STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIII**, Number 1, March 2008

HOMOMORPHISMS BETWEEN JC^*-ALGEBRAS

CHUN-GIL PARK AND THEMISTOCLES M. RASSIAS

Abstract. It is shown that every almost linear mapping $h : \mathcal{A} \to \mathcal{B}$ of a JC^*-algebra $\mathcal{A}$ into a JC^*-algebra $\mathcal{B}$ is a homomorphism when $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all unitaries $u \in \mathcal{A}$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$, and that every almost linear continuous mapping $h : \mathcal{A} \to \mathcal{B}$ of a JC^*-algebra $\mathcal{A}$ of real rank zero to a JC^*-algebra $\mathcal{B}$ is a homomorphism when $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all $u \in \{v \in \mathcal{A} \mid v = v^*, ||v|| = 1, v \text{ is invertible}\}$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$. We moreover prove the Hyers-Ulam stability of homomorphisms in JC^*-algebras. This concept of stability of mappings was introduced for the first time by Th.M. Rassias in his paper [On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. **72** (1978), 297-300].

1. Introduction

Let X and Y be Banach spaces with norms $|| \cdot ||$ and $\| \cdot \|$, respectively. Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Th.M. Rassias [26] introduced the following inequality, that is known as *Cauchy-Rassias inequality* : Assume that there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in X$. Th.M. Rassias [26] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p} ||x||^p$$

Received by the editors: 08.08.2006.

2000 *Mathematics Subject Classification.* 47B48, 39B52, 46L05, 17Cxx.

Key words and phrases. Hyers-Ulam stability, homomorphisms between JC^*-algebras, real rank zero.

for all $x \in X$. This inequality has provided a lot of influence in the development of what is called *generalized Hyers-Ulam stability* of functional equations. Beginning around the year 1980 the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was studied by a number of mathematicians. Găvruta [7] generalized the Rassias' result in the following form: Let G be an abelian group and Y a Banach space. Denote by $\varphi : G \times G \to [0, \infty)$ a function such that

$$\widetilde{\varphi}(x, y) = \sum_{k=0}^{\infty} \frac{1}{2^k} \varphi(2^k x, 2^k y) < \infty$$

for all $x, y \in G$. Suppose that $f : G \to Y$ is a mapping satisfying

$$\|f(x+y) - f(x) - f(y)\| \le \varphi(x, y)$$

for all $x, y \in G$. Then there exists a unique additive mapping $T : G \to Y$ such that

$$\|f(x) - T(x)\| \le \frac{1}{2} \widetilde{\varphi}(x, x)$$

for all $x \in G$. C. Park [15] applied the Găvruta's result to linear functional equations in Banach modules over a C^*-algebra. Various functional equations have been investigated by several authors ([1], [3]-[6], [8]-[12], [16]-[25], [27]-[32]).

Throughout this paper, let $\mathcal{A}$ be a JC^*-algebra with norm $||\cdot||$ and unit e, and $\mathcal{B}$ a JC^*-algebra with norm $\| \cdot \|$ and unit e'. Let $\mathcal{U}(\mathcal{A}) = \{u \in \mathcal{A} \mid uu^* = u^*u = e\}$, $\mathcal{A}_{sa} = \{x \in \mathcal{A} \mid x = x^*\}$, and $I_1(\mathcal{A}_{sa}) = \{v \in \mathcal{A}_{sa} \mid ||v|| = 1, v \text{ is invertible}\}$.

Using the stability methods of linear mappings, we prove that every almost linear mapping $h : \mathcal{A} \to \mathcal{B}$ is a homomorphism when $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$, and that for a JC^*-algebra $\mathcal{A}$ of real rank zero (see [2]), every almost linear continuous mapping $h : \mathcal{A} \to \mathcal{B}$ is a homomorphism when $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all $u \in I_1(\mathcal{A}_{sa})$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$. We moreover prove the Hyers-Ulam stability of homomorphisms in JC^*-algebras.

2. Homomorphisms between JC^*-algebras

The original motivation to introduce the class of nonassociative algebras known as Jordan algebras came from quantum mechanics (see [33]). Let $\mathcal{H}$ be a complex Hilbert space, regarded as the "state space" of a quantum mechanical system. Let $\mathcal{L}(\mathcal{H})$ be the real vector space of all bounded self-adjoint linear operators on $\mathcal{H}$, interpreted as the (bounded) *observables* of the system. In 1932, Jordan observed that $\mathcal{L}(\mathcal{H})$ is a (nonassociative) algebra via the *anticommutator product* $x \circ y := \frac{xy+yx}{2}$. A commutative algebra X with product $x \circ y$ is called a *Jordan algebra* if $x^2 \circ (x \circ y) = x \circ (x^2 \circ y)$ holds.

A complex Jordan algebra $\mathcal{C}$ with product $x \circ y$ and involution $x \mapsto x^*$ is called a JB^*-algebra if $\mathcal{C}$ carries a Banach space norm $\| \cdot \|$ satisfying $\|x \circ y\| \le \|x\| \cdot \|y\|$ and $\|\{xx^*x\}\| = \|x\|^3$. Here $\{xy^*z\} := x \circ (y^* \circ z) - y^* \circ (z \circ x) + z \circ (x \circ y^*)$ denotes the *Jordan triple product* of $x, y, z \in \mathcal{C}$. A unital Jordan C^*-subalgebra of a C^*-algebra, endowed with the anticommutator product, is called a *JC^*-algebra* (see [23]-[25], [33]).

We investigate homomorphisms between JC^*-algebras.

Theorem 1. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ and $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$, for which there exists a function $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y) < \infty, \tag{2.1}$$

$$\|h(\mu x + \mu y) - \mu h(x) - \mu h(y)\| \le \varphi(x, y) \tag{2.2}$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid \ |\lambda| = 1\}$ and all $x, y \in \mathcal{A}$. Assume that

$$\lim_{n \to \infty} \frac{h(2^n e)}{2^n} = e'. \tag{2.3}$$

Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a homomorphism.

Proof. Put $\mu = 1 \in \mathbb{T}^1$. It follows from Găvruta Theorem [7] that there exists a unique additive mapping $H : \mathcal{A} \to \mathcal{B}$ such that

$$\|h(x) - H(x)\| \le \frac{1}{2} \widetilde{\varphi}(x, x) \tag{2.4}$$

for all $x \in \mathcal{A}$. The additive mapping $H : \mathcal{A} \to \mathcal{B}$ is given by

$$H(x) = \lim_{n \to \infty} \frac{1}{2^n} h(2^n x)$$

for all $x \in \mathcal{A}$.

By the assumption, for each $\mu \in \mathbb{T}^1$,

$$\|h(2^n \mu x) - 2\mu h(2^{n-1}x)\| \leq \varphi(2^{n-1}x, 2^{n-1}x)$$

for all $x \in \mathcal{A}$. One can show that

$$\|\mu h(2^n x) - 2\mu h(2^{n-1}x)\| \leq |\mu| \cdot \|h(2^n x) - 2h(2^{n-1}x)\| \leq \varphi(2^{n-1}x, 2^{n-1}x)$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. So

$$\begin{aligned}\|h(2^n \mu x) - \mu h(2^n x)\| \leq & \|h(2^n \mu x) - 2\mu h(2^{n-1}x)\| + \|2\mu h(2^{n-1}x) - \mu h(2^n x)\| \\ \leq & \varphi(2^{n-1}x, 2^{n-1}x) + \varphi(2^{n-1}x, 2^{n-1}x)\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. Thus $2^{-n}\|h(2^n \mu x) - \mu h(2^n x)\| \to 0$ as $n \to \infty$ for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. Hence

$$H(\mu x) = \lim_{n\to\infty} \frac{h(2^n \mu x)}{2^n} = \lim_{n\to\infty} \frac{\mu h(2^n x)}{2^n} = \mu H(x) \tag{2.5}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$.

Now let $\lambda \in \mathbb{C}$ ($\lambda \neq 0$) and M an integer greater than $4|\lambda|$. Then $|\frac{\lambda}{M}| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}$. By [13], Theorem 1, there exist three elements $\mu_1, \mu_2, \mu_3 \in \mathbb{T}^1$ such that $3\frac{\lambda}{M} = \mu_1 + \mu_2 + \mu_3$. So by (2.5)

$$\begin{aligned}H(\lambda x) &= H(\frac{M}{3} \cdot 3\frac{\lambda}{M}x) = M \cdot H(\frac{1}{3} \cdot 3\frac{\lambda}{M}x) = \frac{M}{3}H(3\frac{\lambda}{M}x) \\ &= \frac{M}{3}H(\mu_1 x + \mu_2 x + \mu_3 x) = \frac{M}{3}(H(\mu_1 x) + H(\mu_2 x) + H(\mu_3 x)) \\ &= \frac{M}{3}(\mu_1 + \mu_2 + \mu_3)H(x) = \frac{M}{3} \cdot 3\frac{\lambda}{M}H(x) \\ &= \lambda H(x)\end{aligned}$$

for all $x \in \mathcal{A}$. Hence

$$H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y)$$

for all $\zeta, \eta \in \mathbb{C}(\zeta, \eta \neq 0)$ and all $x, y \in \mathcal{A}$. We have that $H(0x) = 0 = 0H(x)$ for all $x \in \mathcal{A}$. So the unique additive mapping $H : \mathcal{A} \to \mathcal{B}$ is a $\mathbb{C}$-linear mapping.

Since $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$,

$$H(u \circ y) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n u \circ y) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n u) \circ h(y) = H(u) \circ h(y) \tag{2.6}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. By the additivity of H and (2.6),

$$2^n H(u \circ y) = H(2^n u \circ y) = H(u \circ (2^n y)) = H(u) \circ h(2^n y)$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Hence

$$H(u \circ y) = \frac{1}{2^n} H(u) \circ h(2^n y) = H(u) \circ \frac{1}{2^n} h(2^n y) \tag{2.7}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Taking the limit in (2.7) as $n \to \infty$, we obtain

$$H(u \circ y) = H(u) \circ H(y) \tag{2.8}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Since H is $\mathbb{C}$-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements (see [14], Theorem 4.1.7), i.e., $x = \sum_{j=1}^m \lambda_j u_j$ ($\lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(\mathcal{A})$), it follows from (2.8) that

$$\begin{aligned} H(x \circ y) &= H(\sum_{j=1}^m \lambda_j u_j \circ y) = \sum_{j=1}^m \lambda_j H(u_j \circ y) \\ &= \sum_{j=1}^m \lambda_j H(u_j) \circ H(y) = H(\sum_{j=1}^m \lambda_j u_j) \circ H(y) \\ &= H(x) \circ H(y) \end{aligned}$$

for all $x, y \in \mathcal{A}$.

By (2.3) and (2.6),

$$H(y) = H(e \circ y) = H(e) \circ h(y) = e' \circ h(y) = h(y)$$

for all $y \in \mathcal{A}$.

Therefore, the mapping $h : \mathcal{A} \to \mathcal{B}$ is a homomorphism, as desired. □

Corollary 2. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ and $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$, for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\|h(\mu x + \mu y) - \mu h(x) - \mu h(y)\| \leq \theta(||x||^p + ||y||^p)$$

for all $\mu \in \mathbb{T}^1$ *and all* $x, y \in \mathcal{A}$*. Assume that* $\lim_{n\to\infty} \frac{h(2^n e)}{2^n} = e'$*. Then the mapping* $h : \mathcal{A} \to \mathcal{B}$ *is a homomorphism.*

Proof. Define $\varphi(x, y) = \theta(||x||^p + ||y||^p)$ and apply Theorem 1. □

Theorem 3. *Let* $h : \mathcal{A} \to \mathcal{B}$ *be a mapping satisfying* $h(0) = 0$ *and* $h(2^n u \circ y) = h(2^n u) \circ h(y)$ *for all* $u \in \mathcal{U}(\mathcal{A})$*, all* $y \in \mathcal{A}$ *and all* $n \in \mathbb{Z}$*, for which there exists a function* $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ *satisfying* (2.1) *and* (2.3) *such that*

$$\|h(\mu x + \mu y) - \mu h(x) - \mu h(y)\| \leq \varphi(x, y) \tag{2.9}$$

for $\mu = 1, i$ *and all* $x, y \in \mathcal{A}$*. If* $h(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in \mathcal{A}$*, then the mapping* $h : \mathcal{A} \to \mathcal{B}$ *is a homomorphism.*

Proof. Put $\mu = 1$ in (2.9). By the same reasoning as in the proof of Theorem 1, there exists a unique additive mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying (2.4).

By the same reasoning as in the proof of [26], Theorem, the additive mapping $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{R}$-linear.

Put $\mu = i$ in (2.9). By the same method as in the proof of Theorem 1, one can obtain that

$$H(ix) = \lim_{n\to\infty} \frac{h(2^n ix)}{2^n} = \lim_{n\to\infty} \frac{ih(2^n x)}{2^n} = iH(x)$$

for all $x \in \mathcal{A}$.

For each element $\lambda \in \mathbb{C}$, $\lambda = s + it$, where $s, t \in \mathbb{R}$. So

$$\begin{aligned} H(\lambda x) &= H(sx + itx) = sH(x) + tH(ix) = sH(x) + itH(x) = (s + it)H(x) \\ &= \lambda H(x) \end{aligned}$$

for all $\lambda \in \mathbb{C}$ and all $x \in \mathcal{A}$. Thus

$$H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y)$$

for all $\zeta, \eta \in \mathbb{C}$, and all $x, y \in \mathcal{A}$. Hence the additive mapping $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear.

The rest of the proof is the same as in the proof of Theorem 1. □

From now on, assume that $\mathcal{A}$ is a JC^*-algebra of real rank zero, where "real rank zero" means that the set of invertible self-adjoint elements is dense in the set of self-adjoint elements (see [2]).

Now we investigate continuous homomorphisms between JC^*-algebras.

Theorem 4. *Let $h : \mathcal{A} \to \mathcal{B}$ be a continuous mapping satisfying $h(0) = 0$ and $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all $u \in I_1(\mathcal{A}_{sa})$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$, for which there exists a function $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ satisfying* (2.1)*,* (2.2) *and* (2.3)*. Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a homomorphism.*

Proof. By the same reasoning as in the proof of Theorem 1, there exists a unique $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying (2.4).

Since $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all $u \in I_1(\mathcal{A}_{sa})$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$,

$$H(u \circ y) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n u \circ y) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n u) \circ h(y) = H(u) \circ h(y) \tag{2.10}$$

for all $u \in I_1(\mathcal{A}_{sa})$ and all $y \in \mathcal{A}$. By the additivity of H and (2.10),

$$2^n H(u \circ y) = H(2^n u \circ y) = H(u \circ (2^n y)) = H(u) \circ h(2^n y)$$

for all $u \in I_1(\mathcal{A}_{sa})$ and all $y \in \mathcal{A}$. Hence

$$H(u \circ y) = \frac{1}{2^n} H(u) \circ h(2^n y) = H(u) \circ \frac{1}{2^n} h(2^n y) \tag{2.11}$$

for all $u \in I_1(\mathcal{A}_{sa})$ and all $y \in \mathcal{A}$. Taking the limit in (2.11) as $n \to \infty$, we obtain

$$H(u \circ y) = H(u) \circ H(y) \tag{2.12}$$

for all $u \in I_1(\mathcal{A}_{sa})$ and all $y \in \mathcal{A}$.

By (2.3) and (2.10),

$$H(y) = H(e \circ y) = H(e) \circ h(y) = e' \circ h(y) = h(y)$$

for all $y \in \mathcal{A}$. So $H : \mathcal{A} \to \mathcal{B}$ is continuous. But by the assumption that $\mathcal{A}$ has real rank zero, it is easy to show that $I_1(\mathcal{A}_{sa})$ is dense in $\{x \in A_{sa} \mid ||x|| = 1\}$. So for each $w \in \{z \in \mathcal{A}_{sa} \mid ||z|| = 1\}$, there is a sequence $\{\kappa_j\}$ such that $\kappa_j \to w$ as $j \to \infty$

and $\kappa_j \in I_1(\mathcal{A}_{sa})$. Since $H : \mathcal{A} \to \mathcal{B}$ is continuous, it follows from (2.12) that

$$\begin{aligned} H(w \circ y) &= H(\lim_{j\to\infty} \kappa_j \circ y) = \lim_{j\to\infty} H(\kappa_j \circ y) \\ &= \lim_{j\to\infty} H(\kappa_j) \circ H(y) = H(\lim_{j\to\infty} \kappa_j) \circ H(y) \\ &= H(w) \circ H(y) \end{aligned} \tag{2.13}$$

for all $w \in \{z \in \mathcal{A}_{sa} \mid ||z|| = 1\}$ and all $y \in \mathcal{A}$.

For each $x \in \mathcal{A}$, $x = \frac{x+x^*}{2} + i\frac{x-x^*}{2i}$, where $x_1 := \frac{x+x^*}{2}$ and $x_2 := \frac{x-x^*}{2i}$ are self-adjoint.

First, consider the case that $x_1 \neq 0, x_2 \neq 0$. Since $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear, it follows from (2.13) that

$$\begin{aligned} H(x \circ y) &= H(x_1 \circ y + ix_2 \circ y) = H(||x_1||\frac{x_1}{||x_1||} \circ y + i||x_2||\frac{x_2}{||x_2||} \circ y) \\ &= ||x_1||H(\frac{x_1}{||x_1||} \circ y) + i||x_2||H(\frac{x_2}{||x_2||} \circ y) \\ &= ||x_1||H(\frac{x_1}{||x_1||}) \circ H(y) + i||x_2||H(\frac{x_2}{||x_2||}) \circ H(y) \\ &= \{H(||x_1||\frac{x_1}{||x_1||}) + iH(||x_2||\frac{x_2}{||x_2||})\} \circ H(y) = H(x_1 + ix_2) \circ H(y) \\ &= H(x) \circ H(y) \end{aligned}$$

for all $y \in \mathcal{A}$.

Next, consider the case that $x_1 \neq 0, x_2 = 0$. Since $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear, it follows from (2.13) that

$$\begin{aligned} H(x \circ y) &= H(x_1 \circ y) = H(||x_1||\frac{x_1}{||x_1||} \circ y) = ||x_1||H(\frac{x_1}{||x_1||} \circ y) \\ &= ||x_1||H(\frac{x_1}{||x_1||}) \circ H(y) = H(||x_1||\frac{x_1}{||x_1||}) \circ H(y) = H(x_1) \circ H(y) \\ &= H(x) \circ H(y) \end{aligned}$$

for all $y \in \mathcal{A}$.

Finally, consider the case that $x_1 = 0, x_2 \neq 0$. Since $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear, it follows from (2.13) that

$$\begin{aligned} H(x \circ y) &= H(ix_2 \circ y) = H(i||x_2||\frac{x_2}{||x_2||} \circ y) = i||x_2||H(\frac{x_2}{||x_2||} \circ y) \\ &= i||x_2||H(\frac{x_2}{||x_2||}) \circ H(y) = H(i||x_2||\frac{x_2}{||x_2||}) \circ H(y) = H(ix_2) \circ H(y) \\ &= H(x) \circ H(y) \end{aligned}$$

for all $y \in \mathcal{A}$. Hence

$$H(x \circ y) = H(x) \circ H(y)$$

for all $x, y \in \mathcal{A}$.

Therefore, the mapping $h : \mathcal{A} \to \mathcal{B}$ is a homomorphism, as desired. □

Corollary 5. *Let $h : \mathcal{A} \to \mathcal{B}$ be a continuous mapping satisfying $h(0) = 0$ and $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all $u \in I_1(\mathcal{A}_{sa})$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$, for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\|h(\mu x + \mu y) - \mu h(x) - \mu h(y)\| \leq \theta(||x||^p + ||y||^p)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in \mathcal{A}$. Assume that $\lim_{n\to\infty} \frac{h(2^n e)}{2^n} = e'$. Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a homomorphism.

Proof. Define $\varphi(x, y) = \theta(||x||^p + ||y||^p)$ and apply Theorem 4. □

Theorem 6. *Let $h : \mathcal{A} \to \mathcal{B}$ be a continuous mapping satisfying $h(0) = 0$ and $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all $u \in I_1(\mathcal{A}_{sa})$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$, for which there exists a function $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ satisfying (2.1), (2.3) and (2.9). Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a homomorphism.*

Proof. By the same reasoning as in the proof of Theorem 3, there exists a unique $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying (2.4).

The rest of the proof is the same as in the proofs of Theorems 1 and 4. □

3. Stability of homomorphisms in JC^*-algebras

In this section, we prove the generalized Hyers-Ulam stability of homomorphisms in JC^*-algebras.

Theorem 7. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping with $h(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^4 \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y, z, w) := \sum_{j=0}^{\infty} 2^{-j} \varphi(2^j x, 2^j y, 2^j z, 2^j w) < \infty, \tag{3.1}$$

$$\|h(\mu x + \mu y + z \circ w) - \mu h(x) - \mu h(y) - h(z) \circ h(w)\| \le \varphi(x, y, z, w) \tag{3.2}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z, w \in \mathcal{A}$. Then there exists a unique homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that

$$\|h(x) - H(x)\| \le \frac{1}{2} \widetilde{\varphi}(x, x, 0, 0) \tag{3.3}$$

for all $x \in \mathcal{A}$.

Proof. Put $z = w = 0$ in (3.2). By the same reasoning as in the proof of Theorem 1, there exists a unique $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying (3.3). The $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ is given by

$$H(x) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n x) \tag{3.4}$$

for all $x \in \mathcal{A}$.

Let $x = y = 0$ in (3.2). Then we get

$$\|h(z \circ w) - h(z) \circ h(w)\| \le \varphi(0, 0, z, w)$$

for all $z, w \in \mathcal{A}$. Since

$$\frac{1}{2^{2n}} \varphi(0, 0, 2^n z, 2^n w) \le \frac{1}{2^n} \varphi(0, 0, 2^n z, 2^n w),$$

$$\begin{aligned} \frac{1}{2^{2n}} \|h(2^n z \circ 2^n w) - h(2^n z) \circ h(2^n w)\| &\le \frac{1}{2^{2n}} \varphi(0, 0, 2^n z, 2^n w) \\ &\le \frac{1}{2^n} \varphi(0, 0, 2^n z, 2^n w) \end{aligned}$$

for all $z, w \in \mathcal{A}$. By (3.1) and (3.5),

$$\begin{aligned} H(z \circ w) &= \lim_{n\to\infty} \frac{h(2^{2n} z \circ w)}{2^{2n}} = \lim_{n\to\infty} \frac{h(2^n z \circ 2^n w)}{2^n \cdot 2^n} \\ &= \lim_{n\to\infty} (\frac{h(2^n z)}{2^n} \circ \frac{h(2^n w)}{2^n}) = \lim_{n\to\infty} \frac{h(2^n z)}{2^n} \circ \lim_{n\to\infty} \frac{h(2^n w)}{2^n} \\ &= H(z) \circ H(w) \end{aligned}$$

for all $z, w \in \mathcal{A}$. Hence the $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ is a homomorphism satisfying (3.3), as desired. $\square$

Corollary 8. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping with $h(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\|h(\mu x + \mu y + z \circ w) - \mu h(x) - \mu h(y) - h(z) \circ h(w)\| \\ \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z, w \in \mathcal{A}$. Then there exists a unique homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that

$$\|h(x) - H(x)\| \leq \frac{2\theta}{2 - 2^p} ||x||^p$$

for all $x \in \mathcal{A}$.

Proof. Define $\varphi(x, y, z, w) = \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$ and apply Theorem 7. $\square$

Theorem 9. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping with $h(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^4 \to [0, \infty)$ satisfying* (3.1) *such that*

$$\|h(\mu x + \mu y + z \circ w) - \mu h(x) - \mu h(y) - h(z) \circ h(w)\| \leq \varphi(x, y, z, w)$$

for $\mu = 1, i$ and all $x, y, z, w \in \mathcal{A}$. If $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$, then there exists a unique homomorphism $H : \mathcal{A} \to \mathcal{B}$ satisfying (3.3).

Proof. By the same reasoning as in the proof of Theorem 3, there exists a unique $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying (3.3).

The rest of the proof is the same as in the proofs of Theorems 1 and 7. $\square$

References

[1] Baker, J.A., *A general functional equation and its stability*, Proc. Amer. Math. Soc., **133**(2005), 1657-1664.

[2] Brown, L., Pedersen, G., *C^*-algebras of real rank zero*, J. Funct. Anal., **99**(1991), 131-149.

[3] Czerwik, S., *Functional Equations and Inequalities in Several Variables*, World Scientific Publishing Company, New Jersey, London, Singapore and Hong Kong, 2002.

[4] Czerwik, S., *Stability of Functional Equations of Ulam-Hyers-Rassias Type*, Hadronic Press Inc., Palm Harbor, Florida, 2003.

[5] Daróczy, Z., Maksa, G., Páles, Z., *Functional equations involving means and their Gauss composition*, Proc. Amer. Math. Soc., **134**(2006), 521-530.

[6] Faizev, V.A., Rassias, Th.M., Sahoo, P.K., *The space of (ψ, γ)-additive mappings on semigroups*, Trans. Amer. Math. Soc., **354**(2002), 4455-4472.

[7] Găvruta, P., *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., **184**(1994), 431-436.

[8] Haruki, H., Rassias, Th.M., *A new functional equation of Pexider type related to the complex exponential function*, Trans. Amer. Math. Soc., **347**(1995), 3111-3119.

[9] Hyers, D.H., *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A., **27**(1941), 222-224.

[10] Hyers, D.H., Isac, G., Rassias, Th.M., *Stability of Functional Equations in Several Variables*, Birkhäuser, Boston, Basel and Berlin, 1998.

[11] Hyers, D.H., Rassias, Th.M., *Approximate homomorphisms*, Aequationes Math., **44**(1992), 125-153.

[12] Jung, S., *Hyers-Ulam-Rassias stability of Jensen's equation and its application*, Proc. Amer. Math. Soc., **126**(1998), 3137-3143.

[13] Kadison, R.V., Pedersen, G., *Means and convex combinations of unitary operators*, Math. Scand., **57**(1985), 249-266.

[14] Kadison, R.V., Ringrose, J.R., *Fundamentals of the Theory of Operator Algebras*, Elementary Theory, Academic Press, New York, 1983.

[15] Park, C., *On the stability of the linear mapping in Banach modules*, J. Math. Anal. Appl., **275**(2002), 711-720.

[16] Park, C., *Functional equations in Banach modules*, Indian J. Pure Appl. Math., **33**(2002), 1077-1086.

[17] Park, C., *Multi-quadratic mappings in Banach spaces*, Proc. Amer. Math. Soc., **131**(2003), 2501-2504.

[18] Park, C., *Modified Trif's functional equations in Banach modules over a C^*-algebra and approximate algebra homomorphisms*, J. Math. Anal. Appl., **278**(2003), 93-108.

[19] Park, C., *Linear functional equations in Banach modules over a C^*-algebra*, Acta Appl. Math., **77**(2003), 125-161.

[20] Park, C., *On an approximate automorphism on a C^*-algebra*, Proc. Amer. Math. Soc., **132**(2004), 1739-1745.

[21] Park, C., *Lie $*$-homomorphisms between Lie C^*-algebras and Lie $*$-derivations on Lie C^*-algebras*, J. Math. Anal. Appl., **293**(2004), 419-434.

[22] Park, C., *Universal Jensen's equations in Banach modules over a C^*-algebra and its unitary group*, Acta Math. Sinica, **20**(2004), 1047-1056.

[23] Park, C., *Homomorphisms between Poisson JC^*-algebras*, Bull. Braz. Math. Soc., **36**(2005), 79-97.

[24] Park, C., *Homomorphisms between Lie JC^*-algebras and Cauchy-Rassias stability of Lie JC^*-algebra derivations*, J. Lie Theory, **15**(2005), 393-414.

[25] Park, C., *Approximate homomorphisms on JB^*-triples*, J. Math. Anal. Appl., **306** (2005), 375-381.

[26] Rassias, Th.M., *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72**(1978), 297-300.

[27] Rassias, Th.M., *The problem of S.M. Ulam for approximately multiplicative mappings*, J. Math. Anal. Appl., **246**(2000), 352-378.

[28] Rassias, Th.M., *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl., **251**(2000), 264-284.

[29] Rassias, Th.M., *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math., **62**(2000), 23-130.

[30] Rassias, Th.M., Šemrl, P., *On the behavior of mappings which do not satisfy Hyers-Ulam stability*, Proc. Amer. Math. Soc., **114**(1992), 989-993.

[31] Sinopoulos, P., *Wilson's functional equation for vector and matrix functions*, Proc. Amer. Math. Soc., **125**(1997), 1089-1094.

[32] Ulam, S.M., *Problems in Modern Mathematics*, Wiley, New York, 1960.

[33] Upmeier, H., *Jordan Algebras in Analysis*, Operator Theory, and Quantum Mechanics, Regional Conference Series in Mathematics No. 67, Amer. Math. Soc., Providence, 1987.

Department of Mathematics, Chungnam National University,
Daejeon 305-764, South Korea
E-mail address: `cgparkcnu.ac.kr`

Department of Mathematics, National Technical University of Athens,
Zografou Campus, 15780 Athens, Greece
E-mail address: `trassiasmath.ntua.gr`