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A NEW SEQUENCE SPACE DEFINED BY A MODULUS

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Abstract. The idea of difference sequence spaces was introduced by Kızmaz [8] and this concept was generalized by Et and Çolak [6]. In this paper we define the space $\ell(\Delta^m, f, p, q, s)$ on a seminormed complex linear space by using modulus function and we give various properties and some inclusion relations on this space. Furthermore we study some of its properties, solidity, symmetricity etc.

1. **Introduction**

Let w denote the space of all sequences, and let ℓ_∞, c and c_0 be the linear spaces of bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by $\|x\|_\infty = \sup_k |x_k|$, where $k \in \mathbb{N} = \{1, 2, 3, ...\}$, the set of positive integers. Kızmaz [8] defined the sequence spaces

$$X(\Delta) = \{x = (x_k) : (\Delta x_k) \in X\}$$

for $X = \ell_\infty$, c and c_0, where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$.

The sequence spaces $\ell_\infty(\Delta^m)$, $c(\Delta^m)$, $c_0(\Delta^m)$ have been introduced by Et and Çolak [6]. These sequence spaces are BK spaces (Banach coordinate spaces) with norm

$$\|x\|_\Delta = \sum_{i=0}^{m} |x_i| + \|\Delta^m x\|_\infty ,$$

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where $m \in \mathbb{N}, \Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, $\Delta^m x = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$, and so

$$\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}.$$

The operators

$$\Delta^m, \sum\nolimits^m : w \to w$$

are defined by

$$\left(\Delta^1 x\right)_k = \Delta^1 x_k = x_k - x_{k+1}, \qquad \left(\sum\nolimits^1 x\right)_k = \sum_{j=1}^{k-1} x_j, \qquad (k = 0, 1, \ldots),$$

$$\Delta^m = \Delta^1 \circ \Delta^{m-1}, \qquad \sum\nolimits^m = \sum\nolimits^1 \circ \sum\nolimits^{m-1} \qquad (m \geq 2)$$

and

$$\sum\nolimits^m \circ \Delta^m = \Delta^m \circ \sum\nolimits^m = id$$

the identity on w (see [10]).

It is trivial that the generalized difference operator Δ^m is a linear operator. Recently, spectral properties of the difference operator were given by Malafosse [9], Altay and Başar [1].

Subsequently difference sequence spaces have been studied by various authors: (Çolak, Et and Malkowsky [4], Et [5], Mursaleen [8]).

The notion of a modulus function was introduced by Nakano [14] in 1953. We recall that a modulus f is a function from $[0, \infty)$ to $[0, \infty)$ such that

i) $f(x) = 0$ if and only if $x = 0$,

ii) $f(x + y) \leq f(x) + f(y)$, for all $x \geq 0$, $y \geq 0$,

iii) f is increasing,

iv) f is continuous from the right at 0.

It follows that f must be continuous on $[0, \infty)$. A modulus may be bounded or unbounded. For example, $f(x) = x^p$, $(0 < p \leq 1)$ is unbounded and $f(x) = \frac{x}{1+x}$ is bounded. Maddox [12] and Ruckle [16] used a modulus function to construct some sequence spaces.

After then some sequence spaces, defined by a modulus function, were introduced and studied by Bilgin [3], Pehlivan and Fisher [15], Waszak [17], Bhardwaj [2] and many others.

Proposition 1.1. Let f be a modulus and let $0 < \delta < 1$. Then for each $x \geq \delta$ we have $f\ (x) \leq 2f\,(1)\,\delta^{-1}x$, [15].

Let $p = (p_k)$ be a sequence of strictly positive real numbers. Let X be a seminormed space over the field $\mathbb{C}$ of complex numbers with the seminorm q. We introduce the following set of X - valued sequences

$$\ell(\Delta^m, f, p, q, s) = \left\{ x = (x_k) : x_k \in X,\ \sum_{k=1}^{\infty} k^{-s}[f(q(\Delta^m x_k))]^{p_k} < \infty\,,\ s \geq 0 \right\}$$

where f is a modulus. For different seminormed spaces X we get different sequence spaces $\ell(\Delta^m, f, p, q, s)$. Throughout the paper without writing X we use the notation $\ell(\Delta^m, f, p, q, s)$ for any but the same seminormed space X, unless otherwise indicated.

The following inequality will be used throughout this paper.

Let $p = (p_k)$ be a sequence of strictly positive real numbers with $0 < p_k \leq \sup_k p_k = H < \infty$. Then for $a_k, b_k \in \mathbb{C}$, we have

$$|a_k + b_k|^{p_k} \leq C\left\{|a_k|^{p_k} + |b_k|^{p_k}\right\}, \tag{1}$$

where $C = \max\,(1, 2^{H-1})$ (see for instance [11]).

The set $\ell(\Delta^m, f, p, q, s)$ is not a subset of ℓ_∞ for $m \geq 2$, in case $X = \mathbb{C}$ or $X = \mathbb{R}$, the set of real numbers. For this let $X = \mathbb{C}$, $s = 0$, $f\,(x) = x$, $q\,(x) = |x|$ and $p_k = 1$ for all $k \in \mathbb{N}$. If $x_k = k$ for all $k \in \mathbb{N}$, then $(x_k) \in \ell(\Delta^m, f, p, q, s)$ and $(x_k) \notin \ell_\infty$.

Definition 1.2. Let X be a sequence space. Then X is called

a) Solid (or normal) if $(\alpha_k x_k) \in X$ whenever $(x_k) \in X$ for all sequences (α_k) of scalars with $|\alpha_k| \leq 1$,

b) Symmetric if $(x_k) \in X$ implies $\left(x_{\pi(k)}\right) \in X$, where $\pi\,(k)$ is a permutation of $\mathbb{N}$ [7].

Definition 1.3. Let p, q be seminorms on a vector space X. Then p is said to be stronger than q if whenever (x_n) is a sequence such that $p(x_n) \to 0$, then also $q(x_n) \to 0$. If each one stronger then the other one, then p and q are said to be equivalent [18].

Lemma 1.4. Let p and q be seminorms on a linear space X. Then p is stronger than q if and only if there exists a constant $M \geq 0$ such that $q(x) \leq Mp(x)$ for all $x \in X$ [18].

2. Main results

In this section we will give some results on the sequence space $\ell(\Delta^m, f, p, q, s)$, those characterize the structure of the space $\ell(\Delta^m, f, p, q, s)$.

Theorem 2.1. The sequence space $\ell(\Delta^m, f, p, q, s)$ is a linear space over $\mathbb{C}$.

Proof. Let $x, y \in \ell(\Delta^m, f, p, q, s)$. For $\lambda, \mu \in \mathbb{C}$, there exist positive integers M_λ and N_μ such that $|\lambda| \leq M_\lambda$ and $|\mu| \leq N_\mu$. Since f is subadditive, q is a seminorm and Δ^m is linear

$$\begin{aligned}\sum_{k=1}^{\infty} k^{-s}[f(q(\Delta^m(\lambda x_k + \mu y_k)))]^{p_k} &\leq \sum_{k=1}^{\infty} k^{-s}[f(|\lambda|\, q(\Delta^m x_k)) + f(|\mu|\, q(\Delta^m y_k))]^{p_k} \\ &\leq C\,(M_\lambda)^H \sum_{k=1}^{\infty} k^{-s}[f(q(\Delta^m x_k)]^{p_k} + C\,(N_\mu)^H \sum_{k=1}^{\infty} k^{-s}[f(q(\Delta^m y_k)]^{p_k} < \infty.\end{aligned}$$

This proves that $\ell(\Delta^m, f, p, q, s)$ is a linear space.

Theorem 2.2. $\ell(\Delta^m, f, p, q, s)$ is a paranormed space (not totally paranormed), paranormed by

$$g_\Delta(x) = \left\{ \sum_{k=1}^{\infty} k^{-s}\,[f\,(q(\Delta^m x_k))]^{p_k} \right\}^{\frac{1}{M}}$$

where $H = \sup p_k < \infty$ and $M = \max(1, H)$.

Proof. Clearly $g_\Delta(\theta) = 0$ and $g_\Delta(x) = g_\Delta(-x)$, where $\theta = (\theta, \theta, \theta, ...)$ and is the zero of X.

It also follows from (1), Minkowski's inequality and the definition of f that g_Δ is subadditive. Now for a complex number λ, by inequality

$$|\lambda|^{p_k} \leq \max\left(1, |\lambda|^H\right)$$

and the definition of modulus f, we have

$$\begin{aligned} g_\Delta(\lambda x) &= \left(\sum_{k=1}^{\infty} k^{-s}\left[f\left(q\left(\lambda\Delta^m x_k\right)\right)\right]^{p_k}\right)^{\frac{1}{M}} \\ &\leq (1+[|\lambda|])^{\frac{H}{M}}.g_\Delta(x) \end{aligned}$$

where $[|\lambda|]$ denotes the integer part of λ, hence $\lambda \to 0$, $x \to \theta$ imply $\lambda x \to \theta$ and also $x \to \theta$, λ fixed imply $\lambda x \to \theta$.

Now suppose $\lambda_n \to 0$ and x is a fixed point in $\ell(\Delta^m, f, p, q, s)$. Given $\varepsilon > 0$, let K be such that

$$\sum_{k=K+1}^{\infty} k^{-s}\left[f\left(q\left(\Delta^m x_k\right)\right)\right]^{p_k} < \left(\frac{\varepsilon}{2}\right)^M.$$

Hence we have

$$\left(\sum_{k=K+1}^{\infty} k^{-s}\left[f\left(q(\Delta^m x_k)\right)\right]^{p_k}\right)^{\frac{1}{M}} < \frac{\varepsilon}{2}.$$

Since f is continuous on $[0, \infty)$

$$h(t) = \sum_{k=1}^{K} k^{-s}\left[f\left(q((\Delta^m(tx_k)))\right)\right]^{p_k}$$

is continuous at 0. Therefore, there exists $0 < \delta < 1$ such that $|\lambda_n| < \delta$ implies

$$\left(\sum_{k=1}^{K} k^{-s}\left[f\left(q(\lambda_n\Delta^m x_k)\right)\right]^{p_k}\right) < \frac{\varepsilon}{2}.$$

for $n > N$. Hence

$$\left(\sum_{k=1}^{\infty} k^{-s}\left[f\left(q(\lambda_n\Delta^m x_k)\right)\right]^{p_k}\right)^{\frac{1}{M}} < \varepsilon$$

for $n > N$. Therefore $g(\lambda x) \to 0$ as $\lambda \to 0$.

Theorem 2.3. Let f, f_1 and f_2 be modulus functions, q, q_1 and q_2 seminorms and $s, s_1, s_2 \geq 0$ real numbers.

i) If $s>1$, then $\ell\left(\Delta^m, f_1, p, q, s\right) \subseteq \ell\left(\Delta^m, f\circ f_1, p, q, s\right)$,

ii) $\ell\left(\Delta^m, f_1, p, q, s\right) \cap \ell\left(\Delta^m, f_2, p, q, s\right) \subseteq \ell\left(\Delta^m, f_1+f_2, p, q, s\right)$,

iii) $\ell\left(\Delta^m, f, p, q_1, s\right) \cap \ell\left(\Delta^m, f, p, q_2, s\right) \subseteq \ell\left(\Delta^m, f, p, q_1+q_2, s\right)$,

iv) If q_1 is stronger than q_2 then $\ell\left(\Delta^m, f, p, q_1, s\right) \subseteq \ell\left(\Delta^m, f, p, q_2, s\right)$,

v) If $s_1 \le s_2$, then $\ell\left(\Delta^m, f, p, q, s_1\right) \subseteq \ell\left(\Delta^m, f, p, q, s_2\right)$.

Proof. i) Let $x_k \in \ell\left(\Delta^m, f_1, p, q, s\right)$. Let $\varepsilon>0$ and choose δ with $0<\delta<1$ such that $f(t)<\varepsilon$ for $0\le t\le\delta$. Write $t_k = f_1(q(\Delta^m x_k))$ and consider

$$\sum_{k=1}^{\infty} k^{-s}[f(t_k)]^{p_k} = \sum_{1} k^{-s}[f(t_k)]^{p_k} + \sum_{2} k^{-s}[f(t_k)]^{p_k}$$

where the first summation is over $t_k \le \delta$ and the second over $t_k>\delta$. Since f is continuous, we have

$$\sum_{1} k^{-s}[f(t_k)]^{p_k} < \max\left(1, \varepsilon^H\right) \sum_{k=1}^{\infty} k^{-s} \tag{2}$$

and for $t_k>\delta$ we use the fact that

$$t_k < \frac{t_k}{\delta} < 1 + \left[\left|\frac{t_k}{\delta}\right|\right].$$

By the definition of f we have for $t_k>\delta$,

$$f(t_k) \le f(1)[1+\left(\frac{t_k}{\delta}\right)] \le 2f(1)\frac{t_k}{\delta}$$

$$\sum_{2} k^{-s}[f(t_k)]^{p_k} \le \max\left(1, \left(\frac{2f(1)}{\delta}\right)^H\right) \sum_{k=1}^{\infty} k^{-s}[t_k]^{p_k} < \infty. \tag{3}$$

By (2) and (3) we have $\ell\left(\Delta^m, f_1, p, q, s\right) \subseteq \ell\left(\Delta^m, f\circ f_1, p, q, s\right)$.

ii) Let $x = x_k \in \ell\left(\Delta^m, f_1, p, q, s\right) \cap \ell\left(\Delta^m, f_2, p, q, s\right)$. Then using (1) it can be shown that $x_k \in \ell\left(\Delta^m, f_1+f_2, p, q, s\right)$. Hence $\ell\left(\Delta^m, f_1, p, q, s\right) \cap \ell\left(\Delta^m, f_2, p, q, s\right) \subseteq \ell\left(\Delta^m, f_1+f_2, p, q, s\right)$.

iii) The proof of (iii) is similar to the proof of (ii) by using the inequality

$$k^{-s}\left[f(q_1+q_2)\left(\Delta^m x_k\right)\right]^{p_k} \le Ck^{-s}\left[f\left(q_1\left(\Delta^m x_k\right)\right)\right]^{p_k} + Ck^{-s}\left[f\left(q_2\left(\Delta^m x_k\right)\right)\right]^{p_k}$$

where $C = \max\left(1, 2^{H-1}\right)$.

(iv) and (v) follows easily.

We get the following sequence spaces from $\ell(\Delta^m, f, p, q, s)$ by choosing some of the special $p, f,$ and s :

For $f(x) = x$ we get

$$\ell(\Delta^m, p, q, s) = \left\{ x \in w(X) : \sum_{k=1}^{\infty} k^{-s}[q(\Delta^m x_k)]^{p_k} < \infty\, ,\ s \geq 0 \right\};$$

for $p_k = 1$, for all k, we get

$$\ell(\Delta^m, f, q, s) = \left\{ x \in w(X) : \sum_{k=1}^{\infty} k^{-s}[f\left(q(\Delta^m x_k)\right)] < \infty\, ,\ s \geq 0 \right\};$$

for $s = 0$ we get

$$\ell(\Delta^m, f, p, q) = \left\{ x \in w(X) : \sum_{k=1}^{\infty}[f\left(q(\Delta^m x_k)\right)]^{p_k} < \infty \right\};$$

for $f(x) = x$ and $s = 0$ we get

$$\ell(\Delta^m, p, q) = \left\{ x \in w(X) : \sum_{k=1}^{\infty}[q(\Delta^m x_k)]^{p_k} < \infty \right\};$$

for $p_k = 1$, for all k, and $s = 0$ we get

$$\ell(\Delta^m, f, q) = \left\{ x \in w(X) : \sum_{k=1}^{\infty} f\left(q(\Delta^m x_k)\right) < \infty \right\};$$

for $f(x) = x$, $p_k = 1$, for all k, and $s = 0$ we have

$$\ell(\Delta^m, q) = \left\{ x \in w(X) : \sum_{k=1}^{\infty} q(\Delta^m x_k) < \infty \right\}.$$

Corallary 2.4. i) If $s > 1$ then for any modulus f we have

$$\ell(\Delta^m, p, q, s) \subseteq \ell\left(\Delta^m, f, p, q, s\right),$$

ii) If q_1 and q_2 are equivalent seminorms then

$$\ell(\Delta^m, f, p, q_1, s) = \ell\left(\Delta^m, f, p, q_2, s\right),$$

iii) $\ell(\Delta^m, f, p, q) \subseteq \ell\left(\Delta^m, f, p, q, s\right)$,

iv) $\ell(\Delta^m, p, q) \subseteq \ell\left(\Delta^m, p, q, s\right)$,

v) $\ell(\Delta^m, f, q) \subseteq \ell\left(\Delta^m, f, q, s\right)$.

Proof. i) If $f_1(t) = t$ in Theorem 2.3 (i), then the result follows easily.

ii) It follows from Theorem 2.3 (iv).

iii) If we take $s_1 = 0$ and $s_2 = s$ in Theorem 2.3 (vi), then we get $\ell(\Delta^m, f, p, q) \subseteq \ell(\Delta^m, f, p, q, s)$.

iv) If we take $s_1 = 0$, $s_2 = s$, and $f(t) = t$ in Theorem 2.3 (vi), then we get $\ell(\Delta^m, p, q) \subseteq \ell(\Delta^m, p, q, s)$.

v) If we take $s_1 = 0$, $s_2 = s$, and $p_k = 1$ for all k, in Theorem 2.3 (vi) then $\ell(\Delta^m, f, q) \subseteq \ell(\Delta^m, f, q, s)$.

Theorem 2.5. $\ell(\Delta^{m-1}, f, q, s) \subset \ell(\Delta^m, f, q, s)$ for $m \geq 1$ and the inclusion is strict. In general $\ell(\Delta^i, f, q, s) \subset \ell(\Delta^m, f, q, s)$ for all $i = 1, 2, 3, ..., m-1$ and the inclusions are strict.

Proof. Let $x \in \ell(\Delta^{m-1}, f, q, s)$. Then we have

$$\sum_{k=1}^{\infty} k^{-s} f\left(q(\Delta^{m-1} x_k)\right) < \infty. \tag{4}$$

Since $(k+1)^{-s} < k^{-s} \leq 2^s (k+1)^{-s}$ for all $k \in \mathbb{N}$ we get the following inequality

$$k^{-s} f\left(q(\Delta^{m-1} x_{k+1})\right) \leq 2^s (k+1)^{-s} f\left(q(\Delta^{m-1} x_{k+1})\right). \tag{5}$$

(4) and (5) together imply that

$$\sum_{k=1}^{\infty} k^{-s} f\left(q(\Delta^{m-1} x_{k+1})\right) < \infty. \tag{6}$$

Since f is increasing, $f(x+y) \leq f(x) + f(y)$ and q is a seminorm, from (4) and (6) we get

$$\begin{aligned} \sum_{k=1}^{\infty} k^{-s} f\left(q(\Delta^m x_k)\right) &= \sum_{k=1}^{\infty} k^{-s} f\left(q(\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})\right) \\ &\leq \sum_{k=1}^{\infty} k^{-s} f\left(q(\Delta^{m-1} x_k)\right) + \sum_{k=1}^{\infty} k^{-s} f\left(q(\Delta^{m-1} x_{k+1})\right) \ ¡ \ \infty. \end{aligned}$$

Thus $\ell(\Delta^{m-1}, f, q, s) \subset \ell(\Delta^m, f, q, s)$.

In general $\ell(\Delta^i, f, q, s) \subset \ell(\Delta^m, f, q, s)$ for $i = 1, 2, 3, ..., m-1$ and the inclusions are strict. For this consider the following example.

Example 2.1. Let $X = \mathbb{C}$, $f(x) = x$, $q(x) = |x|$, $s = 0$. Consider the sequence $(x_k) = \left(k^{m-1}\right)$. Then $(x_k) \in \ell(\Delta^m, f, q, s)$ but $(x_k) \notin \ell(\Delta^{m-1}, f, q, s)$, since $\Delta^m x_k = 0$ and $\Delta^{m-1} x_k = (-1)^{m-1}(m-1)!$ for all $k \in \mathbb{N}$.

Theorem 2.6. The sequence space $\ell(\Delta^m, f, p, q, s)$ is not solid.

Proof. To show that the space is not solid in general, consider the following example.

Example 2.2. Let $X = \mathbb{C}$, $f(x) = x$, $q(x) = |x|$, $m = 2$, $s = 0$ and $p_k = 1$ for all $k \in \mathbb{N}$. Then $x = (x_k) = (k) \in \ell(\Delta^m, f, p, q, s)$ but $\alpha x = (\alpha_k x_k) \notin \ell(\Delta^m, f, p, q, s)$ for $\alpha_k = (-1)^k$ for all $k \in \mathbb{N}$. Hence $\ell(\Delta^m, f, p, q, s)$ is not solid.

Theorem 2.7. i) Let $0 < t_k \leq r_k < \infty$ for each $k \in \mathbb{N}$. Then

$$\ell(\Delta^m, f, t, q) \subseteq \ell(\Delta^m, f, r, q),$$

ii) $\ell(\Delta^m, f, q) \subseteq \ell(\Delta^m, f, q, s)$,

iii) $\ell(\Delta^m, f, t, q) \subseteq \ell(\Delta^m, f, t, q, s)$.

Proof. i) If $x \in \ell(\Delta^m, f, t, q)$ then, for all sufficiently large k,

$$[f(q(\Delta^m x_k))]^{t_k} \leq 1$$

and so

$$[f(q(\Delta^m x_k))]^{r_k} \leq [f(q(\Delta^m x_k))]^{t_k}.$$

This completes the proof.

The proof of *(ii)* and *(iii)* is trivial.

Theorem 2.8. i) If $0 < p_k \leq 1$ for each $k \in \mathbb{N}$, then $\ell(\Delta^m, f, p, q) \subseteq \ell(\Delta^m, f, q)$,

ii) If $p_k \geq 1$ for all $k \in \mathbb{N}$, then $\ell(\Delta^m, f, q) \subseteq \ell(\Delta^m, f, p, q)$.

Proof. i) If we take $p_k = t_k$ and $r_k = 1$ for all $k \in \mathbb{N}$, in Theorem 2.7 (i), then

$$\ell(\Delta^m, f, p, q) \subseteq \ell(\Delta^m, f, q).$$

ii) If we take $p_k = r_k$ and $t_k = 1$ for all $k \in \mathbb{N}$, in Theorem 2.7 (i), then

$$\ell(\Delta^m, f, q) \subseteq \ell(\Delta^m, f, p, q).$$

Theorem 2.9. The sequence space $\ell(\Delta^m, f, p, q, s)$ is not symmetric.

Proof. To show that the space is not symmetric, consider the following example.

Example 2.3. Let $X = \mathbb{C}$, $f(x) = x$, $q = |x|$, $s = 0$ and $p_k = 1$ for all $k \in \mathbb{N}$. Then the sequence $(x_k) = (k)$ belongs to $\ell(\Delta^m, f, p, q, s)$. Let (y_k) be a rearrangement of (x_k), which is defined as follows:

$$y_k = (x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, x_{49}, x_{10}, ...)$$

then the sequence (y_k) does not belong to $\ell(\Delta^m, f, p, q, s)$.

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