{k} \frac{x^k}{(1+x)^m}$ for any $x \in [0,\infty)$, $k \in \{0,1,\ldots,m\}$, the functionals $A_{m,k} : C_B([0,\infty)) \to \mathbb{R}$, $A_{m,k}(f) = f\left(\frac{k}{m+1-k}\right)$ defined for any $f \in C_B([0,\infty))$ and $k \in \{0,1,\ldots,m\}$. We obtain the Bleimann, Butzer and Hahn operators.

We have

$$\left(T^*_{m,0} L_m\right)(x) = 1,$$

$$\left(T^*_{m,1} L_m\right)(x) = -mx\left(\frac{x}{1+x}\right)^m, \quad m \in \mathbb{N},$$

$$B_0(x) = 1, \quad B_1(x) = 0, \quad B_2(x) = x(1+x)^2, \quad x \in [0,\infty),$$

$$k_2 = 4b(1+b)^2,$$

where $K = [0,b]$, $b > 0$ and $m(0) = 24(1+b)$ (see [14]).

Let $m,n \in \mathbb{N}$. The operator $L_{m,n} : C_B([0,\infty)\times[0,\infty)) \to C_B([0,\infty)\times[0,\infty))$ defined for any function $f \in C_B([0,\infty)\times[0,\infty))$ and any $(x,y) \in [0,\infty)\times[0,\infty)$ by

$$(L_{m,n}f)(x,y) = \tag{3.48}$$
$$= \frac{1}{(1+x)^m(1+y)^n} \sum_{k=0}^{m}\sum_{j=0}^{n} \binom{m}{k}\binom{n}{j} f\left(\frac{k}{m+1-k}, \frac{j}{n+1-j}\right)$$

is named the bivariate operator of Bleimann-Butzer-Hahn type.

Theorem 3.7. *Let* $f : [0,\infty)\times[0,\infty) \to \mathbb{R}$ *be a bivariate function.*

(i) If $(x,y) \in [0,\infty)\times[0,\infty)$ *and* f *admits partial derivatives of second order continuous in a neighborhood of the point* (x,y), *then*

$$\lim_{m\to\infty} m\left[(L_{m,m}f)(x,y) - f(x,y)\right] = \tag{3.49}$$
$$= \frac{x(1+x)^2}{2} f''_{x^2}(x,y) + \frac{y(1+y)^2}{2} f''_{y^2}(x,y).$$

(ii) If f is continuous on $[0,\infty)\times[0,\infty)$ and $b>0$, then

$$|(L_{m,m}f)(x,y)-f(x,y)|\leq \tag{3.50}$$
$$\leq\left[1+8b(1+b)^2+16b^2(1+b)^4\right]\omega_{total}\left(f;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{m}}\right),$$

for any $(x,y)\in[0,b]$, any natural number m, $m\geq 24(1+b)$.

Remark 3.1. From the Theorem 3.2 - 3.7, for (ii) results the uniform convergence of the bivariate operators.

References

[1] Agratini, O., *Aproximare prin operatori liniari*, Presa Universitară Clujeană, Cluj-Napoca, 2000 (Romanian).

[2] Bărbosu, D., *Voronovskaja theorem for Bernstein-Schurer bivariate operator*, Rev. Anal. Numér. Théor. Approx., **33**(2004), no. 1, 19-24.

[3] Bernstein, S.N., *Démonstration du théoreme de Weierstrass fondée sur le calcul de probabilités*, Commun. Soc. Math. Kharkow (2), **13**(1912-1913), 1-2.

[4] Bleimann, G., Butzer, P. L., Hahn, L., *A Bernstein-type operator approximating continuous functions on the semi-axis*, Indag. Math. **42**(1980), 255-262.

[5] Derriennic, M.M., *Sur l'approximation des fonctions intégrables sur* $[0,1]$ *par des polynômes de Bernstein modifiés*, J. Approx. Theory, **31**(1981), 325-343.

[6] DeVore, R. A., Lorentz, G. G., *Constructive Approximation*, Springer Verlag, Berlin, Heidelberg, New York, 1993.

[7] Durrmeyer, J.L., *Une formule d'inversion de la transformée de Laplace: Applications à la théorie des moments*, Thèse de 3e cycle, Faculté des Sciences de l'Université de Paris, 1967.

[8] Kantorovich, L.V., *Sur certain développements suivant les polynômes de la forme de S. Bernstein*, **I, II**, C. R. Acad. URSS (1930), 563-568, 595-600.

[9] Lorentz, G.G., *Bernstein polynomials*, University of Toronto Press, Toronto, 1953.

[10] Lorentz, G.G., *Approximation of functions*, Holt, Rinehart and Winston, New York, 1966.

[11] Pop, O.T., *New properties of the Bernstein-Stancu operators*, Anal. Univ. Oradea, Fasc. Matematica, Tom. **XI** (2004), 51-60.

[12] Pop, O.T., *The generalization of Voronovskaja's theorem for a class of linear and positive operators*, Rev. Anal. Num. Théor. Approx., **34**(2005), no. 1, 79-91.

[13] Pop, O.T., *The Voronovskaja type theorem for the Stancu bivariate operators*, Austral. J. Math. Anal. and Appl., **3**(**2**) (2006), Art. 10, 9 pp. (electronic).

[14] Pop, O.T., *About operator of Bleimann, Butzer and Hahn*, Anal. Univ. Timişoara, **XLIII**, Fasc. 1 (2005), 117-127.

[15] Schurer, F., *Linear positive operators in approximation theory*, Math. Inst. Techn., Univ. Delft. Report, 1962.

[16] Stancu, D.D., *Asupra unei generalizări a polinoamelor lui Bernstein*, Studia Univ. Babeş-Bolyai, Ser. Math.-Phys., **14** (1969), 31-45.

[17] Stancu, D.D., Coman, Gh., Agratini, O., Trîmbiţaş, R., *Analiză numerică şi teoria aproximării*, **I**, Presa Universitară Clujeană, Cluj-Napoca, 2001 (Romanian).

[18] Timan, A.F., *Theory of Approximation of Functions of Real Variable*, New York: Macmillan Co.1963.MR22#8257.

[19] Voronovskaja, E., *Détermination de la forme asymptotique d'approximation des fonctions par les polynômes de M. Berstein*, C. R. Acad. Sci. URSS (1932), 79-85.

National College "Mihai Eminescu"
5 Mihai Eminescu Street
Satu Mare 440014, Romania
Vest University "Vasile Goldiş" of Arad
Branch of Satu Mare
26 Mihai Viteazul Street
Satu Mare 440030, Romania
E-mail address: ovidiutiberiu@yahoo.com, popovidiutiberiu@yahoo.com