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ON RANDOM FIXED POINTS IN RANDOM CONVEX STRUCTURES

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Abstract. In this paper, we present some random fixed point theorems in random convex structures.

1. **Introduction and preliminaries**

Random fixed point theory has received much attention for the last two decades, since the publication of the paper by Bharucha-Reid [2]. Also random best approximation attracted authors after the papers by Sehgal and Singh [15], Papageorgiou [13], Lin [11], and Beg et al. [1].

On the other hand, in the past years, because of practical necessities, the attempts of generalizing the notion of convexity introduced by J. Von Neumann and O. Morgenstern [12], M. Stone [16] were brought up-to-date by S.P. Gudder [5]. Consequently, Gudder (1979) introduced the notion of convex structure and of F-convex set with applications in quantum mechanics, colour vision and petroleum engineering. Subsequently, fixed point theorems for nonexpansive mappings using the convex structures introduced by Gudder was proved by Petrusel [14] and later by Ganguly and Jadhav [6] for approximation theorems.

Again, away from this, Takahashi [17] also introduced a notion of convexity in metric spaces and presented fixed point theorems for nonexpansive mappings. This motivated Guay et al. [7] to discuss the results on convex metric spaces. These

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works alongwith those on random approximations motivated Beg et al. [3] to present random fixed point theorems and related results in random convex metric spaces.

It is a need for further research to study a relationship between convex structures introduced by Takahashi [17] and Gudder [5] respectively. In this vein, we are presenting random fixed point theorems in random convex structures, following Gudder [5], Petrusel [14], Beg and Shahzad [3].

Before we present our theorems, we will introduce some basic preliminaries.

Let (Ω, Σ) be a measurable space, (X, d) a metric space, 2^X the family of all subsets of X, K(X) family of all nonempty compact subsets of X and CB(X) family of all nonempty closed bounded subsets of X.

A mapping $T : \Omega \to 2^X$ is called measurable if for any open subset C of X,

$$T^{-1}(C) = \{\omega \in \Omega : T(\omega) \cap C \neq \phi\} \in \Sigma.$$

A mapping $\xi : \Omega \to$X is said to be a measurable selector of T if ξ is measurable and for any $\omega \in \Omega,\ \xi(\omega) \in T(\omega)$.

A mapping $f : \Omega \times X \to$ X is called a random operator if for any $x \in X$, $f(., x)$ is measurable.

A measurable mapping $\xi : \Omega \to$ X is called a random fixed point of a random multivalued (single valued) operator $T : \Omega \times X \to CB(X) (f : \Omega \times X \to X)$ if for every $\omega \in \Omega,\ \xi(\omega) \in T(\omega,\ \xi(\omega))\ (\xi(\omega) = f(\omega,\ \xi(\omega)))$.

A random operator $T : \Omega \times X \to CB(X)$ is called Lipschitzian if

$$H(T(\omega,\ x),\ T(\omega,\ y)) \leq L(\omega)\, d(x,\ y)$$

for any $x, y \in X$and $\omega \in \Omega$, where $L : \Omega \to [0, \infty)$ is a measurable map and H is the Pompeiu-Hausdorff metric on CB(X), induced by the metric d. When $L(\omega) < 1$, $(L(\omega) = 1)$ for each $\omega \in \Omega$, T is called contraction (nonexpansive).

We present, for the convenience of readers, the following definitions which also appear in Petrusel [14].

Definition 1.1. *Let X be a set and $F : [0, 1] \times X \times X \to X$ a mapping. Then the pair (X, F) forms a convex prestructure.*

Definition 1.2. *Let (X, F) be a convex prestructure. If F satisfies the following conditions:*

1. $F(\lambda, x, F(\mu, y, z)) = F(\lambda+(1-\lambda)\mu, F(\lambda(\lambda+(1-\lambda)\mu)^{-1}, x, y), z)$ *for every* $\lambda, \mu \in [0,1]$ *with* $\lambda + (1-\lambda)\mu \neq 0$ *and* $x, y, z \in X$.
2. $F(\lambda, x, x) = x$ *for any* $x \in X$ *and* $\lambda \in [0,1]$, *then (X, F) forms a semi-convex structure.*

   If (X, F) is a semi-convex structure, then $F(1, x, y) = x$ *for any* $x, y \in X$.

Definition 1.3. *A semi-convex structure (X, F) is said to form a convex structure if F also satisfies the conditions:*

1. $F(\lambda, x, y) = F(1-\lambda, y, x)$ *for every* $\lambda \in [0,1]$, $x, y \in X$
2. *If* $F(\lambda, x, y) = F(\lambda, x, z)$ *for some* $\lambda \neq 0$, $x \in X$, *then* $y = z$.

Definition 1.4. *Let (X, F) be a semi-convex structure. A subset Y of X is called F - semi-starshaped if there exists a* $p \in Y$, *so that for any* $x \in Y$ *and*

$$\lambda \in [0,1],\ F(\lambda, x, p) \in Y.$$

Definition 1.5. *Let (X,F) be a convex structure. A subset Y of X is called:*

1. *F - starshaped if there exists a* $p \in Y$, *so that for any* $x \in Y$ *and*

$$\lambda \in [0,1],\ F(\lambda, x, p) \in Y.$$

2. *F - convex if for any* $u, v \in Y$ *and* $\lambda \in [0,1]$, *we have* $F(\lambda, u, v) \in Y$.

For $F(\lambda, u, v) = \lambda u + (1-\lambda)v$, we obtain the known notions of starshaped and convexity from linear spaces.

Petruşel [14] noted with an example that a set can be a F - semi convex structure without being a convex structure. So, it follows that the results on fixed point theory and on best approximation theory obtained for semi-convex and semi-starshaped structures will be more general than those on F - convex structure.

Definition 1.6 (Random Semi-Convex Structure)**.** *Let*$F : \Omega \times X \times X \times [0,1] \to X$ *be a mapping having the following properties:*

1. *For each* $\omega \in \Omega$, $F(\omega, ., ., .)$ *is a semi-convex structure on X,*

2. *For each* $x, y \in X,\ \lambda \in [0,1],\ F(., x, y, \lambda)$ *is measurable.*

The mapping F is called a random semi-convex structure on X.

Example 1 [5]. The mapping $F : [0,1] \times R_+^* \times R_+^* \to R_+^*$ given by

$$F(\lambda, u, v) = u^\lambda . v^{1-\lambda}$$

together with the set of strict positive real numbers form a convex structure.

Example 2 [14]. The mapping $F : [0,1] \times R \times R \to R$ given by

$$F(\lambda, u, v) = [\lambda u^{2k} + (1-\lambda) v^{2k}]^{1/2k}, k \in N^*$$

together with the set of real numbers form a semi-convex structure without being a convex structure.

2. **Main results**

Theorem 2.1. *Let X be a separable random Banach space with semi-convex structure F, where the mapping* $F : \Omega \times X \times X \times [0,1] \to X$ *satisfies the following conditions:*

1. *F is* ϕ *- contractive relative to the second argument, i.e., there exists a mapping* $\phi : [0,1[\to [0,1[$ *so that:*

$$||F(\omega, x, p, \lambda) - F(\omega, y, p, \lambda)|| \le \phi(\lambda).||x - y||,$$

   for any $x,\ y,\ p \in X$ *and* $\lambda \in [0,1[$ *and* $\omega \in \Omega$.

2. *F is continuous relative to the first argument.*

Let Y be a compact and F - semi-starshaped subset of X and the mapping $T : \Omega \times Y \to Y$ *be nonexpansive random operator. Then T has a random fixed point.*

Proof. Choose $p \in Y$ so that for any $u \in Y$ and $\lambda \in [0,1[$, we have $F(\omega, u, p, \lambda) \in Y$ for each $\omega \in \Omega$. Let $\{K_n\}$ be a sequence of measurable mappings $K_n : \Omega \to (0,1)$ and $K_n(\omega) \to 1$ as $n \to \infty$.

Define the random operator $T_n : \Omega \times Y \to Y$ by

$$T_n(\omega, x) = F(\omega, T(\omega, x), p, K_n(\omega))$$

T_n is, because of F - semi-starshaped of Y, well defined. The operator T_n is a contraction Indeed

$$||T_n(\omega, x_1) - T_n(\omega, x_2)|| = ||F(\omega, T(\omega, x_1), p, K_n(\omega)) - F(\omega, T(\omega, x_2), p, K_n(\omega))||$$

$$\leq \phi(K_n(\omega))||T(\omega, x_1) - T(\omega, x_2)||$$

for all $x, y \in Y$ and $\omega \in \Omega$. By Hans [8], T_n has a unique random fixed point ξ_n.

For each n, define $G_n : \Omega \to K(X)$ by $G_n(w) = Cl\{\xi_i(\omega) : i \geq n\}$.

Define $G : \Omega \to K(X)$ by $G(\omega) = \bigcap_{n=1}^{\infty} G_n(\omega)$. Since G is measurable (see Himmelberg [9], Theorem 4.1), by Kuratowski and Ryll-Nardzewski theorem in [10] we have that G has a measurable selector ξ. Because Y is compact, $\{\xi_n(\omega)\}$ has a subsequence $\{\xi_{n_j}(\omega)\}$ converging to $\xi(\omega)$. By the continuity of T and *F*, $T(\omega, \xi_{n_j}(\omega))$ converges to $T(\omega, \xi(\omega))$. Thus, $T(\omega, \xi(\omega)) = \xi(\omega)$ for each $\omega \in \Omega$.

Next we have the following:

Theorem 2.2. *Let X be a separable random Banach space with a semi-convex structure F,where the mapping $F : \Omega \times X \times X \times [0, 1] \to X$ satisfies the conditions:*

1. *F is ϕ - contractive relative to the second argument.*
2. *F is continuous relative to the first argument.*

Let Y be a weakly compact and F - semi-starshaped subset of X and the mapping $T : \Omega \times Y \to Y$ be nonexpansive and weakly continuous mapping. In these conditions the mapping T has a random fixed point.

Proof. As in Theorem 2.1, define $\{K_n\}$ and the random operator T_n. As before, each Tn is a contraction mapping on Y. Since the weak topology of X is Hausdorff and Y is weakly compact,we have that Y is weakly closed and therefore, strongly closed (See Dotson, Theorem 2 [4]). Hence Y is a complete metric space (with the norm topology of the Banach space X). By Hans [8], T_n has a unique random fixed point $\xi_n \in Y$. By the Eberlein-Smulian [4]theorem, Y is weakly sequentially compact. Thus there is a subsequence $\{\xi_n(\omega)\}$ such that $\xi_{n_j}(\omega) \xrightarrow{\omega} \xi(\omega) \in Y$ (denotes weak convergence). Since T is weakly continuous and F - continuous, we have

$$T(\omega, \xi_{n_j}(\omega)) \xrightarrow{\omega} T(\omega, \xi(\omega))$$

Thus, $T(\omega,\ \xi(\omega)) = \xi(\omega)$. for each $\omega \in \Omega$.

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References

[1] Beg, L. and Shahzad, N., *Random fixed points of random multivalued operators on Polish spaces*, Nonlinear Anal., **20**(7)(1993), 835-847.

[2] Bharucha-Reid, A.T., *Fixed point theorems in probabilistic analysis*, Bull. Amer. Math. Soc., **82**(1976), 641-657.

[3] Beg, L. and Shahzad, N., *Random fixed points and approximations in random convex metric spaces*, J. Appl. Math. Stochastic Anal., **6**(3)(1993), 237-246.

[4] Dotson, Jr. W.G., *On fixed points of nonexpansive mappings in non-convex sets*, Proc. Amer. Math. Soc., **38**(1973), 155-156.

[5] Gudder, S.P., **A general theory of convexity**, Rend. Sem. Mat. Milano, **49**(1979), 89-96.

[6] Ganguly, A. and Jadhav, K.H., *An application of fixed point theorem to approximation theory*, Pure Appl. Math. Sci., XLII No. 1-2, (1995), 19-22.

[7] Guay, M.D., Singh, K.L. and Whitfield, J.H.M., *Fixed point theorems for nonexpansive mappings in convex metric spaces*, Proc. Conf. On Nonlinear Analysis (edited by S.P. Singh and J.H. Burry), Marcel Dekker, Inc., New York, **80**(1982), 179-189.

[8] Hans, O., *Reduzierende Zufallige transformationen*, Czech. Math. J., **7**(1957), 154-158.

[9] Himmelberg, C.J., *Measurable relations*, Fund. Math., **87**(1975), 53-72.

[10] Kuratowski, K. and Ryll-Nardzewski, C., *A general theorem on selectors*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys., **13**(1965), 397-403.

[11] Lin, T.C., *Random approximations and random fixed point theorems for nonself maps*, Proc. Amer. Math. Soc., **103**(1988), 1129-1135.

[12] Neumann, J. Von and Morgenstern, O., *Theory of Games and Economic Behaviour*, Princeton Univ. Press., Princeton, New Jersey, 1944.

[13] Papageorgiou, N.S., *Random fixed point theorems for measurable multifunctions in Banach spaces*, Proc. Amer. Math. Soc., **97**(1986), 507-514.

[14] Petrusel, A., *Starshaped and fixed points, Seminar on fixed point theory*, Babes-Bolyai University (Cluj-Napoca), Faculty of Mathematics and Physics, Research Seminars, Romania, **3**(1987), 19-24.

[15] Sehgal, V.M. and Singh, S.P., *On random approximations and a random fixed point theorem for set-valued mappings*, Proc. Amer. Math. Soc., **95**(1985), 91-94.

[16] Stone, M., *Postulates for a baryoentric calculus*, Ann. Math., **29**(1949), 25-30.

[17] Takahashi, W., *A convexity in metric space and non-expansive mappings*, I, Kodai Math. Sem. Rep., **22**(1970), 142-149.

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