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THE DIRAC EQUATION AND THE NONCOMMUTATIVE HARMONIC OSCILATOR

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Abstract. In this paper we analyzed the Dirac equation using the noncommutative harmonic oscillator. Also we analyzed some particular wave functions cases using this noncommutative operator.

1. **Introduction**

The wave function $\psi(t,x)$ describes the probability distribution in time and space of an particle.

In general the Dirac equation (see [1]), is given by :

$$i\hbar\frac{\partial}{\partial t}\psi(t,x) = H_0\psi(t,x) \qquad (*)$$

Here H_0 represents a differential operator, which is for instance, in the two-dimensional case:

$$H_0 = -i\hbar c\left(\sigma_1\frac{\partial}{\partial x_1} + \sigma_2\frac{\partial}{\partial x_2}\right) + \sigma_3 mc^2.$$

Here $\sigma_1, \sigma_2, \sigma_3$ represent the Pauli matrices, $\hbar$ is the Planck constant and m is the mass of the particle.

The non-commutative harmonic oscilator $Q(x,\partial_x)$ is defined to be the second-order ordinary differential operator:

$$Q(x,\partial_x) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\left(-\frac{\partial_x^2}{2} + \frac{x^2}{2}\right) + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\left(x\partial_x + \frac{1}{2}\right)$$

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$$= \begin{pmatrix} -\frac{\alpha\partial_x^2}{2} + \alpha\frac{x^2}{2} & -x\partial_x - \frac{1}{2} \\ x\partial_x + \frac{1}{2} & -\beta\frac{\partial_x^2}{2} - \beta\frac{x^2}{2} \end{pmatrix}$$

where α, β are two constants, $\alpha, \beta > 0$.

If we change the operator H_0 with $Q(x, \partial_x)$ one obtain a new equation:

$$i\hbar\frac{\partial}{\partial t}\psi(t,x) = Q(x,\partial_x)\psi(t,x) \tag{$**$}$$

Let's call this equation the "*noncommutative Dirac equation*" . In this paper we will analyze this new equation.

2. **Main result**

Theorem 2.1. *For a free particle the noncommutative Dirac equation is:*

$$\hbar\frac{\partial}{\partial t}\psi(t,x) = -\left(x\partial_x + \frac{1}{2}\right)\psi(t,x)$$

Proof. We know that the noncommutative harmonic oscilator is:

$$Q(x,\partial_x) = \begin{pmatrix} -\frac{\alpha\partial_x^2}{2} + \alpha\frac{x^2}{2} & -x\partial_x - \frac{1}{2} \\ x\partial_x + \frac{1}{2} & -\beta\frac{\partial_x^2}{2} - \beta\frac{x^2}{2} \end{pmatrix},$$

then the *noncommutative Dirac equation* is:

$$i\hbar\frac{\partial}{\partial t}\psi(t,x) = \begin{pmatrix} -\frac{\alpha\partial_x^2}{2} + \alpha\frac{x^2}{2} & -x\partial_x - \frac{1}{2} \\ x\partial_x + \frac{1}{2} & -\beta\frac{\partial_x^2}{2} - \beta\frac{x^2}{2} \end{pmatrix}\psi(t,x).$$

Using the matricial representation for the complex numbers,

$$i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

one obtains:

$$\begin{pmatrix} 0 & \hbar\frac{\partial\psi(t,x)}{\partial t} \\ -\hbar\frac{\partial\psi(t,x)}{\partial t} & 0 \end{pmatrix} = \begin{pmatrix} \left(-\frac{\alpha\partial_x^2}{2} + \alpha\frac{x^2}{2}\right)\psi(t,x) & \left(-x\partial_x - \frac{1}{2}\right)\psi(t,x) \\ \left(x\partial_x + \frac{1}{2}\right)\psi(t,x) & \left(-\beta\frac{\partial_x^2}{2} - \beta\frac{x^2}{2}\right)\psi(t,x) \end{pmatrix}$$

Identifying, one obtains: $\hbar\frac{\partial}{\partial t}\psi(t,x) = -\left(x\partial_x + \frac{1}{2}\right)\psi(t,x)$, so the theorem is proved. □

Corollary 2.2. *If the wave function is* $\psi(t,x) = \varphi(x)e^{-\frac{iEt}{\hbar}}$, *where* E *represents the total energy, using the noncommutative Dirac equation, one obtains the total energy:* $E = \frac{1}{\hbar}\left(xp - \frac{i\hbar}{2}\right)$.

Proof.

$$\hbar\frac{\partial}{\partial t}\left(\varphi(x)e^{-\frac{iEt}{\hbar}}\right) = -\left(x\partial_x + \frac{1}{2}\right)\varphi(x)e^{-\frac{iEt}{\hbar}} \Rightarrow$$

$$\hbar\varphi(x)e^{-\frac{iEt}{\hbar}}\left(\frac{-iE}{\hbar}\right) = -\left(x\partial_x + \frac{1}{2}\right)\varphi(x)e^{-\frac{iEt}{\hbar}} \Rightarrow E = \frac{1}{i}\left(x\partial_x + \frac{1}{2}\right).$$

But, using the Schrödinger equation from quantum physics, the impulse is:

$$p = -i\hbar\frac{\partial}{\partial x},$$

so, one obtains:

$$E = \frac{1}{i^2\hbar}\left(i\hbar x\frac{\partial}{\partial x} + \frac{i\hbar}{2}\right) = \frac{1}{\hbar}\left(xp - \frac{i\hbar}{2}\right). \qquad \square$$

Using this expression for total energy, for the wave function, one obtains:

$$\psi(t,x) = \varphi(x)e^{-\left(\frac{it}{\hbar}\frac{1}{\hbar}\left(xp - \frac{i\hbar}{2}\right)\right)} = \varphi(x)e^{-\frac{it}{\hbar^2}xp - \frac{t}{2\hbar}}.$$

Corollary 2.3. *If we consider a plane wave function:* $\psi(t,x) = ce^{\left(-\frac{i}{\hbar}px - \frac{iEt}{\hbar}\right)}$, *using noncommutative Dirac equation, one obtains the total energy:*

$$E = \frac{1}{\hbar}(px + \frac{\hbar}{2i}).$$

Proof.

$$\hbar\frac{\partial}{\partial t}\left(ce^{\left(\frac{ipx}{\hbar} - \frac{iEt}{\hbar}\right)}\right) = -\left(x\partial_x + \frac{1}{2}\right)ce^{\left(\frac{ipx}{\hbar} - \frac{iEt}{\hbar}\right)} \Rightarrow$$

$$\hbar ce^{\left(\frac{ipx}{\hbar} - \frac{iEt}{\hbar}\right)}\left(-\frac{iE}{\hbar}\right) = -\left(x\partial_x + \frac{1}{2}\right)ce^{\left(\frac{ipx}{\hbar} - \frac{iEt}{\hbar}\right)}.$$

So, finally, we obtain:

$$iE = x\partial_x + \frac{1}{2} \Rightarrow E = \frac{1}{i}\left(x\partial_x + \frac{1}{2}\right) = \frac{1}{\hbar}\left(-\frac{i\hbar}{i^2}x\partial_x + \frac{\hbar}{2i}\right) = \frac{1}{\hbar}(px + \frac{\hbar}{2i}). \qquad \square$$

If we replace this expression of the total energy in the wave function, we get:

$$\psi(x,t) = ce^{\frac{i}{\hbar}px - \frac{it}{\hbar}\left(\frac{1}{\hbar}\left(px + \frac{\hbar}{2i}\right)\right)} = ce^{\frac{i}{\hbar}\left(px - \frac{tpx}{\hbar} - \frac{t}{2i}\right)}.$$

Every fermion also has an antifermion. An antiparticle was observed for the first time in 1933, but the idea had been introduced theoretically by Dirac in 1928. We start from the assumption that a particle in free space is described by the de Broglie wave function:

$$\psi(t,x) = N \exp[i(px - Et)/\hbar],$$

with frequency $\nu = \frac{E}{\hbar}$ and wavelenght $\lambda = \frac{\hbar}{p}$. Working nonrelativisticaly the relationship between momentum and energy is : $E = \frac{p^2}{2m}$ and substituting operators one obtains Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t}\psi(t,x) = -\frac{\hbar}{2m} \bigtriangledown^2 \psi(t,x).$$

Relativistically:

$$E^2 = p^2c^2 + m^2c^4,$$

and again substituting operators, we have:

$$-\hbar^2 \frac{\partial^2}{\partial t^2}\psi(t,x) = -\hbar^2 c^2 \bigtriangledown^2 \psi(t,x) + m^2c^4\psi(t,x).$$

This equation is called the Klein-Gordon equation. The solutions of this equation are:

$$\psi(x,t) = N \exp[i(px - Et)/\hbar].$$

Corollary 2.4 *If we consider the deBroglie wave function:*

$$\psi(x,t) = N \exp[i(px - Et)/\hbar],$$

using the noncommutative Dirac equation (**), one obtains the total energy:

$$E = \frac{1}{\hbar}(px + \frac{\hbar}{2i}).$$

Proof. From noncommutative Dirac equation (**), one obtains:

$$\hbar\frac{\partial}{\partial t}(Ne^{-\frac{i(px-Et)}{\hbar}}) = -\left(x\partial_x + \frac{1}{2}\right)Ne^{-\frac{i(px-Et)}{\hbar}} \Rightarrow iE = x\partial_x + \frac{1}{2},$$

so, finally, we obtain:

$$E = \frac{1}{i}\left(x\partial_x + \frac{1}{2}\right) = \frac{1}{\hbar}\left(px + \frac{\hbar}{2i}\right). \qquad \square$$

Then the deBroglie wave function becomes:

$$\psi(x,t) = N \exp[i(px - \frac{1}{i}\left(x\partial_x + \frac{1}{2}\right)t)/\hbar].$$

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