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# A SURROGATE DUAL ALGORITHM FOR QUASICONVEX QUADRATIC PROBLEMS

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**Abstract**. The purpose of this paper is to solve, via a surrogate dual method, a quadratic program where the objectif function is not explicitly given. We apply our study to quasiconvex quadratic programs.

## 1. **Introduction**

In general a quadratic optimization problem can be formulated as:

$$\min\{Q(x) = \frac{1}{2}\, x^T Hx \;+\; c^T x : Ax \leq b, x \geq 0\} \tag{1}$$

where $H$ is a symmetric $n \times n$ matrix, $c \in \mathbb{R}^n$, $A$ is a $m \times n$ matrix and $b \in \mathbb{R}^m$. The computational cost for solving such a problem depends on the properties of the matrix $H$ and the dimensions $m$ and $n$. The convex quadratic problem (i.e. when $H$ is positive semidefinite) is often not more difficult to solve than a linear problem. The non convex case is more difficult, stationary points and local minimums which are not global minimums may exist [15]. In this paper, we are interested in the same quadratic programs (1) with only quasiconvex objective. Historically, the first criteria on the quasiconvex and pseudoconvex quadratic functions were given by Martos [11], Cottle and Ferland [1]. As we will see in the second section, these authors characterize this class of nonconvex quadratic functions with a finite number of conditions, contrary to the classical definitions. Furthermore, Ferland [6] and Schaible [12] independently obtained a characterization of quasiconvex and pseudoconvex quadratic functions on arbitrary solid convex sets. In mathematical programming, the pseudoconvexity of

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the objective is more wished than the quasiconvexity owing to the fact that the conditions of optimality of Karuch-Kuhn-Tucker (K-K-T) are necessary and sufficient to ensure the global minimum of the problem. But by weakening the pseudoconvexity assumptions to the quasiconvex case, these conditions become only necessary, and give only critical points.The third section is devoted to the surrogate duality [3], which is more adapted to quasiconvex programming than Lagrangian duality [8]. Indeed often we obtain an non empty duality gap. In the situation when the surrogate dual can be explicitly computing ( for example $Q$ is strictly convex), this gave rise to interesting numerical treatment[14]; but in the general case the objective function is expressed only in implicit form. Our aim is to give a surrogate dual method in this difficult situation. By taking as a starting point the paper of Dyer [5], we present in the fourth section, an algorithm based on the cutting planes method, well adapted to solve a problem of type (1) with quasiconvex objective function. An example is solved via this algorithm within a small number of iterations.

## 2. **Quasiconvex and pseudoconvex quadratic functions**

In this section, we present criteria in terms of eigenvalues and eigenvectors of the quasi-convex and pseudo-convex quadratic functions defined on a solid convex set, and especially on the positive orthant $\mathbb{R}^n_+$. We note by $intC$ the interior of the set $C$.

### 2.1. **Definitions.** We consider the quadratic function

$$Q(x) = \frac{1}{2}\, x^T H x \ + \ c^T x$$

$$H = (h_{ij})_{i,j=1,..,n} \ \ , \ H \ symmetric, \ \ c = (c_i)_{i=1,..,n}$$

and let $C \subset \mathbb{R}^n$ denote a solid convex set, i.e., $intC \neq \varnothing$.

**Definition 2.1.** The quadratic function $Q$ is said to be quasiconvex [2] on $C$ if,

$$\forall x, y \in C, \ \forall \lambda \in \left]0,1\right[, \quad Q\left((1-\lambda)\,x + \lambda y\right) \leq \max\left(Q\left(x\right), Q\left(y\right)\right). \tag{2}$$

Equivalently, this means that the lower-level sets

$$L_\alpha (Q) = \{x \in C : Q(x) \leq \alpha\}$$

are convex $\forall \alpha \in \mathbb{R}$ [2]. in the smooth case, which is the situation here ($Q$ is quadratic), definition 2.1 becomes

$$\forall x, y \in C, \quad Q(y) \leq Q(x) \quad \implies (y-x)^T \nabla Q(x) \leq 0. \tag{3}$$

**Definition 2.2.** $Q$ is pseudoconvex [10] if,

$$\forall x, y \in C, \quad (y-x)^T \nabla Q(x) \geq 0 \implies Q(y) \geq Q(x) \tag{4}$$

**Definition 2.3.** $Q$ is said to be strictly pseudoconvex if,

$$\forall x, y \in C, x \neq y, \quad (y-x)^T \nabla Q(x) \geq 0 \implies Q(y) > Q(x). \tag{5}$$

It is easy to show that strict pseudoconvexity implies pseudoconvexity, and pseudoconvexity implies quasiconvexity. On the other hand the opposite is not always true. A quasi-convex function (resp. pseudo-convex, strictly quasi-convex) which is not convex is called merely quasi-convex (resp. pseudo-convex, strictly quasi-convex).

2.2. **Finite criteria for a solid convex set.** Denote by $H^\dagger$ the Moore-Penrose pseudoinverse matrix of $H$, and denote by the triple $In(H) = (\mu_+(H), \mu_-(H), \mu_0(H))$ the inertia of the matrix $H$, where $\mu_+(H)$, $\mu_-(H)$ and $\mu_0(H)$ denote respectively the numbers of positive, negative and null eigenvalues of $H$. There exist a $n \times n$ diagonal matrix $D$ and $n \times n$ matrix $P$ such that $H = P^t H P$, $P^t P = I$ and let $(d_i)$ where $i = 1, ..., n$ the i-th diagonal entry of $D$. We denote by $U = \{y : \langle Dy, y\rangle \leq 0\}$ and by $T$ the set $T = P^t U$. It is known that the quadratic function is convex if and only if $\mu_-(H) = 0$. we look at the merely quasiconvex and pseudoconvex case. The caracterization of generalized convex quadratic functions in terms of spectral properties is given by the following theorem

**Theorem 2.1.**[4] *A nonconvex quadratic function*

$$Q(x) = \frac{1}{2}\, x^T H x \; + \; c^T x$$

*is quasiconvex (resp. pseudoconvex) on a solid convex* $C \subset \mathbb{R}^n$ *if and only if* *(i) $H$ has one and only one negative eigenvalue, i.e., $\mu_-(H) = 1$;*

*(ii) $c \in H(\mathbb{R}^n)$;*

*(iii) $C - H^\dagger c \subset T$ or $C - H^\dagger c \subset -T$ ( $C - H^\dagger c \subset intT$ or $C - H^\dagger c \subset -intT$).*

It is also seen in [4] that $T$ and $-T$ (resp. $intT$ and -$intT$) are the maximal domains of quasiconvexity (pseudoconvexity) of $Q$. the algorithm presented in section 4 is applied to the problem (1) with general quasiconvex objective $Q$ and constraint included in the maximal set of quasiconvexity. The fact that the constraints in Problem (1) below to the posif orthant more attention is given to the case $C = \mathbb{R}^n_+$.

2.3. **Finite criteria for nonnegative orthant.** We give below criteria for quasiconvex and pseudoconvex quadratic functions defined on $\mathbb{R}^n_+$ making the definitions (2), (3), (4) and (5) much more practical, this criteria can be derived by specializing the general result in theorem (2.1). We note that a quadratic function is quasiconvex on $\mathbb{R}^n$ if and only if it is convex on $\mathbb{R}^n$, and contrary to the convex functions the quasiconvex functions can be quasiconvex on a convex subset of $\mathbb{R}^n$ without being it on all the space $\mathbb{R}^n$.

**Theorem 2.2.** [11] and [1] *The quadratic function $Q$ is merely quasiconvex (esp. merely pseudoconvex)on $\mathbb{R}^n_+$ (on $int\mathbb{R}^n_+$) if and only if*

*(i) $H \leq 0$; i.e. $h_{ij} \leq 0$ $\forall i, j = 1, ..., n$.*

*(ii) $c \leq 0$; i.e. $c_i \leq 0$ $\forall i = 1, ...., n$.*

*(iii) $H$ has exactly one and only one eigenvalue, i.e.,$\mu_-(H) = 1$;*

*(iv) $c^T H^\dagger c \leq 0$.*

**Remark 2.1.** We note that the condition (iv) of theorem (2.1) imposes that the component $c_k$ of the vector $c$ is necessarily equal to 0 if the line $h_k$ of the matrix $H$ is null. Furthermore, if $H$ is nonsingular, then $Q$ is strictly pseudo-convex if and only if (i), (ii), (iii) and (iv) are checked, this last condition can be replaced by the condition $c^T H^{-1} c \leq 0$ . With true statement, if the quadratic function $Q$ is quasi-convex on $\mathbb{R}^n_+$, and if we suppose moreover that $c \neq 0$, then $Q$ is always pseudo-convex on $\mathbb{R}^n_+ - \{0\}$. This result can be found in [1].

**Exemple 2.1.** Consider the function

$$Q_1(x) = -\frac{1}{2}(x_1 + x_2)^2 - x_1 - x_2$$

where $H_1 = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$, et $c_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$. The two eigenvectors of $H_1$ are $\lambda_1 = -2$ et $\lambda_2 = 0$. $Q_1$is not convex ($H_1$is not positive semidefinite), we can remark that the vector $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ satisfies the condition (iv) of theorem (2.2), then $Q_1$ is merely quasiconvex on $\mathbb{R}^2_+$, and with remark (2.1) $Q$ is also merely pseudo-convex on $\mathbb{R}^2_+ - \{0\}$.

Pseudo-convexity is wished in mathematical programming, since the conditions of optimality of K-K-T become necessary and sufficient. This makes it possible to solve our problem with the various algorithms using the system of K-K-T (method of Lemke, methods of interior points...). Problems appear when the function $Q$ is merely quasiconvex, in such a situation the algorithm of the section 4 can be registered.

It is significant to also announce that the conditions (i) and (ii) of theorem (2.2) are not restrictive, because if we want to solve a problem of minimization with objective $Q$, (iii) and (iv) are checked but $(h_{ij} \geq 0 \forall i, j = 1, ..., n)$ and $(c_i \geq 0 \forall i = 1, ..., n)$ on a compact polyhedral. Thus we will have to solve the following problem:

$$\min \left\{ Q(x) : Ax - b = 0, x \in \mathbb{R}^n_+ \right\} = -\max \left\{ -Q(x) : Ax - b = 0, x \in \mathbb{R}^n_+ \right\}$$

then we have, a maximization problem of a quasiconvex function, where the solution is characterized by the following proposition:

**Proposition 2.2.** *Let $C$ be a polyhedral compact set of $\mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}$ a continuous quasiconvex function on $C$. Consider the problem to maximize $f$ on $C$. An optimal solution $\widetilde{x}$ to the problem then exists, where $\widetilde{x}$ is an extreme point of $C$.*

*Proof.* $f$ attains its maximum at $\widetilde{x} \in C$. Let $x_1, x_2, ...., x_k$ the extreme points of $C$, assumes that $f(\widetilde{x}) > f(x_j)$ for all $j = 1, ..., k$. By definition $\widetilde{x} = \sum_{j=1}^{k} \lambda_j x_j$ where $\sum_{j=1}^{k} \lambda_j = 1$ and $\lambda_j \geq 0$ for $j = 1, ..., k$.

Since $f(\widetilde{x}) > f(x_j)$ for every $j$, then

$$f(\widetilde{x}) > \max_{1\leq\ j\ \leq k} f(x_j) = \alpha$$

or $f$ is quasiconvex, then

$$f(\widetilde{x}) = f\left(\sum_{j=1}^{k} \lambda_j x_j\right) \leq \max_{1\leq\ j\leq\ k} f(x_j) = \alpha$$

hence the contradiction, so there exists necessarily $j_0 \in \{1, .., k\}$ such that $f(\widetilde{x}) = f(x_{j_0})$. □

## 3. Surrogate duality for the quasiconvex programming

Return now to our problem (1), for every $u$ belonging to a compact $X$ set in $\mathbb{R}_+^m$, we define:

$$X(u) = \left\{x \in X : u^\top (Ax - b) \leq 0\right\} \tag{6}$$

and the dual function

$$s(u) = \min\{Q(x) : x \in X(u)\}. \tag{7}$$

then the problem

$$(SP) \qquad s^* = \sup\left\{s(u) : u \in \mathbb{R}_+^m\right\} \tag{8}$$

is called the surrogate dual problem associated with the primal problem (1). it is clear that $s(tu) = s(u)\ \ \forall\ u \in X$ et $\forall t > 0$. This property simplifies the formulation of the problem (SP) which can be rewritten as:

$$(SP) \qquad s^* = \sup\left\{s(u) : u \in \mathbb{R}_+^m, \|u\|_1 = 1\right\}$$

where $\|\ \ \|_1$, is the norm 1 of $\mathbb{R}^m$, the problem (SP) becomes:

$$(SP) \qquad s^* = \sup\{s(u) : u \in \triangle\} \tag{9}$$

where $\triangle = \left\{u \in \mathbb{R}_+^m : \sum_{i=1}^{m} u_i = 1\right\}$ is the simplex of $\mathbb{R}_+^m$.

The following result is a deduction of two theorems. The first is due to Luenberger [9] and the second to Greenberg and Pierskalla [8].

**Proposition 3.1.** *The function s is continuous and quasiconcave (i.e. -s is quasiconvex) on the simplex* $\triangle$.

If we note by $v(P)$ the value of the primal problem, we check the weak duality easily ($s^* \leq v(P)$). Luemberger [9] has shown that if $v(P)$ is finite then, there exists $\widetilde{u} \in \triangle$ such that

$$v(P) = s^* = s(\widetilde{u}) = \max\{s(u) : u \in \triangle\}. \tag{10}$$

The fundamental reason for choosing the surrogate duality is that it produces a strong duality (the duality gap $v(P) - s^*$=0), this is due of course to the historical result of Luemberger. In addition, we can always associate the Lagrangian dual problem to (1)

$$(LDP) \qquad L^* = \sup\{\min\{Q(x) + \lambda^\top (Ax - b) : x \in \mathbb{R}^n_+\} : \lambda \in \mathbb{R}^m_+\}$$

It is important to notice that the objective function of our problem is not necessarily pseudo-convex, from where the possibility of having a non nulle Lagrangean duality gap ($v(P) - L^* \neq 0$). In addition, if we manage to calculate by a means or another a multiplier of Lagrange, this last can be a good point of initialization for the algorithm to present in the preceding section. In the article of Dyer [5] we find the proposition quoted below which makes in evidence what we have just said.

**Proposition 3.2.** *If $\overline{\lambda}$ is a Lagrange multiplier, and $\overline{\overline{\lambda}} = \frac{\overline{\lambda}}{\|\overline{\lambda}\|_1}$. then, we have always*

$$s\left(\overline{\overline{\lambda}}\right) \geq L^*$$

*moreover exactly one of the situations below holds:*

*(i) $s\left(\overline{\overline{\lambda}}\right) \geq L^*$.*

*(ii) $s\left(\overline{\overline{\lambda}}\right) = L^*$ but every neighbourhood of $\overline{\overline{\lambda}}$ in $\triangle$, contains a point $u$ such that $s(u) > L^*$.*

*(iii) $s\left(\overline{\overline{\lambda}}\right) = L^* = s^*$.*

For that follows we consider the set

$$G(\alpha) = A(L_\alpha(Q)) - b = \left\{g = Ax - b : Q(x) \leq \alpha, \forall x \in \mathbb{R}^n_+\right\}$$

and it's polar set

$$G^\oplus(\alpha) = \left\{u \in \triangle : g^\top u \geq 0, \forall g \in G(\alpha)\right\}$$
$$= \left\{u \in \triangle : (Ax - b)^\top u \geq 0, Q(x) \leq \alpha, \forall x \in \mathbb{R}^n_+\right\}$$

these two sets will be fundamental for the characterization of the solution $\overline{u}$ of $s(.)$.

**Proposition 3.3.** *$s^*$ is the minimum number $\alpha$ such that $intG^{\oplus}(\alpha) = \varnothing$.*

*Proof.* If $\alpha < s^*$, then there exists $u$ such that $s(u) > \alpha$. Which is equivalent to $X(u) \cap L_\alpha(Q) = \varnothing$, which is again true if and only if $g^T u > 0$ for all $g \in G(\alpha)$, but this is equivalent to say that $u \in intG^{\oplus}(\alpha)$, we conclude that if $\alpha < s^*$ then $intG^{\oplus}(\alpha) \neq \varnothing$. If now $\alpha \geq s^*$, we get necessarily for all $u \in intG^{\oplus}(\alpha)$, $s(u) > \alpha \geq s^*$, which is impossible, hence $intG^{\oplus}(\alpha) = \varnothing$. □

## 4. **An algorithm for a quasiconvex quadratic problem**

The method of resolution suggested here is a dual method, it is a question of finding the point $\widetilde{u}$ which solves the surrogate dual problem, and which will give the value of the primal problem thus $s^*$ and a solution $x(\widetilde{u})$, if it is feasible it is the optimal solution of the primal problem. When the quadratic function $Q$ is strictly convex (i.e., $H$ is positive definite), for the following problem

$$\min\{Q(x) = \frac{1}{2}\, x^\top Hx \ + \ c^\top x : Ax \leq b, x \in \mathbb{R}^n\}$$

we can calculate explicitly the dual function $s(.)$, which can be formulated as

$$s(u) = \frac{1}{2}\frac{\left(u^\top \left(AH^{-1}c + b\right)\right)^2}{u^\top AH^{-1}A^\top u} - \frac{1}{2}c^T H^{-1}c$$

see [14] for more detail. Unfortunately, it is not the case for problem(1) with $Q$ only quasiconvex.

The algorithm described below gives to each iteration $k$ the point $s_k$ the element of the sequence $(s_k)_k$ which will have to converge towards the optimal value $s^*$, each point $s_k$, is equal to $s(u_k)$ if

$$X\left(u^k\right) \cap L_{s_{k-1}}(Q) = \varnothing \tag{11}$$

else take the value $s_{k-1}$.

The formula (11) lead us to the resolution of the problem with a single constraint

$$(NLP)_k \qquad s(u_k) = \min\left\{Q(x) : (u^k)^\top (Ax - b) \leq 0, x \in \mathbb{R}^n_+\right\}. \tag{12}$$

The point $u_k \in intU_k$, this set will have the property to contain $intG^{\oplus}(u_k)$ at each iteration $k$, considering the proposition (3.3) the algorithm will stop at the first $k$ such that $intU_k = \varnothing$, and this is true if the radius $r_k$ of $U_k$ becomes negative.

The set $U_k = U_{k-1} \cap \left\{u \in \triangle : u^\top(Ax^k - b) \geq 0\right\}$, where $x_k$ is the optimal solution of $(NLP)_k$ , if this last admits a solution in this step of the iteration $k$, and in this case, like noted above, $s_{k-1}$ is increased with the value $s_k = s\left(u^k\right) = Q\left(x^k\right)$. Otherwise $x^k$ is any feasible solution of $(NLP)_k$. In each iteration $k$ we add a cutting plane, defined by the hyperplane $H^k = \left\{u \in \mathbb{R}^n : (u)^\top A\left(x^k - b\right) = 0\right\}$.

Let us note by $g^k$ the vector $Ax^k - b$. The Euclidean distance between the point $u$ of $U_k$ and its border is equal to $r_k(u) = \dfrac{u^\top g^k}{\gamma_k}$, where $\gamma_k = \sqrt{(g^k)^\top g^k - \frac{1}{m}(e^\top g^k)^2}$ and $e^\top = (1, ...., 1)$.

The radius of $U_k$ is given by

$$r_k^* = \max\left\{r_k(u) : u \in U_k\right\},$$

we can check that the $intU_k \neq \varnothing$ if and only if $r_k^* > 0$.

It is not difficult to see that

$$(LP)_k \qquad r_k^* = \max\left\{r : u^\top g^k - \gamma_k r \geq 0, , u \in \triangle\right\} \tag{13}$$

the problem $(LP)_k$ is linear, it is considered at each iteration $k$ and its resolution by a classical method such as the simplex method will give the solution $\left(\overline{u}^k, r_k^*\right)$. At each iteration $k$ the choice of $u^{k+1}$ depends on a parameter of convergence $\theta \in ]0,1]$ fixed at the beginning, the number $\alpha_k \in ]0,1]$ calculated at each iteration $k$, the point $\overline{u}^k$solution of the linear problem $(LP)_k$ and the point $u^k$ who should not belong to the the interior of $U_k$ in the iteration $k$. The point $u^{k+1}$ must be sufficiently distant from the boundary of $U_k$, then for any boundary point $u$, $u_{k+1} = \theta\overline{u}_k + (1-\theta)\,u \in intU_k$ but for $\overline{u}^k$ and $u^k$ it is easy to find a boundary point $u$ of $U_k$, let us choose it as

$$\alpha_k\overline{u}^k + (1-\alpha_k)\,u^k \tag{14}$$

where $\alpha \in [0,1[$ is given by

$$\alpha_k = \frac{-(u^k)^T g^k}{(\overline{u}^k) g^k - (u^k)^T g^k} \tag{15}$$

we replace (15) in (14), the point $u^{k+1}$ can be taken as

$$u^{k+1} = (1-\beta_k)\,\overline{u}^k + \beta_k u^k \quad whith \quad \beta_k = (1-\alpha_k)(1-\theta) \tag{16}$$

and will have the property to belong to the $intU_k$, and if the parameter $\theta$ is quite selected the continuity of the dual function $s(.)$ will give an accepted variation from the point $u^k$ to the point $u^{k+1}$ which will ensure a growth moderated towards the optimal value $s^*$. We give the steps of the algorithm at each iteration $k$ and the convergence result.

**The Algorithm**

■ step 0:

$k = 1, 0 < \theta \le 1$ let $\varepsilon > 0$ the tolerance, and a given $u_1$.

■ step 1:

Resolution of the nonlinear problem $(NLP)_k$

$$(NLP)_k : \quad s\left(u^k\right) = \min\left\{Q(x) \ : \left(u^k\right)^\top (Ax-b) \le 0, x \in IR^n_+\right\}$$

♦ If $(NLP)_k$ has a solution $x^k$ and if $s\left(u^k\right) \ge s_{k-1}$

$$s_k = s\left(u^k\right) = Q(x^k)$$

♦ else consider any feasible solution $x^k$ of $(NLP)_k$ and put

$$s_k = s_{k-1}$$

compute

$$g^k, \quad \gamma_k, \quad \beta_k$$

■ step 2:

Resolution of the linear problem $(PL)_k$

$$(PL)_k : \ r_k^* = \max\left\{r : \ \sum_{i=1}^n u_i g_i^l \ - \gamma_k r \ge 0, \ l = 1,...,k, \ \sum_{i=1}^m u_i = 1, \quad u \ge 0\right\}$$

♦ if $r_k^* < \varepsilon$!then stop.

♦ Else consider the solution $\left(r_k^*, \overline{u}^k\right)$ of $(PL)_k$, and compute the vector of the simplex $\triangle$

$$u^{k+1} = (1-\beta_k)\,\overline{u}^k + \beta_k u^k$$

■ step 3:

$k = k+1$ . Go to step 1.

The convergence result is presented in the following proposition

**Proposition 4.1.** *The sequence of points $(s_k)_k$ generated by the algorithm will become stationary and take the value $s^*$ from a certain rank, or $\lim_{k\to\infty} s_k = s^*$.*

*Proof.* By construction the sequence $(s_k)_k$ is nondecreasing and is majored by $s^*$ where $s_k \leq s^* \forall k$, thus either it becomes stationary starting from a certain rank, or it converges towards a limit. The nondecreasing of the sequence $(s_k)_k$ gives the following inclusion $\left\{x^l\right\} \subseteq L_{s_k}(Q)\ \ \forall l \leq k$, this leads us to say that $G^{\oplus}(s_k) \subseteq U_k\ \forall k$.

If $r_k \leq 0$ for a certain $k$ then, $intU_k = \varnothing$ and consequently $intG^{\oplus}(s_k) = \varnothing$. But the proposition (3.3) implies that $s_k \geq s^* \forall k$, thus necessarily $s_k = s^*$. Let us show now that if $s_k$ converges to a limit $\widetilde{s}$ then necessarily $\widetilde{s} = s^*$. To be done let us show initially that $r_k$ tends inevitably to 0. Let us suppose that $r_k > 0\ \forall k$. At the iteration $k$, $u^l \notin intU_k$ for all $l < k$, since for all $x^l$ we have $intU_k \subseteq \left\{u : u^T g^l > 0\right\}$ and $u_l^T g^k \leq 0$, and hence the Euclidean distance between $u_k$ and $u_l$ , $\|u_k - u_l\| \geq r_k^*$ . The sequence $\left(u^k\right)_k$ admits a value of adherence since it is contained in the simplex $\triangle$, the sequence $(r_k)_k$ is convergent towards a limit since it is nonincreasing and lowerbounded by 0, then $\forall \eta > 0\ \ \exists N \in \mathbb{N}$ such that for all $k > l \geq N$, we have $\|u_k - u_l\| < \eta$, from where $r_k < \eta$, this shows that $\lim_{k\to\infty} r_k = 0$.

Let us suppose now that $\lim_{k\to\infty} s_k = \widetilde{s} < s^*$, then $s_k \leq \widetilde{s}$ for any $k$, we deduce that $G^{\oplus}(\widetilde{s}) \subseteq U_k\ \forall k$, but from the proposition (3.3) we deduced that $intG^{\oplus}(\widetilde{s}) \neq \varnothing$, and hence this set contains a point $\widehat{u}$ of distance $\widehat{r} > 0$ from the boundary of $intG^{\oplus}(\widetilde{s})$, and there will be $r_k^* \geq \widehat{r}\ \forall k$, which gives that $\lim_{k\to\infty} r_k^* \geq \widehat{r} > 0$, this contradiction show that $\lim_{k\to\infty} s_k = s^*$. □

**Example.** The algorithm given above can be applied to general quasiconvex programming, but for illustration we consider the counterexample of Martos given in

[11], where some (not all ) primal convex quadratic algorithms fail to solve it.

$$\min\{Q_2(x) = 1/2\ x^T H_2 x : A_2(x) \leq b; x = (x_1, x_2) \geq 0\}$$

where

$$H_2 = \begin{bmatrix} -1 & -2 & -7 \\ -2 & 0 & 0 \\ -7 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 16 \\ 12 \end{bmatrix}$$

the optimal solution of this problem is $(5, 0, 6)$ , and $-222.5$ is the optimum value.

The convergence parameter $\theta$ is set equal to 0.25, for this example a relatively small value would not work better, we take for the starting point $u^1$ the center of the simplex $(1/2, 1/2)$, at each iteration the quantities $\alpha_k$, $\beta_k$ are as defined in (15) and (16). The implementation is proposed in the Matlab environment, at each iteration we use the two functions of Matlab *quadprog* and *linprog* for the problem $(NLP)_k$ and $(LP)_k$ respectively. The following table gives the evolution of the sequence $(s_k)_k$ for this example.

| iteration $k$ | $u_k$ | $s_k$ | $x^k$ | $g^k$ | $r_k$ |
|---|---|---|---|---|---|
| 1 | (1/2,1/2) | -256.11 | (7.84,0,4.12) | (3.79,3.79) | 0.71 |
| 2 | (0.62,0.37) | -232.34 | (6.29,0,4.82) | (1.41,-2.35) | 0.53 |
| 3 | (0.63,0.36) | -224.53 | (5.52,0,5.41) | (0.46,-1.18) | 0.19 |
| 4 | (0.65,0.34) | -222.55 | (5.08,0,5.89) | (0.06,-0.21) | 0.08 |
| 5 | (0.66,0.33) | -222.50 | (5.01,0,6.02) | (0.04,-5.98) | 0.00 |

after five iterations we get $s_4 \simeq -222.5$, the corresponding surrogate multiplier $u_4 = (0.66, 0.33)$ and the solution $x^4 \simeq (5, 0, 6)$.

**Conclusion.** The computing experiences that we have done for several examples with general quasiconvex programming, shows that if we get at hand a good subroutine to solve at each iteration the problem $(NLP)_k$ with a single constraint this algorithm converges to the optimal value, it is the case in non linear quadratic programming, which is explains our choice. The question of how we can compute a global minimum of a nonlinear program is always very difficult, but in this context we get at least a tool that lead's to the optimal value.

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