STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume LIII, Number 2, June 2008

ON A LIMIT THEOREM FOR FREELY INDEPENDENT RANDOM VARIABLES

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Abstract. A direct proof of Voiculescu's addition theorem for freely independent real-valued random variables, using resolvents of self-adjoint operators, is given. The addition theorem leads to a central limit theorem for freely independent, identically distributed random variables of finite variance is given.

1. Introduction

The concept of independent random variables lies at the heart of classical probability. Via independent sequences it leads to the Gauss and Poisson distribution. Classical, commutative independence of random variables amounts to a factorisation property of probability spaces.

At the opposite, non-comutative extreme Voiculescu discovered in 1983 the notion of *free independence* of random variables, which corresponds to a *free* product of von Neumann algebras [3]. He showed that this notion leads naturally to analogues of the Gauss and Poisson distributions, very different in form from the classical ones [3] and [5]. For instance the free analoque of the Gauss curve is a semi-ellipse.

In this paper we consider the addition problem: Which is the probability distribution μ of the sum X_1+X_2 of two freely independent random varibles, given the distribution μ_1 and μ_2 of the summands? This problem was solved by Voiculescu in 1986 for the case of bounded, not necessarily self-adjoint random variables, relying on the existence of all the moments of the probability distributions μ_1 and μ_2 ([4]). Later this problem

Received by the editors: 25.11.2005.
2000 *Mathematics Subject Classification.* 62E10, 60F05, 60G50.
Key words and phrases. Cauchy transform, free random variables.

was solve by Hans Massen in 1992 for the case of self-adjoint random variables with finite variance. The result is an explicit calculation procedure for the free convolution product of two probability distributions. In this procedure a central role is played by the Cauchy transform $G(z)$ of a distribution μ, which equals the expectation of the resolvent of the associated operator X. If we take X self-adjoint, μ is a probability measure on $\mathbb{R}$ and we may write:

$$G(z) := \int_{-\infty}^{\infty} \frac{\mu \,\mathrm{d}x}{z - x} = E\left((z - X)^{-1}\right)$$

This formula points at a direct way to find the free convolution product of μ_1 and μ_2. This article consists of four sections. The first contains some preliminaries on free independence. In the second we gather some facts about Cauchy transforms. In three section it is shown that $F_1 \otimes F_2 = E\left((z - (\overline{X_1 + X_2}))^{-1}\right)^{-1}$, where X_1 and X_2 are freely independent random variables with distributions μ_1 and μ_2 respectively, and the bar denotes operator closure. The last section contains the central limit theorem.

2. **Free independence of random variables**

By a random variable we shall mean a self-adjoint operator X on a Hibert space $\mathcal{H}$ in which a particular unit vector ξ has been singled out. Via the functional calculus of spectral theory such an operator determines an embedding ι_X of the commutative C^*-algebra $C(\overline{\mathbb{R}})$ of continuous functions on the one-point compactification $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ of $\mathbb{R}$ to be bounded operators on $\mathcal{H}$:

$$\iota_X(f) = f(X)$$

We shall consider the spectral measure μ of X, which is determined by

$$< \xi, \iota_X(f)\xi >= \int_{-\infty}^{\infty} f(x)\mu \,\mathrm{d}x \qquad \left(f \in C(\overline{\mathbb{R}})\right)$$

as the *probability distribution* of X and we shall think of $< \xi, \iota_X(f)\xi >$ as the *expectation value* of the (complex-valued) random variable $f(X)$, which is a bounded normal operator on $\mathcal{H}$.

Definition 2.1. The random variables X_1 and X_2 on $(\mathcal{H}, \xi)$ are said to be *freely independent* if for all $n \in \mathbb{N}$ and all alternating sequences $i_1, i_2, ..., i_n$ such that $i_1 \neq i_2 \neq i_3 \neq ... \neq i_n$ and for all $f_k \in C(\overline{\mathbb{R}})$, $k = \overline{1, n}$ one has

$$< \xi, f_k(X_{i_k})\xi >= 0 \implies < \xi, f_1(X_{i_1}) f_2(X_{i_2}) ... f_n(X_{i_n})\xi >= 0$$

3. **The reciprocal Cauchy transform**

We consider the expectation values of functions $f \in C(\overline{\mathbb{R}})$ of the form

$$f(x) = \frac{1}{z - x} \ , \qquad (\Im(z) > 0)$$

In particular they play a key role in the addition of freely independent random variables.

For the complex plane $\mathbb{C}$ denote $\mathbb{C}^+ = \{z \in \mathbb{C} : \Im(z) > 0\}$ the upper half-plane, $\mathbb{C}^- = \{z \in \mathbb{C} : \Im(z) < 0\}$ the lower half-plane. If μ is a finite positive measure on $\mathbb{R}$, then its Cauchy transform

$$G(z) := \int_{-\infty}^{\infty} \frac{\mu \, \mathrm{d}x}{z - x} \ , \qquad (\Im(z) > 0) \ ,$$

is a holomorphic function ($G : \mathbb{C}^+ \to \mathbb{C}^+$) with the property

$$\limsup_{y \to \infty} y \, |G(iy)| < \infty \tag{1}$$

Conversely every holomorphic function $\mathbb{C}^+ \to \mathbb{C}^-$ with this property is the Cauchy transform of some finite positive measure on $\mathbb{R}$, and the lim sup in (1) equals $\mu(\mathbb{R})$. The inverse correspondence is given by Stieltjes' inversion formula:

$$\mu(B) = -\frac{1}{\pi} \lim_{\epsilon \downarrow 0} \int_B \Im(G(x + i\epsilon) \, \mathrm{d}x$$

valid for all Borel sets $B \in \mathbb{R}$ for which $\mu(\partial B) = 0$ ([1]).

We shall be mainly interested in the *reciprocal Cauchy transform*

$$F(z) = \frac{1}{G(z)}$$

The coresponding classes of reciprocal Cauchy transforms of probability measures with finite variance and zero mean will be denoted by $\mathcal{F}_0^2$.

The next proposition characterises the class $\mathcal{F}_0^2$.

Proposition 3.1. [2] *Let F be a holomorphic function $G : \mathbb{C}^+ \to \mathbb{C}^+$. Then the following statements are equivalent:*

(a): *F is the reciprocal Cauchy transform of a probability measure on $\mathbb{R}$ with finite variance and zero mean:*

$$\int_{-\infty}^{\infty} x^2 \mu \,\mathrm{d}x < \infty \qquad \textit{and} \qquad \int_{-\infty}^{\infty} x\mu \,\mathrm{d}x = 0 \ ;$$

(b): *There exists a finite positive measure ρ on $\mathbb{R}$ such that for all $z \in \mathbb{C}^+$:*

$$F(z) = z + \int_{-\infty}^{\infty} \frac{\rho \,\mathrm{d}x}{x - z} \ ;$$

(c): *There exists a positive number C such that for all $z \in \mathbb{C}^+$:*

$$|F(z) - z| \leq \frac{C}{\Im(z)}$$

Moreover, the variance σ^2 of μ in (a), the total weight $\rho(\mathbb{R})$ of ρ in (b) and the (smallest possible) constant C in (c) are all equal.

Proof. For the proof it is useful to introduce the function $C_F : (0, \infty) \to \mathbb{C}$

$$y \longmapsto y^2 \left(\frac{1}{F(iy)} - \frac{1}{iy} \right) = \frac{iy}{F(iy)} \left(F(iy) - iy \right))$$

In case F is the reciprocal Cauchy transform of some probability measure μ on $\mathbb{R}$, the limiting behaviour of $C_F(y)$ as $y \to \infty$ gives information on the integrals $\int x^2 \mu \,\mathrm{d}x$ and $\int x\mu \,\mathrm{d}x$. Indeed one has

$$C_F(y) = y^2 \int_{-\infty}^{\infty} \left(\frac{1}{iy - 1} - \frac{1}{iy} \right) \mu \,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{-xy^2 + ix^2 y}{x^2 + y^2} \mu \,\mathrm{d}x$$

The function $y \mapsto \Im(C_F(y))$ is nondecreasing and

$$\begin{aligned}\sup_{y>0} y\,\Im(C_F(y)) &= \lim_{y\to\infty} y\,\Im(C_F(y)) \qquad (2)\\ &= \lim_{y\to\infty}\int_{-\infty}^{\infty}\frac{y^2}{x^2+y^2}\,x^2\mu\,\mathrm{d}x = \int_{-\infty}^{\infty} x^2\mu\,\mathrm{d}x < \infty\end{aligned}$$

On the other side, by the dominated convergence theorem,

$$\int_{-\infty}^{\infty} x\mu\,\mathrm{d}x = \lim_{y\to\infty}\int_{-\infty}^{\infty}\frac{y^2}{x^2+y^2}\,x\mu\,\mathrm{d}x = -\lim_{y\to\infty}\Re(C_F(y)) \qquad (3)$$

(a)⇒(b). If $F \in \mathcal{F}_0^2$, then by (2) and (3) both the real and the imaginary part of $C_F(y)$ tends to zero as $y \to \infty$. How $C_F(y) = \frac{iy}{F(iy)}\,(F(iy) - iy))$, then $iC_F(y) = \frac{iy^2}{F(iy)} - y$ and $|C_F(y)| = y\left|\frac{iy}{F(iy)} - 1\right|$. But $\lim\limits_{y\to\infty} C_F(y) = 0$, it follows that

$$\lim_{y\to\infty}\frac{F(iy)}{iy} = 1$$

Therefore

$$\begin{aligned}\sigma^2 &= \lim_{y\to\infty} y\Im(C_F(y)) = \lim_{y\to\infty} y\,|C_F(y)| \qquad (4)\\ &= \lim_{y\to\infty} y\left|\frac{iy}{F(iy)}\right||F(iy) - iy| = \lim_{y\to\infty} y\,|F(iy) - iy| < \infty\end{aligned}$$

This condition says that the function $z \mapsto F(z) - z$ satisfies (1) and is therefore the Cauchy transform of some finite positive measure ρ on $\mathbb{R}$ with $\rho(\mathbb{R}) = \sigma^2$. This proves (b).

(b)⇒(c). If F is of the form (b), then

$$|F(z) - z| = \left|\int_{-\infty}^{\infty}\frac{\rho\,\mathrm{d}x}{x - z}\right| \le \int_{-\infty}^{\infty}\frac{\rho\,\mathrm{d}x}{|z - x|} \le \frac{\rho(\mathbb{R})}{\Im(z)} \qquad (5)$$

where C it may be equal with $\rho(\mathbb{R})$, whatever is $z \in \mathbb{C}^+$.

(c)⇒(a). Since $F : \mathbb{C}^+ \to \mathbb{C}^+$ is holomorphic, it can written in Nevanlinna's integral form [1]:

$$F(z) = a + bz + \int_{-\infty}^{\infty}\frac{1 + xz}{x - z}\,\tau\,\mathrm{d}x \qquad (6)$$

where $a, b \in \mathbb{R}$ with $b \geq 0$ and τ is a finite positive measure. Putting $z = iy$, $y > 0$, we find that

$$\begin{aligned} y\,\Im(F(iy) - iy) &= y\,\Im\left(a + iby + \int_{-\infty}^{\infty} \frac{1 + ixy}{x - iy}\,\tau\,\mathrm{d}x - iy\right) \\ &= y^2\left[(b-1) + \int_{-\infty}^{\infty} \frac{x^2+1}{x^2+y^2}\,\tau\,\mathrm{d}x\right] \end{aligned}$$

As $y \to \infty$, the integral tends to zero. By the assumptiom (c), the whole expression must remain bounded, which can be the case if $b = 1$. But then by (6), F must increase the imaginary part:

$$\Im(F(z)) \leq \Im(z)$$

Moreover, (c) implies that $F(z)$ and z can be brought arbitrarily close together, so by [2], proposition 2.1 F is the reciprocal Cauchy transform of some probability measure μ on $\mathbb{R}$.

Again by (c) this measure μ must have the properties

$$\int_{-\infty}^{\infty} x^2\mu\,\mathrm{d}x \leq \lim_{y\to\infty}\sup y\,|C_F(y)| = \lim_{y\to\infty}\sup y\,|F(iy) - iy| \leq y\,\frac{C}{\Im(iy)} = C$$

and

$$\int_{-\infty}^{\infty} x\mu\,\mathrm{d}x = -\lim_{y\to\infty}\Re(C_F(y)) = 0$$

The fact that

$$\sigma^2 \geq \rho(R) \geq C \geq \sigma^2$$

is clear from the above; this these three numbers must be equal. □

We now presents one lemma about invertibility of reciprocal Cauchy transforms of measures and certain related functions, to be called φ-functions. The lemma act in opposite directions; from reciprocal Cauchy transforms of probability measures to φ-functions and vice versa.

Lemma 3.1. [2] *Let $C > 0$ and let $\varphi : \mathbb{C}^+ \to \mathbb{C}^-$ be analytic with*

$$|\varphi(z)| \leq \frac{C}{\Im(z)}$$

Then the function $K : \mathbb{C}^+ \to \mathbb{C}^+$, $K(u) = u + \varphi(u)$ takes every value in $\mathbb{C}^+$ precisely once. The inverse $K^{-1} : \mathbb{C}^+ \to \mathbb{C}^+$ thus defined is of class $\mathcal{F}_0^2$ with variance $\sigma^2 \leq C$.

4. **The addition theorem**

We now formulate the main theorem of this section, namely the addition theorem.

Theorem 4.1. [2] *Let X_1 and X_2 be freely independent random variables on some Hilbert space $\mathcal{H}$ with distinguished vector ξ, cyclic for X_1 and X_2. Suppose that X_1 and X_2 have distributions μ_1 and μ_2 with variances σ_1^2 and σ_2^2. Then the closure of the operator*

$$X = X_1 + X_2$$

defined on Dom $(X_1) \cap$ Dom (X_2) *is self-adjoint and its probability distribution μ on $(\mathcal{H}, \xi)$ is given by*

$$\mu = \mu_1 \otimes \mu_2$$

where $\otimes$ is the free convolution product.
In particular in the region $\left\{ z \in \mathbb{C} \mid \Im(z) > 2\sqrt{\sigma_1^2 + \sigma_2^2} \right\}$ the φ-functions related to μ, μ_1 and μ_2 satisfy

$$\varphi = \varphi_1 + \varphi_2$$

The proof of this theorem is given in [2] where show that $< \xi, (z - \overline{X})^{-1}\xi >^{-1} = (F_1 \otimes F_2)(z)$ for all $z \in \mathbb{C}^+$.

5. **A free limit theorem**

In this section, we prove that sums of large numbers of frelly independent random variables of finite variance tend to certain distribution different to semiellipse distribution. The semiellipse distribution was first encountered by Wigner [6] when a studying spectra of large random matrices. The distribution obtained by author is defined by:

$$b_\sigma(x) = \frac{\sigma^2}{\pi(x^2+\sigma^4)}$$

where the graphics representation is in figure 1 for $\sigma_i = 1, 4, 10, 25, 50, 100$, $i = \overline{1,6}$. We remark that $b_\sigma(x)$ is the Cauchy distribution $Cau(0, \sigma^2)$.

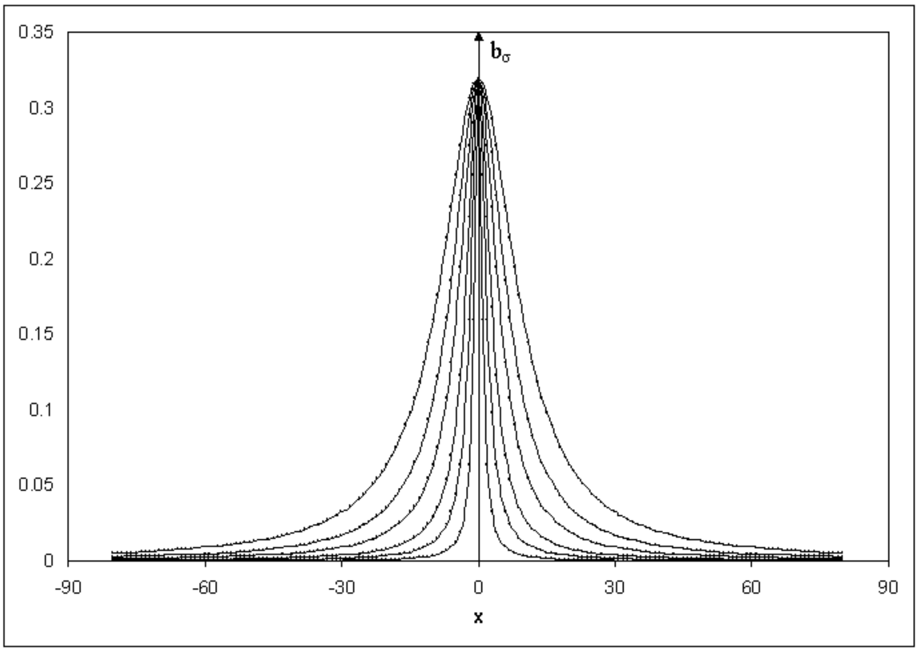

FIGURE 1. The graphics representation of distribution b_σ

Lemma 5.1. *The distribution* b_σ *has the following* φ*-function:*

$$\varphi(u) = -i\sigma^2 \tag{7}$$

Proof. We know that the inverse of the function $K_\sigma : \mathbb{C}^+ \to \mathbb{C}^+$, $K_\sigma(u) = u - i\sigma^2$ is the function $F_\sigma \in \mathcal{F}_0^2$. This is

$$F_\sigma : \mathbb{C}^+ \to \mathbb{C}^+,\ F_\sigma(z) = z + i\sigma^2$$

But this is the reciprocal Cauchy transform of b_σ by Stieltjes' inversion formula

$$\lim_{\epsilon\searrow 0} -\frac{1}{\pi}\Im\left(\frac{1}{F(x+i\epsilon)}\right) = b_\sigma(x)$$

Indeed:

$$\begin{aligned}\lim_{\epsilon\searrow 0} -\frac{1}{\pi}\Im\left(\frac{1}{F(x+i\epsilon)}\right) &= \lim_{\epsilon\searrow 0} -\frac{1}{\pi}\Im\left(\frac{1}{x+i(\epsilon+\sigma^2)}\right)\\ &= \lim_{\epsilon\searrow 0} -\frac{1}{\pi}\Im\left(\frac{x-i(\epsilon+\sigma^2)}{x^2+(\epsilon+\sigma^2)^2}\right)\\ &= \frac{1}{\pi}\cdot\frac{\sigma^2}{x^2+\sigma^4}\end{aligned}$$

□

We now formulate the free central limit theorem. We denote by $D_\lambda\mu$ its dilation by a factor λ for a probability measure μ on $\mathbb{R}$:

$$D_\lambda\mu(A) = \mu(\lambda^{-1}A)\ ,\ (A\subset\mathbb{R}\text{ measurable})$$

Theorem 5.1. *Let μ be a probability measure on $\mathbb{R}$ with mean 0 and variance σ^2, and for $n\in\mathbb{N}^*$ let*

$$\mu_n = \underbrace{D_{1/n}\mu\otimes\ldots\otimes D_{1/n}\mu}_{n-\ \textit{times}}$$

Then

$$\lim_{n\to\infty}\mu_n = b_\sigma$$

Proof. Let F, $\widetilde{F}_n$ and F_n denote the reciprocal Cauchy transforms of μ, $D_n\mu$ and μ_n respectively. Denote the associated φ-functions by φ, $\widetilde{\varphi}_n$ and φ_n. Let as in the proof of lemma 5.1, F_σ denote the reciprocal Cauchy transform of b_σ. By the continuity theorem 2.5 in [2] it suffices to show that for some $M>0$ and all $z\in\mathbb{C}_M^+$:

$$\lim_{n\to\infty} F_n(z) = F_\sigma(z)$$

or is equivalent with

$$\lim_{n\to\infty} K_\sigma\circ F_n(z) = z \tag{8}$$

Now, fix $z \in \mathbb{C}_M^+$ and put $u_n = F_n(z)$ and $z_n = \widetilde{F}_n^{-1}(u_n)$. Then $z - u_n = \varphi_n(u_n)$ and $z_n - u_n = \widetilde{\varphi}_n(u_n)$. Hence by an n-fold application of the addition theorem 4.1,

$$z - u_n = n(z_n - u_n)$$

Note that also

$$|z - u_n| \leq \frac{\sigma^2}{M} \ , \ \Im(u_n) > M$$

with respect to lemma 3.1.

By the property $F_{D_\lambda \mu}(z) = \lambda F(\lambda^{-1} z)$ and the integral reprezentation of F in accord to proposition 3.1,(b), we have:

$$\begin{aligned} z - u_n &= n(z_n - u_n) = n(z_n - \widetilde{F}_n(z_n)) \\ &= n(z_n - n^{-1}F(nz_n)) = nz_n - F(nz_n) \\ &= \int_{-\infty}^{+\infty} \frac{\rho \, \mathrm{d}x}{nz_n - x} \end{aligned}$$

Hence

$$\begin{aligned} |z - K_\sigma \circ F_n(z)| &= |z - K_\sigma(u_n)| = \left| z - u_n + i\sigma^2 \right| \\ &= \int_{-\infty}^{+\infty} \left| \frac{1}{nz_n - x} + i\sigma^2 \right| \rho \, \mathrm{d}x \end{aligned}$$

The integrand on the right hand side is uniformly bounded and tends to zero pointwise as n tends to infinity. □

Remark 5.1. First note that every φ-function goes like $-i\sigma^2$ high above the real line. Indeed we have $z = F^{-1}(u) \approx u$ and

$$\varphi(u) = K(u) - u = F^{-1}(u) - u = \underbrace{z - F(z)}_{\varphi(z)} \approx -i\sigma^2$$

Now, due to the scaling law $\varphi_{D_\lambda \mu}(u) = \lambda \varphi(\lambda^{-1} u)$ and by proposition 3.1 we obtain

$$\varphi_n(u) = n\widetilde{\varphi}_n(u) = n\varphi_{D_{\frac{1}{n}}\mu}(u) = n \cdot \frac{1}{n} \ \varphi(nu) \to -i\sigma^2 \ , \ (n \to \infty)$$

In [3], the author to use in place of b_σ Cauchy distribution, the distribution defined by

$$b_\sigma(x) = \begin{cases} \frac{1}{2\sqrt{2}\pi x}\sqrt{\sqrt{1+16x^2\sigma^4}-1} & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \leq 0 \end{cases}$$

where the dilation of probability measure has a factor $\lambda = n$.

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