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MONOTONE INTERPOLANT BUILT WITH SLOPES OBTAINED BY LINEAR COMBINATION

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Abstract. Slopes needed to obtain a monotone piecewise cubic Hermite interpolant are constructed. These slopes are obtained local as linear combination of the slopes of the line segments joining the data.

The most used methods to construct a monotone interpolant to monotone data is to insert new points between two adjacent knots, respectively to give the slopes needed to build the piecewise interpolant. The paper of Fritsch-Carlson [3] refers to necessary and sufficient condition to obtain a monotone cubic interpolant. There is also discussed a nonlocal algorithm to built the adequate slopes. We use the domain given there and we propose a local method to compute the slopes necessary to built a monotone piecewise cubic interpolant.

Let $\pi : x_1 < x_2 < ... < x_n$ be a partition of the interval $I = [x_1, x_n]$. Let $\{f_i : i = 1, ..., n\}$ be a given set of monotone data values at the partition points (knots): $f_i \leq f_{i+1}$ or $f_i \geq f_{i+1}$, $i = 1, ..., n-1$. The goal is to construct a monotone piecewise cubic function $p \in C^1(I)$ that interpolate the given data. In each subinterval $[x_i, x_{i+1}]$ the function p is the cubic Hermite interpolant that interpolates the points $(x_i, f_i), (x_{i+1}, f_{i+1})$ and with the endslopes d_i, d_{i+1} which will be determined later. Let $\Delta_i = (f_{i+1} - f_i)/h_i$ be the slope of the line segment joining the data $(x_i, f_i), (x_{i+1}, f_{i+1})$ where $h_i = x_{i+1} - x_i$. Let $\alpha = \frac{d_i}{\Delta_i}, \beta = \frac{d_{i+1}}{\Delta_i}$ be the ratios of the endpoint derivates to the slope of the secant line.

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In [3] it was proved that the piecewise cubic interpolant is monotone on each $[x_i, x_{i+1}]$ if and only if:

$$(\alpha, \beta) \in \mathcal{M} \tag{1}$$

where the monotonicity region $\mathcal{M}$ is depicted in Figure 1.

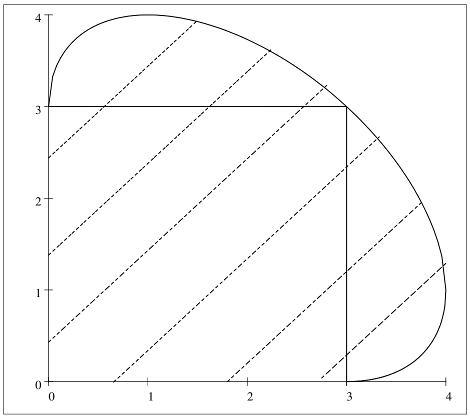

FIGURE 1. The region $\mathcal{M}$ (dashed) with the square $\mathcal{S} = [0,3] \times [0,3]$ inside

As domain we use a subregion of $\mathcal{M}$ bounded by the four lines $\alpha = 0, 3$ and $\beta = 0, 3$:

$$\mathcal{S} = [0,3] \times [0,3] \,.$$

We build the slopes d_i as a linear combination of the adjacent Δ_{i-1}, Δ_i :

$$d_i = (1 - \lambda_i)\,\Delta_{i-1} + \lambda_i \Delta_i,\; i = 2, ..., n-1. \tag{2}$$

Such a linear combination was also proposed by Akima in [1] with

$$\lambda_i = \frac{|\Delta_{i-1} - \Delta_{i-2}|}{|\Delta_{i+1} - \Delta_i| + |\Delta_{i-1} - \Delta_{i-2}|},\; i = 3, ..., n-2$$

but this method fails to preserve everywhere the monotonicity. Another local method proposed in [4] use the harmonic mean of the Δ_{i-1}, Δ_i.

We search the admissible values of the parameter λ_i according to relation (1), such that:

$$\left(\frac{d_i}{\Delta_{i-1}}, \frac{d_i}{\Delta_i}\right) \in [0,c] \times [0,c] \tag{3}$$

with $c \in [0,3]$. The value $c = 0$, discussed also in [6], produce a slightly flat interpolant.

The condition (3) is equivalent with the following two inequalities:

$$0 \leq \frac{(1-\lambda_i)\,\Delta_{i-1} + \lambda_i \Delta_i}{\Delta_{i-1}} \leq c \tag{4}$$

$$0 \leq \frac{(1-\lambda_i)\,\Delta_{i-1} + \lambda_i \Delta_i}{\Delta_i} \leq c. \tag{5}$$

From (4) and (5) we obtain:

$$-\Delta_{i-1} \leq \lambda_i\,(\Delta_i - \Delta_{i-1}) \leq (c-1)\,\Delta_{i-1} \tag{6}$$

$$-\Delta_{i-1} \leq \lambda_i\,(\Delta_i - \Delta_{i-1}) \leq c\Delta_i - \Delta_{i-1} \tag{7}$$

If $\Delta_i - \Delta_{i-1} \neq 0$ the admissible interval for λ_i becomes:

$$-\frac{\Delta_{i-1}}{\Delta_i - \Delta_{i-1}} \leq \lambda_i \leq \frac{(c-1)\,\Delta_{i-1}}{\Delta_i - \Delta_{i-1}},\ \text{if } \Delta_i - \Delta_{i-1} > 0, \tag{8}$$

$$\frac{c\Delta_i - \Delta_{i-1}}{\Delta_i - \Delta_{i-1}} \leq \lambda_i \leq -\frac{\Delta_{i-1}}{\Delta_i - \Delta_{i-1}},\ \text{if } \Delta_i - \Delta_{i-1} < 0. \tag{9}$$

If $\Delta_i - \Delta_{i-1} = 0$, then λ_i have no influence on $d_i : d_i = \Delta_i$.

For $\lambda_i = -\frac{\Delta_{i-1}}{\Delta_i - \Delta_{i-1}}$ the slope $d_i = 0$ and, although this value is admissible, the interpolant becomes flat. It seems reasonable to impose that the slope $d_i \geq \min\{\Delta_{i-1}, \Delta_i\}$. That's mean:

$$0 \leq \lambda_i,\ \text{if } \Delta_i - \Delta_{i-1} > 0,$$

$$\lambda_i \leq 1,\ \text{if } \Delta_i - \Delta_{i-1} < 0.$$

So, we restrict the relations (8) and (9) to:

$$0 \leq \lambda_i \leq \frac{(c-1)\,\Delta_{i-1}}{\Delta_i - \Delta_{i-1}},\ \text{if } \Delta_i - \Delta_{i-1} > 0, \tag{10}$$

$$\frac{c\Delta_i - \Delta_{i-1}}{\Delta_i - \Delta_{i-1}} \leq \lambda_i \leq 1,\ \text{if } \Delta_i - \Delta_{i-1} < 0. \tag{11}$$

The inequalities (10),(11) are consistent if $0 \leq \frac{(c-1)\Delta_{i-1}}{\Delta_i - \Delta_{i-1}}$ and $\frac{c\Delta_i - \Delta_{i-1}}{\Delta_i - \Delta_{i-1}} \leq 1$, which are equivalent with $c \geq 1$. So we impose:

$$c \in [1, 3].$$

To fix the value of λ_i in the admissible interval given in (10),(11) we use a convex combination between the ends of these intervals:

$$\lambda_i = \begin{cases} (1 - w_i)\, 0 + w_i \frac{(c-1)\Delta_{i-1}}{\Delta_i - \Delta_{i-1}}, & \text{if } \Delta_i - \Delta_{i-1} > 0, \\ (1 - v_i) + v_i \frac{c\Delta_i - \Delta_{i-1}}{\Delta_i - \Delta_{i-1}}, & \text{if } \Delta_i - \Delta_{i-1} < 0 \end{cases}$$

equivalent with

$$\lambda_i = \begin{cases} \frac{\Delta_{i-1}}{\Delta_i - \Delta_{i-1}} w_i \,(c-1)\ , & \text{if } \Delta_i - \Delta_{i-1} > 0, \\ \frac{1}{\Delta_i - \Delta_{i-1}} \left((1 + (c-1)\, v_i)\, \Delta_i - \Delta_{i-1} \right), & \text{if } \Delta_i - \Delta_{i-1} < 0. \end{cases}$$

Then from (2) follows for the slopes:

$$d_i = \begin{cases} (1 + (c-1)\, w_i)\, \Delta_{i-1}, & \text{if } \Delta_i - \Delta_{i-1} \geq 0, \\ (1 + (c-1)\, v_i)\, \Delta_i\ , & \text{if } \Delta_i - \Delta_{i-1} < 0, \end{cases} \tag{12}$$

We would like that the value d_i depends not only on the slope of line segment but also on the relative spacing of x_i and f_i-values. For this reason we use the length of the line segments (in $||\cdot||_1$ norm)

$$l_i = |x_{i+1} - x_i| + |f_{i+1} - f_i|$$

and we choose the weights w_i, v_i as follow:

$$w_i = \left(1 - \frac{\Delta_{i-1}}{\Delta_i}\right) \frac{1}{1 + \frac{l_{i-1}}{l_i}} \in [0, 1], \tag{13}$$

$$v_i = \left(1 - \frac{\Delta_i}{\Delta_{i-1}}\right) \frac{1}{1 + \frac{l_i}{l_{i-1}}} \in [0, 1]. \tag{14}$$

The proposed values are based on the following idea:

- if Δ_i is close to Δ_{i-1} then naturally d_i must be also close to this value; the first therm in (13),(14) care about this because $\left(1 - \frac{\Delta_{i-1}}{\Delta_i}\right) \simeq 0$ $\left(\left(1 - \frac{\Delta_i}{\Delta_{i-1}}\right) \simeq 0\right)$ so $d_i \simeq \Delta_{i-1} \simeq \Delta_i$.

- if Δ_i is not close to Δ_{i-1} ($\Delta_i \gg \Delta_{i-1}$, or $\Delta_i \ll \Delta_{i-1}$) then the slope d_i must be close to the value Δ_{i-1} if $l_{i-1} > l_i$, respectively close to Δ_i if $l_{i-1} < l_i$. The second therm in (13),(14) have the function to meet this requirement.

The slopes at end points are computed using a formula for numerical differentiation (the three-point formula), but inside of the admissible values. This values coresponds to the slopes of the parabola built on three consecutive points.

The rate of convergence of the derivative is in general $O(h)$, but for $c = 2$ and uniformly spaced data, the rate becomes $O\left(h^2\right)$.

Theorem 1. *Let $(x_i)_{i=1}^n$ a uniformly spaced data $x_{i+1} - x_i = h,\ i = 1, ..., n-1,$ and let $f \in C^3[a, b]$ be a monotone incresing function with:*

$$f_i = f(x_i).$$

Then for $c = 2$ the values (12) gives $O\left(h^2\right)$ approximation to $f'(x_i)$:

$$f'(x_i) - d_i = O\left(h^2\right).$$

Proof. If $\Delta_i - \Delta_{i-1} \geq 0$, then $d_i = (1 + w_i)\Delta_{i-1}$, where $\Delta_{i-1} = \frac{f_i - f_{i-1}}{x_i - x_{i-1}}$, so using a Taylor formula we get:

$$f_{i-1} = f(x_i - h) = f(x_i) - hf'(x_i) + \frac{h^2 f''(x_i)}{2} - \frac{h^3 f'''(\xi_i)}{6},\ \xi_i \in (x_{i-1}, x_i)$$

consequently

$$d_i = (1 + w_i)\frac{1}{h}\left(hf'(x_i) - \frac{h^2 f''(x_i)}{2} + \frac{h^3 f'''(\xi_i)}{6}\right).$$

To compute w_i we use also the expansion:

$$f_{i+1} = f(x_i + h) = f(x_i) + hf'(x_i) + \frac{h^2 f''(x_i)}{2} + \frac{h^3 f'''(\theta_i)}{6},\ \theta_i \in (x_i, x_{i+1}).$$

So we obtain for the difference:

$$f'(x_i) - d_i = \frac{E}{3\left(12\left(f_i' + 1\right) + \left(f_i''' + f_{i+1}'''\right)h^2\right)\left(6f_i' + 3f_i''h + f_i'''h^2\right)}h^2 \tag{15}$$

with

$$E = -f_i'''^2 f_{i+1}''' h^4 + 3 f_i''' \ f_i'' \left(f_i''' - 2 f_{i+1}'''\right) h^3 +$$
$$+ 3 \left(f_{i+1}'''^2 - 3 f_i' f_i''' f_{i+1}''' - 3 f_i''' f_{i+1}''' - f_i' f_i'''^2 + 3 \ f_i''^2 \left(2 f_i''' - f_{i+1}'''\right)\right) h^2 +$$
$$+ 9 \ f_i'' \left(3 f_i''' - 5 f_{i+1}''' + f_i' f_i''' - 3 f_i' f_{i+1}''' + 3 \ f_i''^2\right) h -$$
$$- 18 \left(f_i' \left(f_i' + 1\right) \left(f_i''' + f_{i+1}'''\right) - 3 \ f_i''^2 \left(f_i' + 2\right)\right)$$

where

$$f_i' = f'\left(x_i\right), \ f_i'' = f''\left(x_i\right), \ f_i''' = f'''\left(\xi_i\right), \ f_{i+1}''' = f'''\left(\theta_i\right).$$

The case $\Delta_i - \Delta_{i-1} < 0$ can be treated in the same manner and we obtain:

$$f'\left(x_i\right) - d_i = \frac{F}{3\left(12\left(f_i' + 1\right) + \left(f_i''' + f_{i+1}'''\right) h^2\right)\left(-6 f_i' + 3 f_i'' h - f_i''' h^2\right)} h^2$$

with

$$\begin{aligned} F \quad = \quad & -f_i'''^2 f_{i+1}''' h^4 + 3 f_i''' \ f_i'' \left(f_i''' - 2 f_{i+1}'''\right) h^3 + \\ & +3 \left(f_{i+1}'''^2 - 3 f_i' f_i''' f_{i+1}''' - 3 f_i''' f_{i+1}''' - f_i' f_{i+1}'''^2 + 3 \ f_i''^2 \left(2 f_i''' - f_{i+1}'''\right)\right) h^2 + \\ & 9 \ f_i'' \left(3 f_i''' - 5 f_{i+1}''' + f_i' f_i''' - 3 f_i' f_{i+1}''' + 3 \ f_i''^2\right) h + \\ & -18 \left(f_i' \left(f_i' + 1\right) \left(f_i''' + f_{i+1}'''\right) - 3 \ f_i''^2 \left(f_i' + 2\right)\right). \end{aligned}$$

□

Corollary 2. *If $c = 2$ the cubic Hermite interpolant with slopes (12) gives an $O\left(h^3\right)$ approximation to f for uniformly spaced data.*

For the particular value $c = 2$ the slope d_i fulfill another (reasonable) properties, namely it's value don't break through the maximum between Δ_{i-1} and Δ_i.

Proposition 3. *If $c = 2$ the slopes d_i given in (12) satisfy:*

$$\min\left\{\Delta_{i-1}, \Delta_i\right\} \le d_i \le \max\left\{\Delta_{i-1}, \Delta_i\right\}. \tag{16}$$

Proof. The inequality:

$$\min\left\{\Delta_{i-1}, \Delta_i\right\} \le d_i$$

was already used.

Admit now that $\Delta_i - \Delta_{i-1} \geq 0$, then we must prove that:

$$d_i \leq \Delta_i$$

equivalent with:

$$(1 + w_i)\,\Delta_{i-1} \leq \Delta_i.$$

Substituting (13) it follows:

$$\left(1 - \frac{\Delta_{i-1}}{\Delta_i}\right) \frac{1}{1 + \frac{l_{i-1}}{l_i}} \leq \frac{\Delta_i}{\Delta_{i-1}} - 1$$

equivalent with:

$$\frac{1}{1 + \frac{l_{i-1}}{l_i}} \leq \frac{\Delta_i}{\Delta_{i-1}}$$

which is true because the left side is lower, while the right side is greater than 1.

The case $\Delta_i - \Delta_{i-1} \leq 0$ can be treated similarly. □

Remark 4. *The property (16) hold for* $c \in [1, 2]$.

As example we use the data from [1]:

x_i	0	2	3	5	6	8	9	11	12	14	15
f_i	10	10	10	10	10	10	10.5	15	50	60	85

The cubic Hermite interpolant for $c = 2$ respectively for $c = 3$ are represented in Figure 2.

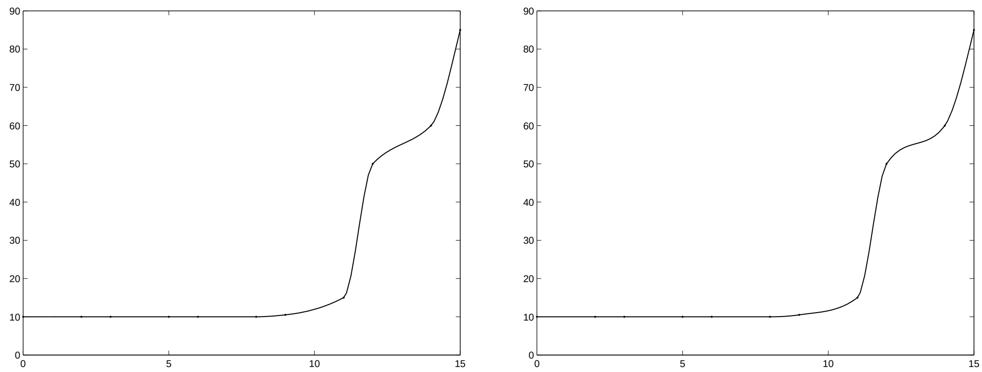

FIGURE 2. The monotone interpolant for $c = 2$ (left) and $c = 3$ (right)

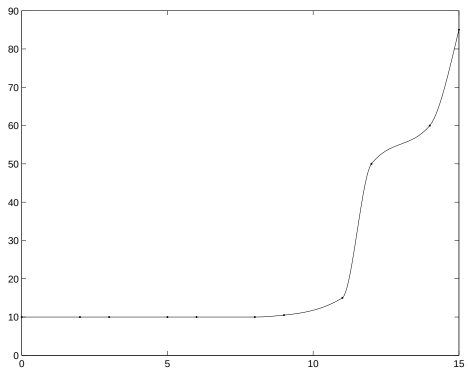

FIGURE 3. The piecewise cubic Hermite interpolating polynomial-pchip

By comparison we have represented in Figure 3 the cubic interpolant using the MATLAB's specialized function *pchip*. Those slopes d_i are computed using a weighted average of Δ_{i-1}, Δ_i.

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