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QUASI-INTERPOLATORY AND INTERPOLATORY SPLINE OPERATORS: SOME APPLICATIONS

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Abstract. In this paper we consider quasi-interpolatory spline operators that satisfy some interpolation conditions. We give some applications of these operators constructing approximating integral operators and numerically solving Volterra integral equations of the second kind. We prove convergence results for the constructed methods and we perform numerical examples and comparisons with other spline methods.

1. Introduction

It is known that quasi-interpolatory operators play a main role in the approximation of data and functions, in the numerical solution of integrals or, more in general, of integral equations. Interpolatory operators also are very important in function approximation theory and there exists a wide literature on such two class of operators. In the last years, in [11], a method for constructing a quasi-interpolatory operator with interpolation properties, has been presented giving a general convergence theorem and in [7] a new class of operators, which are refinable, quasi-interpolatory and that satisfy some interpolation conditions has been studied. In this paper we consider a quasi-interpolatory spline operator that satisfies some interpolation conditions (qi-i operator) and we propose some its applications; for example, we construct a collocation method for solving a second kind linear Volterra integral equation

$$f(x) = g(x) + \int_0^x k(x,s)f(s)ds, \quad x \in [0,X] \tag{1.1}$$

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with $k(x, s) = k(x - s)$ and $k \in C(0, X] \cap L_1[0, X]$. The integral equation (1.1) has a unique solution $f \in C[0, X]$ if g is a continuous function in $[0, X]$, but the derivatives of this solution can be unbounded at $x = 0$; then graded grids used in the partition of $[0, X]$ reflect the possible singular behaviour of the derivative of the exact solution near $x = 0$. For example, in [2] and in [3], a collocation method using graded meshes and piecewise polynomials, for weakly singular Volterra integral equations, has been considered. In [6] and in [8] collocation methods based on spline functions have been studied for numerically solving (1.1). In [8] a method based on projector splines has been used in a suitable, first subinterval of $[0, X]$ combined with a Simpson's rule in the last part of $[0, X]$. In [6] nodal splines that are quasi-interpolatory and interpolatory (with the number of interpolation points that increases when the number knots increases) has been considered for numerically solving (1.1). The collocation method of this paper, based on qi-i spline operators of order $m \geq 2$, has several good properties as a low computational complexity and good performance when the solution of (1.1) is a continuous function. The results obtained are comparable with those obtained by using nodal splines or projector splines. Moreover, with the collocation method of this paper, we can opportunely choose the interpolation points of the qi-i spline, that can be different from the partition knots and we can obtain directly by a linear system, the value of f on such points without a successive evaluation of the approximation of f (as required, instead, if we use only projector splines). The approximate solution error obtained will converges to zero at the same rate as the quasi-interpolatory spline error.

This paper is organized as follows. In Section 2 we give definition and properties of qi-i spline operators on graded meshes and we give convergence results. In Section 3 we define an approximating integral operator based on qi-i spline operators and we analize its main properties and convergence. In Section 4 we describe a collocation method for Volterra integral equation of second kind based on the approximating integral operator of Section 3 and we give convergence results. In Section 5 we give some numerical results and comparisons with rules based on projector splines and on nodal splines.

2. Qi-i spline operators

We give the definition and the main properties of qi-i spline operators.

Let $s \geq 0$ be a given positive integer and consider the partition of $[a, b]$

$$\Delta_s := \{a = y_0 < y_1 < \ldots < y_s < y_{s+1} = b\} \tag{2.1}$$

in $s+1$ subintervals $[y_k, y_{k+1})$, with $h_k = y_{k+1} - y_k$, $k = 0, 1, ..., s$. We shall assume that $\max\limits_{0 \leq k \leq s} h_k \to 0$ as $s \to \infty$.

We say that the sequence of partitions $\{\Delta_s,\ s = 1, 2, ...\}$ is locally uniform ($l.u.$) if there exists a constant $R \geq 1$ such that

$$\frac{1}{R} \leq \frac{y_{i+1} - y_i}{y_{j+1} - y_j} \leq R,\ j = i \pm 1,\ \forall i.$$

We consider the sequence of partitions Δ_s obtained by using graded meshes (see for example [2]) of the form

$$y_i = \left(\frac{i}{s+1}\right)^r \cdot (b-a) + a,\ 0 \leq i \leq s+1,\ r \geq 1. \tag{2.2}$$

In [6] has been proved that the sequence $\{\Delta_s\}$ is $l.u.$. Let m be a given positive integer and $n = m + s$; we denote by Δ_s^e the extended partition of Δ_s defined as

$$\Delta_s^e := \{a = x_1 = ... = x_m < x_{m+1} < \ldots < x_{m+s} < x_{n+1} = ... = x_{n+m} = b\}$$

where $x_i = y_0$, $x_{n+i} = y_{s+1}$, $i = 1, ..., m$, $x_{m+j} = y_j$, $j = 1, ..., s$.

We denote by $I\!P_l$ the set of polynomials of degree $\leq l$. The space of polynomial splines of order m with simple knots $y_1, y_2, ..., y_s$ and $S_m(\Delta_s) \subset C^{m-2}[a, b]$ is defined by:

$$S_m(\Delta_s) := \left\{ \begin{array}{l} s : s(x) = s_k(x) \in I\!P_{m-1},\ x \in [y_k, y_{k+1}),\ k = 0, 1, ..., s; \\ D^j s_{k-1}(y_k) = D^j s_k(y_k),\ j = 0, 1, ..., m-2,\ k = 1, 2, ..., s. \end{array} \right\}. \tag{2.3}$$

The set of normalized B-splines of order m, B_{im} , $i = 1, 2, ..., n$, constitutes a basis for $S_m(\Delta_s)$ [10].

We define the following quasi-interpolatory and interpolatory operator applied to a function $f \in C[a, b]$ ([11], [7]):

$$T_n f := Q_n f + Uf - UQ_n f \tag{2.4}$$

where Q_n is the quasi-interpolating operator defined as

$$Q_n f(x) := \sum_{i=1}^{n} (\lambda_i f) B_{im}(x) = \sum_{i=1}^{n} [\sum_{j=1}^{m} v_{ij} f(\tau_{ij})] B_{im}(x) \qquad (2.5)$$

with $x \in [a, b]$,

$$v_{ij} = \sum_{r=j}^{m} \frac{\alpha_{ir}}{\prod\limits_{s=1, s\neq j}^{r} (\tau_{ij} - \tau_{is})}, \ r = 2, ..., m$$

and

$$\alpha_{ir} = \frac{(m-r)!}{(m-1)!} \sum \prod_{l=1}^{r-1} (x_{v_l} - \tau_{il})$$

where the sum is extended over all choices of distinct $v_1, ..., v_{r-1}$ from $i+1, ..., i+m-1$ and is set equal to 1 when $r = 1$; the τ_{ij} are m distinct points opportunely chosen in $[x_i, x_{i+m}]$, $i = 1, ..., n$ [4]. A possible distribution for $\{\tau_{ij}\}$ is the following

$$\tau_{ij} = x_{\rho_i} + j \frac{x_{\rho_i+1} - x_{\rho_i}}{k}, \ k = \begin{cases} m, & \text{if } \rho_i \neq n \\ m+1, & \text{if } \rho_i = n \end{cases}, \ i = 1, ..., n, \ j = 1, ..., m$$

with $[x_{\rho_i}, x_{\rho_i+1}] \subseteq [x_i, x_{i+m}]$, $m \leq \rho_i \leq n$.

The interpolating operator U is defined as:

$$U f(x) := \sum_{k=1}^{l} c_k(f) \overline{B}_{km}(x) = \sum_{k=1}^{l} [\sum_{h=1}^{l} \overline{b}_{kh}^{-1} f(t_h)] \overline{B}_{km}(x) \qquad (2.6)$$

where $\overline{B}_{km}(x)$, $k = 1, ..., l$ are normalized B-splines constituting a basis for the spline space $S_m(\Delta_{\overline{s}})$, $\overline{s} = l - m$; l is a fixed integer and t_k, $k = 1, ..., l$ are l distinct interpolation points with $t_k \in (\overline{x}_k, \overline{x}_{k+m})$ where $\overline{x}_k$, $\overline{x}_{k+m}$ belong to the extended partition $\Delta_{\overline{s}}^{e}$. The coefficients c_k have been obtained by imposing the interpolation conditions $U f(t_h) = f(t_h)$, $h = 1, ..., l$ and $\overline{b}_{kh}^{-1}$, h, $k = 1, ..., l$ denote the coefficients of the inverse matrix $\overline{B}_{\underline{t}}^{-1}$ of $\overline{B}(\underline{t}) = \begin{bmatrix} \overline{B}_{1m}(t_1) & ... & \overline{B}_{lm}(t_1) \\ \vdots & & \vdots \\ \overline{B}_{1m}(t_l) & ... & \overline{B}_{lm}(t_l) \end{bmatrix}$.

Remark 2.1. *We observe that the inverse matrix $\overline{B}_{\underline{t}}^{-1}$ exists because (theorem 4.63 in* [10]*), choosing the distinct interpolation points t_k in $(\overline{x}_k, \overline{x}_{k+m})$, $k = 1, ..., l$, we obtain $\overline{B}(\underline{t})$ not singular.*

Then we can write

$$UQ_nf(x) := \sum_{k=1}^{l}\sum_{h=1}^{l}\overline{b}_{kh}^{-1}Q_nf(t_h)\overline{B}_{km}(x). \tag{2.7}$$

By using (2.5), (2.6) and (2.7), the operator (2.4) can be written in the form:

$$\begin{aligned} T_nf(x) \quad &:= \sum_{i=1}^{n}\sum_{j=1}^{m} v_{ij}f(\tau_{ij})B_{im}(x) \\ &+ \sum_{k=1}^{l}\sum_{h=1}^{l}\overline{b}_{kh}^{-1}\left[f(t_h) - Q_nf(t_h)\right]\overline{B}_{km}(x). \end{aligned} \tag{2.8}$$

The operator (2.8) is quasi-interpolatory and interpolatory on the knots t_k, $k = 1, ..., l$ [7]. In fact: $T_nf(t_k) := Q_nf(t_k)+Uf(t_k)-UQ_nf(t_k) = Q_nf(t_k)+f(t_k)-Q_nf(t_k) = f(t_k)$ and $T_np(x) := Q_np(x)+Up(x)-UQ_np(x) = p(x)+Up(x)-Up(x) = p(x)$, where $p(x)$ is a polynomial of degree less or equal to $m-1$.

We observe that we can use a vectorial notation for the operator (2.8) that will be very useful in the following Sections. We set the following column vectors:

$\underline{t}= [t_1, ..., t_l]^T \in R^l$,

$\underline{\tau}= [\tau_{11}, ..., \tau_{1m}, ..., \tau_{n1}, ..., \tau_{nm}]^T \in R^{n\cdot m}$ and

$\underline{\xi}= [\underline{\tau}; \underline{t}] \in R^{n\cdot m+l}$.

We will use the notation: $f(\underline{z}) = [f(z_1), ..., f(z_r)]^T$, where $\underline{z}= [z_1, ..., z_r]^T$ is a column vector with the elements belonging to $[a, b]$ and we will indicate $B_{im}(x)$ with $B_i(x)$, $\overline{B}_{im}(x)$ with $\overline{B}_i(x)$, where $x \in [a, b]$. If we denote with:

$$B(x) = \begin{bmatrix} B_1(x), & ..., & B_n(x) \end{bmatrix},\ \overline{B}(x) = \begin{bmatrix} \overline{B}_1(x), & ..., & \overline{B}_l(x) \end{bmatrix}, \tag{2.9}$$

$$B(\underline{t}) = \begin{bmatrix} B_1(t_1) & ... & B_n(t_1) \\ \vdots & & \vdots \\ B_1(t_l) & ... & B_n(t_l) \end{bmatrix},\ \overline{B}(\underline{t}) = \begin{bmatrix} \overline{B}_1(t_1) & ... & \overline{B}_l(t_1) \\ \vdots & & \vdots \\ \overline{B}_1(t_l) & ... & \overline{B}_l(t_l) \end{bmatrix}, \tag{2.10}$$

$$V = \begin{bmatrix} v_{11} & \dots & v_{1m} & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & v_{21} & \dots & v_{2m} & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & v_{n1} & \dots & v_{nm} \end{bmatrix} \tag{2.11}$$

then we can write

$$Q_n f(x) = B(x) V f(\underline{\tau}), \tag{2.12}$$

$$U f(x) = \overline{B}(x) \overline{B}_{\underline{t}}^{-1} f(\underline{t}), \tag{2.13}$$

$$U Q_n f(x) = \overline{B}(x) \overline{B}_{\underline{t}}^{-1} B(\underline{t}) V f(\underline{\tau}) \tag{2.14}$$

and

$$T_n f(x) = M_n(x) f(\underline{\xi}) \tag{2.15}$$

where, considering (2.4), (2.12), (2.13) and (2.14) $M_n(x)$ is the row vector:

$$M_n(x) = \left[\ \left[B(x) - \overline{B}(x) \overline{B}_{\underline{t}}^{-1} B(\underline{t}) \right] V, \quad \overline{B}(x) \overline{B}_{\underline{t}}^{-1} \ \right] \in R^{n \cdot m + l}.$$

By (2.15), we can see that T_n is a linear operator. We recall that the norm of a bounded operator $F : C[0, X] \to C[0, X]$ can be defined as

$$\|F\| = \sup_{\|h\| \leq 1} \|F h\| .$$

The following proposition holds:

Proposition 2.1. *The operator T_n in (2.15) is a bounded operator for all n in $[a, b]$ and $\forall f \in C[a, b]$.*

Proof. T_n is a linear operator and so it suffices to prove that $\forall f \in C[a, b]$ and $\forall n$, exists a constant α such that

$$\|T_n f\|_\infty \leq \alpha \|f\|_\infty$$

where $\|g\|_\infty = \max_{x \in [a,b]} |g(x)|$, $g \in C[a, b]$.

By (2.4) we can write

$$|T_n f| \leq |Q_n f| + |U f| + |U Q_n f| ;$$

in [10] (Theorem 6.22) has been proved that $Q_n f$ is bounded; by (2.5), (2.6), (2.7) it easy also to get

$$|Q_n f(x)| \leq \|V\|_\infty \|f\|_\infty , \tag{2.16}$$

$$|U f(x)| \leq \left\| \overline{B}_{\underline{t}}^{-1} \right\|_\infty \|f\|_\infty , \tag{2.17}$$

$$|U Q_n f| \leq \|V\|_\infty \left\| \overline{B}_{\underline{t}}^{-1} \right\|_\infty \|f\|_\infty \tag{2.18}$$

with $\|V\|_\infty = \max_{1\le i\le n}\sum_{j=1}^{m}|v_{ij}| \le mD$ for all n ([4]) and D independent from n, $\left\|\overline{B}_{\underline{t}}^{-1}\right\|_\infty = \max_{1\le k\le n}\sum_{h=1}^{l}\left|\overline{b}_{kh}^{-1}\right|$ bounded because $\overline{B}_{\underline{t}}^{-1}$ is independent from n. The thesis follows, by setting $\alpha = \|V\|_\infty + \left\|\overline{B}_{\underline{t}}^{-1}\right\|_\infty + \|V\|_\infty\left\|\overline{B}_{\underline{t}}^{-1}\right\|_\infty$. □

We can give, now, the following

Theorem 2.1. *Let $f \in C[a,b]$. For the qi-i spline operator (2.15), the following relation holds*

$$\|f - T_n f\|_\infty \le C_1\,\|f - Q_n f\|_\infty \tag{2.19}$$

where $C_1 = 1 + \left\|\overline{B}_{\underline{t}}^{-1}\right\|_\infty$ and

$$\lim_{n\to\infty}\|f - T_n f\|_\infty = 0. \tag{2.20}$$

Proof. Considering that we can write $T_n f(x) = Q_n f(x) + U(f - Q_n f)(x)$ and (2.17) holds, the proof is similar to the proof of Theorem 4.1 in [7].

By Lemma 3.3 in [4], (2.20) follows. □

3. **An approximating integral operator**

Let $[a,b] \equiv [0,X]$ and K the following integral operator:

$$Kf(x) := \int_0^x k(x,s)f(s)ds,\ \ k \in C(0,X] \cap L_1[0,X]; \tag{3.1}$$

we consider the approximating operator KT_n

$$KT_n f(x) := \int_0^x k(x,s)T_n f(s)ds \tag{3.2}$$

that, by (2.4) and (2.8) we can write

$$\begin{aligned} KT_n f(x) &= KQ_n f(x) + KUf(x) - KUQ_n f(x) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} v_{ij} f(\tau_{ij}) KB_i(x) \\ &+ \sum_{k=1}^{l}\sum_{h=1}^{l} \overline{b}_{kh}^{-1}\left[f(t_h) - Q_n f(t_h)\right] K\overline{B}_k(x) \end{aligned} \tag{3.3}$$

where

$$KB_i(x) = \int_0^x k(x,s)B_i(s)ds,\ \ i = 1,...,n$$

and

$$K\overline{B}_k(x) = \int_0^x k(x,s)\overline{B}_k(s)ds,\ k = 1, ..., l.$$

By using the same vectorial notation used for the T_n operator, we can set:

$$\begin{aligned} KB(x) &= \left[\begin{array}{ccc} KB_1(x), & ..., & KB_n(x) \end{array} \right] = \\ &= \left[\begin{array}{ccc} \int_0^x k(x,s)B_1(s)ds, & ..., & \int_0^x k(x,s)B_n(s)ds \end{array} \right], \end{aligned} \tag{3.4}$$

$$\begin{aligned} K\overline{B}(x) &= \left[\begin{array}{ccc} K\overline{B}_1(x), & ..., & K\overline{B}_l(x) \end{array} \right] = \\ &= \left[\begin{array}{ccc} \int_0^x k(x,s)\overline{B}_1(s)ds, & ..., & \int_0^x k(x,s)\overline{B}_l(s)ds \end{array} \right]; \end{aligned} \tag{3.5}$$

then

$$\begin{cases} KQ_nf(x) & = KB(x)Vf(\underline{\tau}) \\ KUf(x) & = K\overline{B}(x)\overline{B}_{\underline{t}}^{-1}f(\underline{t}) \\ KUQ_nf(x) & = K\overline{B}(x)\overline{B}_{\underline{t}}^{-1}B(\underline{t})Vf(\underline{\tau}) \end{cases} \tag{3.6}$$

and

$$KT_nf(x) = A_n(x)f(\underline{\xi}) \tag{3.7}$$

where $A_n(x)$ is the row vector

$$A_n(x) = \left[\begin{array}{cc} [KB(x) - K\overline{B}(x)\overline{B}_{\underline{t}}^{-1}B(\underline{t})]V, & K\overline{B}(x)\overline{B}_{\underline{t}}^{-1} \end{array} \right] \in R^{n\cdot m+l}.$$

We observe that, by (3.7), the operator KT_n is a linear operator. Now we can define

$$\widetilde{k}(x,s) = \begin{cases} k(x,s) & \text{if } 0 \le s \le x, \\ 0 & \text{if } s > x; \end{cases} \tag{3.8}$$

if $\widetilde{k}(x,s)$ satisfies:

$$\begin{array}{l} 1)\ \widetilde{k}(x,s) \text{ is Riemann integrable in the variable } s \in [0,X], \\ 2)\ \lim\limits_{x'\to x} \int_0^X \left|\widetilde{k}(x',s) - \widetilde{k}(x,s)\right| ds = 0,\ x',\ x \in [0,X], \\ 3)\ \max\limits_{x\in[0,X]} \int_0^X \left|\widetilde{k}(x,s)\right| ds < \infty \end{array} \tag{3.9}$$

then we can say that $\int_0^X \left|\widetilde{k}(x,s)\right| ds = \int_0^x |k(x,s)|\, ds$, is a compact operator on $C[0,X]$.

Proposition 3.1. *Let KT_n be the operator (3.7) and the hypoteses (3.9) hold. Then KT_n is a bounded operator for all n, on $[0,X]$.*

Proof. KT_n is a linear operator and so it suffices to prove that $\forall f \in C[0,X]$ and $\forall n$, a constant β exists such that

$$\|KT_n f\|_\infty \le \beta \|f\|_\infty .$$

By (3.3), we can write

$$|KT_n f| \le |KQ_n f| + |KUf| + |KUQ_n f| .$$

$\max\limits_{x\in[0,X]} \int_0^x |k(x,s)|\, ds \le L$ because (3.9) holds; recalling that $\forall\ x \in [0,X]$ and $\forall n$ we have $\sum\limits_{i=1}^{n} B_i(x) = \sum\limits_{i=1}^{l} \overline{B}_i(x) = 1$, we obtain

$$\begin{aligned} |KQ_n f(x)| &= \left|\sum_{i=1}^{n}\sum_{j=1}^{m} v_{ij} f(\tau_{ij}) KB_i(x)\right| \le \|f\|_\infty \sum_{i=1}^{n} |KB_i(x)| \sum_{j=1}^{m} |v_{ij}| \\ &\le \|f\|_\infty \|V\|_\infty \int_0^x |k(x,s)| \sum_{i=1}^{n} |B_i(s)|\, ds \le L \|V\|_\infty \|f\|_\infty , \end{aligned}$$

$$\begin{aligned} |KUf(x)| &\le \left|\sum_{k=1}^{l}\sum_{h=1}^{l} \overline{b}_{kh}^{-1} f(t_h) K\overline{B}_k(x)\right| \le \|f\|_\infty \sum_{k=1}^{l} \left|K\overline{B}_k(x)\right| \sum_{h=1}^{l} \left|\overline{b}_{kh}^{-1}\right| \\ &\le \|f\|_\infty \left\|\overline{B}_{\underline{t}}^{-1}\right\|_\infty \int_0^x |k(x,s)| \sum_{k=1}^{l} \left|\overline{B}_k(s)\right| ds \le L \left\|\overline{B}_{\underline{t}}^{-1}\right\|_\infty \|f\|_\infty \end{aligned}$$

and

$$|KUQ_n f| \le L \|V\|_\infty \left\|\overline{B}_{\underline{t}}^{-1}\right\|_\infty \|f\|_\infty$$

with $\|V\|_\infty$ and $\left\|\overline{B}_{\underline{t}}^{-1}\right\|_\infty$ bounded (see proof of Proposition 2.1). The thesis follows, by setting $\beta = L\alpha = L(\|V\|_\infty + \left\|\overline{B}^{-1}(\underline{t})\right\|_\infty + \left\|\overline{B}^{-1}(\underline{t})\right\|_\infty \|V\|_\infty)$. $\square$

Theorem 3.1. *Let $f \in C[0,X]$ and $k(x,s)$ such that (3.9) holds. Then*

$$\|K(f - T_n f)\|_\infty \leq C_2 \|f - T_n f\|_\infty \,, \ \forall \ n \tag{3.10}$$

where $C_2 = \max\limits_{x\in[0,X]} \int_0^x |k(x,s)|\, ds$ and

$$\lim_{n\to\infty} \|K(f - T_n f)\|_\infty = 0. \tag{3.11}$$

Proof. $\|K(f - T_n f)\|_\infty = \max\limits_{x\in[0,X]} \left|\int_0^x k(x,s)(f(s) - T_n f(s))ds\right|$
$\leq \|f - T_n f\|_\infty \max\limits_{x\in[0,X]} \int_0^x |k(x,s)|\, ds$

and by (3.9) we have $\max\limits_{x\in[0,X]} \int_0^x |k(x,s)|\, ds < \infty$.

By Theorem 2.1 we have that $\|f - T_n f\|_\infty \to 0$ as $n \to \infty$ and by (3.10), (3.11) follows. $\square$

4. **A collocation method and convergence results**

Consider the equation (1.1); substituting there $T_n f$ for f in the integral, we obtain

$$f(x) = g(x) + \int_0^x k(x,s) T_n f(s) ds + r_n(x) \tag{4.1}$$

with

$$r_n(x) = \int_0^x k(x,s)\left(f(s) - T_n f(s)\right) ds = K(f(x) - T_n f(x)) \tag{4.2}$$

the residual term obtained approximating f by $T_n f$ in the integral. If we do not consider the error term, the (4.1) becomes

$$f_n(x) = g(x) + K T_n f_n(x), \tag{4.3}$$

or equivalently

$$\begin{aligned} f_n(x) \quad &= g(x) + \sum_{i=1}^{n}\sum_{j=1}^{m} v_{ij} f_n(\tau_{ij}) K B_i(x) \\ &+ \sum_{k=1}^{l}\sum_{h=1}^{l} \overline{b}_{kh}^{-1}\left[f_n(t_h) - Q_n f_n(t_h)\right] K \overline{B}_k(x) \end{aligned} \tag{4.4}$$

if we collocate the equation (4.3) in the vector $\underline{\xi}$ defined in Section 2, considering (3.7), we obtain the linear system

$$(Id - A_n(\underline{\xi}))f_n(\underline{\xi}) = g(\underline{\xi})$$

where Id is the identity matrix of order $nm + l$ and $A_n(\underline{\xi}) = [A_n(\xi_1), ..., A_n(\xi_{nm+l})]$ $\in R^{(n \cdot m+l)\times(n \cdot m+l)}$.

When we have solved the linear system just written, the value $f_n(x)$, $x \in [0, X]$, can be recovered by (4.4). Substracting (4.3) from (4.1) we obtain

$$f(x) - f_n(x) = KT_n(f(x) - f_n(x)) + r_n(x),$$

from which, considering also (4.2)

$$(I - KT_n)\,(f(x) - f_n(x)) = K(f(x) - T_n f(x)). \tag{4.5}$$

We can prove now, the following proposition:

Proposition 4.1. *Let $I - KT_n$ the operator in (4.5) and $k(x, s)$ such that (3.9) holds. For all n sufficiently large, $n \geq n_0$ with n_0 an integer > 0, the operator $(I - KT_n)^{-1}$ exists and*

$$\sup_{n \geq n_0} \left\| (I - KT_n)^{-1} \right\| \leq L < \infty.$$

Proof. By Proposition 3.1, KT_n is a bounded operator. We observe that the operators K in (3.1) and KT_n in (3.2) can be written as

$$\widetilde{K} f(x) := \int_0^X \widetilde{k}(x, s) f(s) ds,$$

$$\widetilde{K} T_n f(x) := \int_0^X \widetilde{k}(x, s) T_n f(s) ds$$

with $\widetilde{k}(x, s)$ defined in (3.8). Then $I - KT_n = I - \widetilde{K}T_n$. Following the proof of the Theorem 1. in [6] we can write

$$I - \widetilde{K}T_n = (I - \widetilde{K}) \left[I - (I - \widetilde{K})^{-1} (\widetilde{K}T_n - \widetilde{K}) \right].$$

Considering that T_n and $\widetilde{K}T_n$ are bounded operators and taking in account (3.11), the proof is similar to that one in [6] (theorem 1). $\square$

Theorem 4.1. *We consider the equation (1.1) . Let $f \in C[0, X]$ and $k(x, s)$ such that (3.9) holds. Then f_n in (4.3) exists and is unique $\forall\ n \geq n_0$ where n_0 is an integer > 0; moreover f_n converges uniformly to the solution f that is*

$$\lim_{n\to\infty} \|f_n - f\|_\infty = 0$$

and there results

$$\|f_n - f\|_\infty \leq C_3 \|f - T_n f\|_\infty$$

where $C_3 = C_2 \sup_{n\geq n_0} \left\|(I - KT_n)^{-1}\right\|$ and $C_2 = \max_{x\in[0,X]} \int_0^x |k(x, s)|\, ds$.

Proof. By using (4.5), Proposition 4.1 and Theorem 3.1 the thesis follows. □

5. **Numerical applications and comparisons**

We consider now some numerical results obtained by applying our collocation method to (1.1). The results have been compared with those obtained by a collocation method based on projector splines and with those proposed in [6].

We have considered the following equations of type (1.1):

$$\begin{cases} f(x) = & \sqrt{x} + \frac{1}{2}\pi x - \int_0^x (x - s)^{-\frac{1}{2}} f(s) ds,\ x \in [0, 1], \\ \\ f(x) = & \sqrt{x} \qquad \text{is the solution.} \end{cases} \tag{5.1}$$

and

$$\begin{cases} f(x) = & \frac{1}{\sqrt{1+x}} + \frac{\pi}{8} - \frac{1}{4} \arcsin \frac{1-x}{1+x} - \frac{1}{4} \int_0^x (x - s)^{-\frac{1}{2}} f(s) ds,\ x \in [0, 1], \\ \\ f(x) = & \frac{1}{\sqrt{1+x}} \qquad \text{is the solution.} \end{cases} \tag{5.2}$$

From Table 1 to Table 4 we have indicated with $\mathbf{e}_N$, $\mathbf{e}_P$ and $\mathbf{e}_{QI-I}$, the absolute error evaluated in $\mathbf{x}$, respectively obtained by the method in [6], by the collocation method that use the projector splines and by our method that in these examples takes the value $l = 10$ for the interpolatory spline. The methods use graded partitions of the form (2.2) with $r = 1$ and $r = 2$.

In Tables 1, 2 and 3 the values of x, of n and of m are chosen as in the examples in [6] in order to compare the results.

The results in Table 4 obtained with $r = 2$ for (5.2) and not shown in [6], confirm the theorical convergence results of our method.

Table 1

f(x)=$\sqrt{x}$, g(x)=$\sqrt{x}+\frac{1}{2}\pi$x, k=λ(x-s)$^{-1/2}$, λ=-1

r = 1, m = 3									
	n = 11			**n = 21**			**n = 41**		
x	$\mathbf{e}_N$	$\mathbf{e}_P$	$\mathbf{e}_{QI-I}$	$\mathbf{e}_N$	$\mathbf{e}_P$	$\mathbf{e}_{QI-I}$	$\mathbf{e}_N$	$\mathbf{e}_P$	$\mathbf{e}_{QI-I}$
0	-	0	0	-	0	0	-	0	0
0.01	5.8e-2	6.0e-3	5.2e-4	4.3e-2	2.2e-3	3.1e-4	2.4e-2	5.9e-4	2.6e-5
0.51	9.9e-4	1.6e-4	7.1e-5	3.3e-4	5.0e-5	6.3e-6	1.1e-4	1.7e-5	8.7e-6
1	4.2e-4	7.5e-5	3.5e-5	1.4e-4	2.3e-5	3.6e-6	4.9e-5	7.7e-6	3.7e-6

Table 2

f(x)=$\sqrt{x}$, g(x)=$\sqrt{x}+\frac{1}{2}\pi$x, k=λ(x-s)$^{-1/2}$, λ=-1

r = 2, m = 3									
	n = 11			**n = 21**			**n = 41**		
x	$\mathbf{e}_N$	$\mathbf{e}_P$	$\mathbf{e}_{QI-I}$	$\mathbf{e}_N$	$\mathbf{e}_P$	$\mathbf{e}_{QI-I}$	$\mathbf{e}_N$	$\mathbf{e}_P$	$\mathbf{e}_{QI-I}$
0	-	0	**0**	-	0	0	-	0	0
0.01	1.7e-3	7.1e-5	1.1e-4	2.8e-4	9.9e-5	1.4e-5	3.0e-5	2.3e-5	6.4e-6
0.51	2.3e-5	1.7e-4	7.7e-6	9.4e-6	2.1e-5	4.6e-7	2.2e-6	2.8e-6	1.8e-8
1	1.2e-4	6.0e-5	9.2e-6	1.2e-5	8.4e-6	6.1e-7	2.4e-6	1.2e-6	2.1e-7

Table 3

f(x)=$\frac{1}{\sqrt{1+x}}$, g(x)=$\frac{1}{\sqrt{1+x}}+\frac{\pi}{8}$-$\frac{1}{4}$ arcsin($\frac{1-x}{1+x}$),k=λ(x-s)$^{-1/2}$, λ=-$\frac{1}{4}$

r = 1, m = 4									
	n = 11			**n = 21**			**n = 41**		
x	$\mathbf{e}_N$	$\mathbf{e}_P$	$\mathbf{e}_{QI-I}$	$\mathbf{e}_N$	$\mathbf{e}_P$	$\mathbf{e}_{QI-I}$	$\mathbf{e}_N$	$\mathbf{e}_P$	$\mathbf{e}_{QI-I}$
0	-	0	0	-	0	0	-	0	0
0.1	1.0e-6	3.9e-7	6.4e-8	3.7e-8	1.5e-8	1.3e-9	1.5e-9	8.3e-10	9.7e-10
0.4	3.0e-7	4.3e-8	2.1e-9	1.4e-8	1.6e-8	5.1e-9	4.7e-10	1.2e-9	4.8e-10
1	2.1e-7	1.0e-7	1.1e-8	8.0e-9	9.2e-9	8.0e-10	9.5e-10	6.5e-10	1.4e-10

Table 4

f(x)=$\frac{1}{\sqrt{1+x}}$, g(x)=$\frac{1}{\sqrt{1+x}}+\frac{\pi}{8}$-$\frac{1}{4}$ arcsin($\frac{1-x}{1+x}$),k=λ(x-s)$^{-1/2}$, λ=-$\frac{1}{4}$

r = 2, m = 4						
	n = 11		**n = 21**		**n = 41**	
x	$\mathbf{e}_P$	$\mathbf{e}_{QI-I}$	$\mathbf{e}_P$	$\mathbf{e}_{QI-I}$	$\mathbf{e}_P$	$\mathbf{e}_{QI-I}$
0	0	0	0	0	0	0
0.01	1.9e-9	2.7e-10	2.7e-12	4.7e-12	9.8e-13	3.2e-13
0.1	1.0e-7	2.5e-8	4.0e-9	1.3e-9	1.7e-10	1.4e-11
0.4	9.5e-7	7.4e-8	3.8e-8	5.1e-9	1.8e-9	2.2e-10
0.51	1.4e-6	2.3e-7	5.0e-8	1.0e-8	2.3e-9	3.9e-10
1	1.3e-6	7.7e-8	5.8e-8	8.0e-10	3.2e-9	1.6e-10

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