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ON THE LIPSCHITZ EXTENSION CONSTANT FOR A COMPLEX-VALUED LIPSCHITZ FUNCTION

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Abstract. In order to show that the Lipschitz constant for the extension of a complex-valued Lipschitz function cannot generally be 1, one can use the following example (see *Lipschitz Algebras*, by N. Weaver, World Scientific, Singapore, 1999, p. 18, Example 1.5.7): Let $X = \{e,\ p_1,\ p_2,\ p_3\}$ be a metric space such that $d(p_i, p_j) = 1$, for all distinct $i,\ j \in \{1,\ 2,\ 3\}$ and $d(e, p_i) = \frac{1}{2}$, for all $i \in \{1,\ 2,\ 3\}$ and let $X_0 = \{p_1,\ p_2,\ p_3\}$ be a subset of X. An isometric map of X_0 into $\mathbb{C}$ can be extended to X with an increase in the Lipschitz constant of at least $\frac{2}{\sqrt{3}}$, this constant being attained for the function that takes e to the circumcenter of the triangle formed by the points $f(p_i)$, for all $i \in \{1,\ 2,\ 3\}$. The purpose of this article is to show that we can loosen somewhat the conditions imposed on d, namely we show that considering a metric space $X = \{e,\ p_1,\ p_2,\ p_3\}$ such that $d(e, p_i) + d(e, p_j) = d(p_i, p_j)$, for all distinct $i,\ j \in \{1,\ 2,\ 3\}$, the above increase in the Lipschitz constant for the extended Lipschitz function is preserved.

1. **Introduction**

The problem of the extension of a Lipschitz function is a central one in the theory of Lipschitz functions. There are a lot of results in this direction (see for example [1-17]).

One of the main problems which is not completely answered in Lipschitz analysis is the following:

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Given two metric spaces X and Y and a subset X_0 of X under what conditions can we extend a Lipschitz function $f_0 : X_0 \to Y$ to a Lipschitz function $f : X \to Y$ with only a multiplicative loss in the Lipschitz constant and such that $f/X_0 = f_0$?

The problem is of particular interest especially when we take Y to be $\mathbb{R}$ or $\mathbb{C}$ (the so called scalar-valued Lipschitz functions). Under this hypothesis one fundamental theorem that we will now state works. (See [18] for more details.)

Theorem. If we take X to be a metric space and X_0 a subset of X then:

i) For any $f_0 : X_0 \to \mathbb{R}$ there exists $f : X \to \mathbb{R}$ such that $f/X_0 = f_0$ and $L(f) = L(f_0)$;

ii) For any $f_0 : X_0 \to \mathbb{C}$ there exists $f : X \to \mathbb{C}$ such that $f/X_0 = f_0$ and $L(f) \leq \sqrt{2} \cdot L(f_0)$, where by $L(f)$ we denoted the Lipschitz constant of f.

As one can see, in the real case one can extent the Lipschitz function to the whole space with no multiplicative loss in the Lipschitz constant whatsoever, whereas in the complex case we can only say that $L(f) \leq \sqrt{2} \cdot L(f_0)$. What is then the best constant that we can use for this inequality?

One step in this direction was taken by Kirszbraun when he proved the following

Theorem. If X is a subspace of $\mathbb{R}^n$ (for some $n \in \mathbb{N}^*$) equipped with the inherited Euclidean metric then the function $f_0 : X_0 \to \mathbb{C}$ can be extended to all of X without increasing its Lipschitz number. (See [7])

Could the constant we search for be 1? As we shall see in the following example (taken from [18, p. 18]) the answer is no. (The reason for which the constant is 1 in Kirszbraun's theorem has to do with the fact that in this particular case the space X is Euclidean.)

Example. Let $X = \{e, p_1, p_2, p_3\}$ be a four element set and let d be a distance on X such that $d(p_i, p_j) = 1$, for all distinct i, $j \in \{1, 2, 3\}$ and $d(e, p_i) = \frac{1}{2}$, for all $i \in \{1, 2, 3\}$. Now let $X_0 = \{p_1, p_2, p_3\}$ and $f_0 : X_0 \to \mathbb{C}$ be an isometric map. f_0 therefore takes the points p_1, p_2, p_3 to the vertices of an equilateral triangle. The Lipschitz extension of f_0 to the whole of X with the smallest Lipschitz constant will be the one taking e to the center of the triangle as one can easily see. In this case the constant we search for is $\frac{2}{\sqrt{3}}$.

2. **Main Result**

Could this constant still work for a more general setting? The answer is yes. As in the above example let $X = \{r,\ a,\ b,\ c\}$ be a four element set and let d be a distance on X such that $d(r,a) + d(r,b) = d(a,b)$ and the analogues. Given an isometric map f_0 taking a, b, c to the points A, B, C of the complex plane we claim that we can extend it to X such that $L(f) \leq \frac{2}{\sqrt{3}}$. Let us notice that by taking $d(a,b) = d(b,c) = d(c,a) = 1$ and $d(r,a) = d(r,b) = d(r,c) = \frac{1}{2}$ we get the example already mentioned.

Let us now restate the problem. For an extension f of f_0 to X to exist it is necessary and sufficient that there exists a value for $f(r)$ and a constant k such that $|f(r) - f(x)| \leq k \cdot d(r,x)$, for all $x \in X_0$. Or, from another point of view, it is necessary and sufficient that there exists a constant k such that by expanding the discs $D(f(a), d(r,a))$, $D(f(b), d(r,b))$, $D(f(c), d(r,c))$ by a factor of k there intersection will not be empty. From the main theorem we stated before it is clear that such a value of k exists and $k \leq \sqrt{2}$. It is also quite obvious that the smallest constant for these circles does exist and is attained when the circles have exactly one point in common. If one could prove that this happens for a constant of at most $\frac{2}{\sqrt{3}}$ then one would get that this is the smallest possible constant when we pass from X_0 to X.

Given that $AB = d(a,b) = d(r,a) + d(r,b)$ we can see that the circles $C(A, d(r,a))$ and $C(B, d(r,b))$ are tangent. The same goes for the other pairs. For briefness we will take $d(r,a) = r_A$, $d(r,b) = r_B$, $d(r,c) = r_C$. Let A', B' and C' be the points where the three tangent circles touch each other. Let also M be the point of intersection for the lines AA', BB', CC' and $B"$ be the intersection point of the segment BM with the circle $C(B, r_B)$. If one can prove for example that $\frac{\|BM\|}{\|BB"\|} \leq \frac{2}{\sqrt{3}}$ then one would get that M belongs to the disc $D(B, \frac{2}{\sqrt{3}} \cdot r_B)$. By doing the same for A and C one obtains that the point M belongs to all the three discs $D(A, \frac{2}{\sqrt{3}} \cdot r_A)$,

$D(B, \frac{2}{\sqrt{3}} \cdot r_B)$, $D(C, \frac{2}{\sqrt{3}} \cdot r_C)$ and therefore the minimum constant we search for will be less than or equal to $\frac{2}{\sqrt{3}}$.

Let us now prove that indeed $\frac{\|BM\|}{\|BB"\|} \leq \frac{2}{\sqrt{3}}$. Consider the origin of the plane to be B and let the Ox axis be the one containing C. One can easily see that $\overrightarrow{BA} = \left\|\overrightarrow{BA}\right\| \cos B \cdot \overrightarrow{i} + \left\|\overrightarrow{BA}\right\| \sin B \cdot \overrightarrow{j}$. (Here we denote by $\overrightarrow{i}$ and $\overrightarrow{j}$ the unity vectors of the axes Ox and Oy respectively.)

Given that $\frac{B'A}{B'C} = \frac{r_A}{r_C}$, we have

$$\overrightarrow{BB'} = \frac{r_C}{r_A + r_C} \cdot \overrightarrow{BA} + \frac{r_A}{r_A + r_C} \cdot \overrightarrow{BC} =$$

$$= (\frac{r_C(r_A + r_B)\cos B}{r_A + r_C} + \frac{r_A(r_B + r_C)}{r_A + r_C}) \cdot \overrightarrow{i} + \frac{r_C(r_A + r_B)\sin B}{r_A + r_C} \cdot \overrightarrow{j}.$$

By applying Menelaos' Theorem to the triangle CBB' and the line $A - M - A'$ we obtain the equality:

$$\frac{A'B}{A'C} \cdot \frac{AC}{AB'} \cdot \frac{MB'}{MB} = 1.$$

Expressing $\frac{BM}{B'M}$ from here we get

$$\frac{BM}{B'M} = \frac{r_B}{r_C} \cdot \frac{r_A + r_C}{r_A} = \frac{r_B(r_A + r_C)}{r_A r_C}.$$

Or equivalently

$$\frac{BM}{BB'} = \frac{r_B(r_A + r_C)}{\sum r_A r_B}.$$

Given the expression of $\overrightarrow{BB'}$ above we have

$$\overrightarrow{BM} = (\frac{r_B r_C(r_A + r_B)\cos B}{\sum r_A r_B} + \frac{r_A r_B(r_B + r_C)}{\sum r_A r_B}) \cdot \overrightarrow{i} + \frac{r_B r_C(r_A + r_B)\sin B}{\sum r_A r_B} \cdot \overrightarrow{j}.$$

From here

$$\frac{\|BM\|}{\|BB"\|} = \frac{\sqrt{\frac{[r_B r_C(r_A+r_B)\cos B + r_A r_B(r_B+r_C)]^2 + [r_B r_C(r_A+r_B)\sin B]^2}{[\sum r_A r_B]^2}}}{r_B},$$

or

$$\frac{\|BM\|}{\|BB"\|} = \frac{\sqrt{[r_B r_C(r_A + r_B)\cos B + r_A r_B(r_B + r_C)]^2 + [r_B r_C(r_A + r_B)\sin B]^2}}{r_B \sum r_A r_B}.$$

After simplifying both in the numerator and denominator by r_B we get

$$\frac{\|BM\|}{\|BB"\|} = \frac{\sqrt{[r_C(r_A+r_B)\cos B + r_A(r_B+r_C)]^2 + [r_C(r_A+r_B)\sin B]^2}}{\sum r_A r_B},$$

that is, we have to prove the inequality

$$\frac{\sqrt{{r_C}^2(r_A+r_B)^2 + {r_A}^2(r_B+r_C)^2 + 2r_Ar_C(r_A+r_B)(r_B+r_C)\cos B}}{\sum r_A r_B} \le \frac{2}{\sqrt{3}}.$$

From the cosine theorem in triangle ABC one gets that

$$2(r_A+r_B)(r_B+r_C)\cos B = (r_A+r_B)^2 + (r_B+r_C)^2 - (r_A+r_C)^2.$$

By replacing this in the above expression, the inequality becomes

$$\frac{\sqrt{{r_C}^2(r_A+r_B)^2 + {r_A}^2(r_B+r_C)^2 + r_Ar_C(r_A+r_B)^2 + r_Ar_C(r_B+r_C)^2 - r_Ar_C(r_A+r_C)^2}}{\sum r_A r_B} \le \frac{2}{\sqrt{3}}.$$

Squaring we get

$$\frac{(r_A+r_B)^2 r_C(r_A+r_C) + (r_B+r_C)^2 r_A(r_A+r_C) - r_Ar_C(r_A+r_C)^2}{(\sum r_Ar_B)^2} \le \frac{4}{3},$$

or

$$\frac{(r_A+r_C)({r_B}^2(r_A+r_C) + 4r_Ar_Br_C)}{(\sum r_Ar_B)^2} \le \frac{4}{3}.$$

Factoring out we get

$$3{r_B}^2(r_A+r_C)^2 + 12r_Ar_Br_C(r_A+r_C) \le 4{r_B}^2(r_A+r_C)^2 + 4{r_A}^2{r_C}^2 + 8r_Ar_Br_C(r_A+r_C),$$

or

$$4r_Ar_Br_C(r_A+r_C) \le {r_B}^2(r_A+r_C)^2 + 4{r_A}^2{r_C}^2,$$

which is nothing more than a trivial case of AM-GM inequality. Notice also that the equality is attained for

$$r_B = \frac{2r_Ar_C}{r_A+r_C}.$$

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