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SOME RESULTS ON TRANSFORMATIONS GROUPS OF N-LINEAR CONNECTIONS IN THE 2-TANGENT BUNDLE

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Abstract. In the present paper we study the transformations for the coefficients of an N-linear connection (definition 1.1) on the tangent bundle of order two, T^2M, by a transformation of a nonlinear connection in T^2M. We prove that the set $\mathcal{T}$ of these transformations together with composition of mappings isn't a group. But we give some groups of $\mathcal{T}$, which keep invariant a part of components of the local coefficients of an N-linear connection. We also determine the transformation laws of the torsion and curvature tensor fields, with respect to the transformations of the group $\mathcal{T}_N \subset \mathcal{T}$.

1. The N-and JN-linear connections on tangent bundle of order two

Let M be a real C^∞-manifold with n dimensions and (T^2M, π, M) its 2-tangent bundle, [1]. The local coordinates on 3n-dimensional manifold T^2M are denoted by $(x^i, y^{(1)i}, y^{(2)i}) = (x, y^{(1)}, y^{(2)}) = u$, $(i = 1, 2, ...n)$.

Let $\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^{(1)i}}, \frac{\partial}{\partial y^{(2)i}}\right)$ be the natural basis of the tangent space TT^2M at the point $u \in T^2M$ and let us consider the natural 2-tangent structure on T^2M, $J : \chi(T^2M) \to \chi(T^2M)$ given by

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^{(1)i}},\ J\left(\frac{\partial}{\partial y^{(1)i}}\right) = \frac{\partial}{\partial y^{(2)i}}, J\left(\frac{\partial}{\partial y^{(2)i}}\right) = 0. \tag{1.1}$$

We denote with N a nonlinear connection on T^2M with the local coefficients $(\underset{1}{N}{}^i{}_j, \underset{2}{N}{}^i{}_j)$ $(i, j = 1, 2, ..., n)$, [7], [8].

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Hence, the tangent space of T^2M in the point $u \in T^2M$ is given by the direct sum of the linear vector spaces:

$$T_nT^2M = N_0(u) \oplus N_1(u) \oplus V_2(u),\ \forall u \in T^2M. \tag{1.2}$$

An adapted basis to the direct decomposition (1.2) is given by

$$\left\{ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \frac{\delta}{\delta y^{(2)i}} \right\}, \tag{1.3}$$

where:

$$\begin{aligned} \tfrac{\delta}{\delta x^i} &= \tfrac{\partial}{\partial x^i} - \underset{1}{N}{}^j{}_i \tfrac{\partial}{\partial y^{(1)j}} - \underset{2}{N}{}^j{}_i \tfrac{\partial}{\partial y^{(2)j}}, \\ \tfrac{\delta}{\delta y^{(1)i}} &= \tfrac{\partial}{\partial y^{(1)i}} - \underset{1}{N}{}^j{}_i \tfrac{\partial}{\partial y^{(2)j}}, \\ \tfrac{\delta}{\delta y^{(2)i}} &= \tfrac{\partial}{\partial y^{(2)i}}. \end{aligned} \tag{1.4}$$

Let us consider the dual basis of (1.3):

$$\{dx^i, \delta y^{(1)i}, \delta y^{(2)i}\}, \tag{1.5}$$

where

$$\begin{aligned} \delta x^i &= dx^i, \\ \delta y^{(1)i} &= dy^{(1)i} + \underset{1}{N}{}^i{}_j dx^j, \\ \delta y^{(2)i} &= dy^{(2)i} + \underset{1}{N}{}^i{}_j dy^{(1)j} + (\underset{2}{N}{}^i{}_j + \underset{1}{N}{}^i{}_m \underset{1}{N}{}^m{}_j) dx^j. \end{aligned} \tag{1.6}$$

Definition 1.1. *([1]-[3]) A linear connection D on T^2M, $D : \chi(T^2M) \times \chi(T^2M) \to \chi(T^2M)$ is called an N-linear connection on T^2M if it preserves by parallelism the horizontal and vertical distributions N_0, N_1 and V_2 on T^2M.*

An N-linear connection D on T^2M is characterized by its coefficients in the adapted basis (1.3) in the form:

$$\begin{aligned} &D_{\frac{\delta}{\delta x^k}} \tfrac{\delta}{\delta x^j} = \underset{(00)}{L}{}^i{}_{jk} \tfrac{\delta}{\delta x^i},\ D_{\frac{\delta}{\delta x^k}} \tfrac{\delta}{\delta y^{(1)j}} = \underset{(10)}{L}{}^i{}_{jk} \tfrac{\delta}{\delta y^{(1)i}},\ D_{\frac{\delta}{\delta x^k}} \tfrac{\partial}{\partial y^{(2)j}} = \underset{(20)}{L}{}^i{}_{jk} \tfrac{\partial}{\partial y^{(2)i}}, \\ &D_{\frac{\delta}{\delta y^{(1)k}}} \tfrac{\delta}{\delta x^j} = \underset{(01)}{C}{}^i{}_{jk} \tfrac{\delta}{\delta x^i}, D_{\frac{\delta}{\delta y^{(1)k}}} \tfrac{\delta}{\delta y^{(1)j}} = \underset{(11)}{C}{}^i{}_{jk} \tfrac{\delta}{\delta y^{(1)i}}, D_{\frac{\delta}{\delta y^{(1)k}}} \tfrac{\partial}{\partial y^{(2)j}} = \underset{(21)}{C}{}^i{}_{jk} \tfrac{\partial}{\partial y^{(2)i}}, \\ &D_{\frac{\partial}{\partial y^{(2)k}}} \tfrac{\delta}{\delta x^j} = \underset{(02)}{C}{}^i{}_{jk} \tfrac{\delta}{\delta x^i}, D_{\frac{\partial}{\partial y^{(2)k}}} \tfrac{\delta}{\delta y^{(1)j}} = \underset{(12)}{C}{}^i{}_{jk} \tfrac{\delta}{\delta y^{(1)i}}, D_{\frac{\partial}{\partial y^{(2)k}}} \tfrac{\partial}{\partial y^{(2)j}} = \underset{(22)}{C}{}^i{}_{jk} \tfrac{\partial}{\partial y^{(2)i}}. \end{aligned} \tag{1.7}$$

The system of nine functions

$$D\Gamma(N) = (\underset{(00)}{L^i{}_{jk}}, \underset{(10)}{L^i{}_{jk}}, \underset{(20)}{L^i{}_{jk}}, \underset{(01)}{C^i{}_{jk}}, \underset{(11)}{C^i{}_{jk}}, \underset{(21)}{C^i{}_{jk}}, \underset{(02)}{C^i{}_{jk}}, \underset{(12)}{C^i{}_{jk}}, \underset{(22)}{C^i{}_{jk}}), \tag{1.8}$$

are called the **coefficients** of the N-linear connection D.

Generally, an N-linear connection $D\Gamma(N)$ on T^2M is not compatible with the natural 2-tangent structure J given by (1.1).

Definition 1.2. *An N-linear connection D on T^2M is called JN-linear connection if it is absolute parallel with respect to D:*

$$D_X J = 0, \ \ \forall X \in \chi(T^2M). \tag{1.9}$$

Theorem 1.1. (Gh. Atanasiu, [1]) *A JN-linear connection on T^2M is characterized by the coefficients $JD\Gamma(N)$ given by (1.8), where*

$$\begin{aligned} \underset{(00)}{L^i{}_{jk}} &= \underset{(10)}{L^i{}_{jk}} = \underset{(20)}{L^i{}_{jk}} \ (= L^i{}_{jk}), \\ \underset{(01)}{C^i{}_{jk}} &= \underset{(11)}{C^i{}_{jk}} = \underset{(21)}{C^i{}_{jk}} \ (= \underset{(1)}{C^i{}_{jk}}), \\ \underset{(02)}{C^i{}_{jk}} &= \underset{(12)}{C^i{}_{jk}} = \underset{(22)}{C^i{}_{jk}} \ (= \underset{(2)}{C^i{}_{jk}}). \end{aligned} \tag{1.10}$$

It results that a $JD\Gamma(N)$- linear connection on T^2M has three essentially coefficients:

$$JD\Gamma(N) = (L^i{}_{jk}, \underset{(1)}{C^i{}_{jk}}, \underset{(2)}{C^i{}_{jk}}). \tag{1.11}$$

Obvious, the geometrical theory on 2-tangent bundle (T^2M, π, M) with the N- linear connection [1]-[3], [15], generalize on that with the JN-linear connection (cf.with R. Miron and Gh. Atanasiu [5]-[8]; see, also M. Purcaru [12], [13]).

In the following we use the N-linear connections, only.

2. The set of transformations of N-linear connections

Let $D\Gamma(N)$ be an N-linear connection on T^2M with the coefficients given by (1.8). If $\bar{N}$ is another nonlinear connection on T^2M with the coefficients $(\underset{1}{\bar{N}^i}_j, \underset{2}{\bar{N}^i}_j)$, $(i, j = 1, 2, ..., n)$, then there exists the uniquely determined tensor fields $\underset{\beta}{A^i}_j, \in \tau^1_1(T^2M)$ such that:

$$\underset{\beta}{\bar{N}^i}_j = \underset{\beta}{N^i}_j - \underset{\beta}{A^i}_j, \quad (\beta = 1, 2). \tag{2.1}$$

Conversely, if $\underset{\beta}{N^i}_j$ and $\underset{\beta}{A^i}_j$, are fixed, then $\underset{\beta}{\bar{N}^i}_j$, $(\beta = 1, 2)$, given by (2.1) is a nonlinear connection.

Let us suppose that the mapping $N \to \bar{N}$ is given by (2.1) and we denote:

$$\frac{\bar{\delta}}{\delta x^i} = \frac{\partial}{\partial x^i} - \underset{1}{\bar{N}^j}_i \frac{\partial}{\partial y^{(1)j}} - \underset{2}{\bar{N}^j}_i \frac{\partial}{\partial y^{(2)j}}, \frac{\bar{\delta}}{\delta y^{(1)i}} = \frac{\partial}{\partial y^{(1)i}} - \underset{1}{\bar{N}^j}_i \frac{\partial}{\partial y^{(2)j}}, \frac{\bar{\delta}}{\delta y^{(2)i}} = \frac{\partial}{\partial y^{(2)i}}.$$

It follows first of all that the transformations (2.1) preserve the coefficients $\underset{(\alpha 2)}{C^i}_{jk}$, $(\alpha = 0, 1, 2)$.

Taking in account the fact that:

$$\frac{\bar{\delta}}{\delta x^i} = \frac{\delta}{\delta x^i} + \underset{1}{A^j}_i \frac{\partial}{\partial y^{(1)j}} + \underset{2}{A^j}_i \frac{\partial}{\partial y^{(2)j}}, \quad \frac{\bar{\delta}}{\delta y^{(1)i}} = \frac{\delta}{\delta y^{(1)i}} + \underset{1}{A^j}_i \frac{\partial}{\partial y^{(2)j}}, \frac{\bar{\delta}}{\delta y^{(2)i}} = \frac{\delta}{\delta y^{(2)i}},$$

it follows:

$$D_{\frac{\bar{\delta}}{\delta x^k}} \frac{\bar{\delta}}{\delta y^{(2)j}} = D_{\frac{\bar{\delta}}{\delta x^k}} \frac{\partial}{\partial y^{(2)j}} = \underset{(20)}{\bar{L}^i}_{jk} \frac{\partial}{\partial y^{(2)i}} = D_{(\frac{\delta}{\delta x^k} + \underset{1}{A^l}_k \frac{\partial}{\partial y^{(1)l}} + \underset{2}{A^l}_k \frac{\partial}{\partial y^{(2)l}})} \frac{\partial}{\partial y^{(2)j}} =$$

$$= D_{\frac{\delta}{\delta x^k}} \frac{\partial}{\partial y^{(2)j}} + \underset{1}{A^l}_k D_{(\frac{\delta}{\delta y^{(1)l}} + \underset{1}{N^m}_l \frac{\partial}{\partial y^{(2)m}})} \frac{\partial}{\partial y^{(2)j}} + \underset{2}{A^l}_k D_{\frac{\partial}{\partial y^{(2)l}}} \frac{\partial}{\partial y^{(2)j}} =$$

$$= \underset{(20)}{L^i}_{jk} \frac{\partial}{\partial y^{(2)i}} + \underset{1}{A^l}_k \underset{(21)}{C^i}_{jl} \frac{\partial}{\partial y^{(2)i}} + \underset{1}{A^l}_k \underset{1}{N^m}_l \underset{(22)}{C^i}_{jm} \frac{\partial}{\partial y^{(2)i}} + \underset{2}{A^l}_k \underset{(22)}{C^i}_{jl} \frac{\partial}{\partial y^{(2)i}} =$$

$$= (\underset{(20)}{L^i}_{jk} + \underset{1}{A^l}_k \underset{(21)}{C^i}_{jl} + \underset{1}{A^l}_k \underset{1}{N^m}_l \underset{(22)}{C^i}_{jm} + \underset{2}{A^l}_k \underset{(22)}{C^i}_{jl}) \frac{\partial}{\partial y^{(2)i}}.$$

$$D_{\frac{\bar{\delta}}{\delta y^{(1)k}}} \frac{\bar{\delta}}{\delta y^{(2)j}} = D_{\frac{\bar{\delta}}{\delta y^{(1)k}}} \frac{\partial}{\partial y^{(2)j}} = \underset{(21)}{\bar{C}^i}_{jk} \frac{\partial}{\partial y^{(2)i}} = D_{(\frac{\delta}{\delta y^{(1)k}} + \underset{1}{A^l}_k \frac{\partial}{\partial y^{(2)l}})} \frac{\partial}{\partial y^{(2)j}} =$$

$$= D_{\frac{\delta}{\delta y^{(1)k}}} \frac{\partial}{\partial y^{(2)j}} + \underset{1}{A^l}_k D_{\frac{\partial}{\partial y^{(2)l}}} \frac{\partial}{\partial y^{(2)j}} = \underset{(21)}{C^i}_{jk} \frac{\partial}{\partial y^{(2)i}} + \underset{1}{A^l}_k \underset{(22)}{C^i}_{jl} \frac{\partial}{\partial y^{(2)i}} =$$

$$= (\underset{(21)}{C^i}_{jk} + \underset{1}{A^l}_k \underset{(22)}{C^i}_{jl}) \frac{\partial}{\partial y^{(2)i}}.$$

Therefore the change we are looking for is:

$$\begin{cases} \underset{(\alpha 0)}{\bar{L}^i}_{jk} = \underset{(\alpha 0)}{L^i}_{jk} + \underset{1}{A^l}_k \underset{(\alpha 1)}{C^i}_{jl} + \underset{1}{A^l}_k \underset{1}{N^m}_l \underset{(\alpha 2)}{C^i}_{jm} + \underset{2}{A^l}_k \underset{(\alpha 2)}{C^i}_{jl} , \\ \underset{(\alpha 1)}{\bar{C}^i}_{jk} = \underset{(\alpha 1)}{C^i}_{jk} + \underset{1}{A^l}_k \underset{(\alpha 2)}{C^i}_{jl} , \\ \underset{(\alpha 2)}{\bar{C}^i}_{jk} = \underset{(\alpha 2)}{C^i}_{jk} , (\alpha = 0, 1, 2). \end{cases} \tag{2.2}$$

So, we have proved:

Proposition 2.1. *The transformation (2.1) of nonlinear connections imply the transformations (2.2) for the coefficients $D\Gamma(N)$ of the N-linear connection D.*

Theorem 2.1. *Let N and $\bar{N}$ be two nonlinear connections, with the coefficients $(\underset{1}{N^i}_j, \underset{2}{N^i}_j), (\underset{1}{\bar{N}^i}_j, \underset{2}{\bar{N}^i}_j)$-respectively. If*

$$D\Gamma(N) = (\underset{(00)}{L^i}_{jk}, \underset{(10)}{L^i}_{jk}, \underset{(20)}{L^i}_{jk}, \underset{(01)}{C^i}_{jk}, \underset{(11)}{C^i}_{jk}, \underset{(21)}{C^i}_{jk}, \underset{(02)}{C^i}_{jk}, \underset{(12)}{C^i}_{jk}, \underset{(22)}{C^i}_{jk})$$

and

$$D\bar{\Gamma}(\bar{N}) = (\underset{(00)}{\bar{L}^i}_{jk}, \underset{(10)}{\bar{L}^i}_{jk}, \underset{(20)}{\bar{L}^i}_{jk}, \underset{(01)}{\bar{C}^i}_{jk}, \underset{(11)}{\bar{C}^i}_{jk}, \underset{(21)}{\bar{C}^i}_{jk}, \underset{(02)}{\bar{C}^i}_{jk}, \underset{(12)}{\bar{C}^i}_{jk}, \underset{(22)}{\bar{C}^i}_{jk})$$

are two N-, respectively $\bar{N}$-linear connections on the differentiable manifold T^2M, then there exists only one system of tensor fields

$$(\underset{1}{A^i}_j, \underset{2}{A^i}_j, \underset{(00)}{B^i}_{jk}, \underset{(10)}{B^i}_{jk}, \underset{(20)}{B^i}_{jk}, \underset{(01)}{D^i}_{jk}, \underset{(11)}{D^i}_{jk}, \underset{(21)}{D^i}_{jk}, \underset{(02)}{D^i}_{jk}, \underset{(12)}{D^i}_{jk}, \underset{(22)}{D^i}_{jk}),$$

such that:

$$\begin{cases} \underset{\beta}{\bar{N}^i}_j = \underset{\beta}{N^i}_j - \underset{\beta}{A^i}_j, \\ \underset{(\alpha 0)}{\bar{L}^i}_{jk} = \underset{(\alpha 0)}{L^i}_{jk} + \underset{1}{A^l}_k \underset{(\alpha 1)}{C^i}_{jl} + \underset{1}{A^l}_k \underset{1}{N^m}_l \underset{(\alpha 2)}{C^i}_{jm} + \underset{2}{A^l}_k \underset{(\alpha 2)}{C^i}_{jl} - \underset{(\alpha 0)}{B^i}_{jk} , \\ \underset{(\alpha 1)}{\bar{C}^i}_{jk} = \underset{(\alpha 1)}{C^i}_{jk} + \underset{1}{A^l}_k \underset{(\alpha 2)}{C^i}_{jl} - \underset{(\alpha 1)}{D^i}_{jk} , \\ \underset{(\alpha 2)}{\bar{C}^i}_{jk} = \underset{(\alpha 2)}{C^i}_{jk} - \underset{(\alpha 2)}{D^i}_{jk} , (\alpha = 0, 1, 2;\ \beta = 1, 2). \end{cases} \tag{2.3}$$

Proof. The first equality (2.3) determines uniquely the tensor fields $\underset{\beta}{A}{}^i{}_j, (\beta = 1, 2)$. Since $\underset{(01)}{C}{}^i{}_{jk}, \underset{(11)}{C}{}^i{}_{jk}, \underset{(21)}{C}{}^i{}_{jk}, \underset{(02)}{C}{}^i{}_{jk}, \underset{(12)}{C}{}^i{}_{jk}$ *and* $\underset{(22)}{C}{}^i{}_{jk}$ are tensor fields, the second equation (2.3) determines uniquely the tensor fields $\underset{(00)}{B}{}^i{}_{jk}$, $\underset{(10)}{B}{}^i{}_{jk}$ *and* $\underset{(20)}{B}{}^i{}_{jk}$. Similarly the third and the fourth equation (2.3) determine the tensor fields $\underset{(01)}{D}{}^i{}_{jk}$, $\underset{(11)}{D}{}^i{}_{jk}$, $\underset{(21)}{D}{}^i{}_{jk}$ and $\underset{(02)}{D}{}^i{}_{jk}$, $\underset{(12)}{D}{}^i{}_{jk}$, $\underset{(22)}{D}{}^i{}_{jk}$, respectively.

Conversely, we have

Theorem 2.2. *If*

$$D\Gamma(N) = (\, \underset{(00)}{L}{}^i{}_{jk},\ \underset{(10)}{L}{}^i{}_{jk},\ \underset{(20)}{L}{}^i{}_{jk},\ \underset{(01)}{C}{}^i{}_{jk},\ \underset{(11)}{C}{}^i{}_{jk},\ \underset{(21)}{C}{}^i{}_{jk},\ \underset{(02)}{C}{}^i{}_{jk},\ \underset{(12)}{C}{}^i{}_{jk},\ \underset{(22)}{C}{}^i{}_{jk}\,)$$

are local coefficients of an N-linear connection D and

$$(\underset{1}{A}{}^i{}_j, \underset{2}{A}{}^i{}_j,\ \underset{(00)}{B}{}^i{}_{jk},\ \underset{(10)}{B}{}^i{}_{jk},\ \underset{(20)}{B}{}^i{}_{jk},\ \underset{(01)}{D}{}^i{}_{jk},\ \underset{(11)}{D}{}^i{}_{jk},\ \underset{(21)}{D}{}^i{}_{jk},\ \underset{(02)}{D}{}^i{}_{jk},\ \underset{(12)}{D}{}^i{}_{jk},\ \underset{(22)}{D}{}^i{}_{jk}\,)$$

is a system of tensor fields, then:

$$D\bar{\Gamma}(\bar{N}) = (\, \underset{(00)}{\bar{L}}{}^i{}_{jk},\ \underset{(10)}{\bar{L}}{}^i{}_{jk},\ \underset{(20)}{\bar{L}}{}^i{}_{jk},\ \underset{(01)}{\bar{C}}{}^i{}_{jk},\ \underset{(11)}{\bar{C}}{}^i{}_{jk},\ \underset{(21)}{\bar{C}}{}^i{}_{jk},\ \underset{(02)}{\bar{C}}{}^i{}_{jk},\ \underset{(12)}{\bar{C}}{}^i{}_{jk},\ \underset{(22)}{\bar{C}}{}^i{}_{jk}\,)$$

given by (2.3) are local coefficients of an $\bar{N}$-linear connections $\bar{D}$.

The system of tensor fields

$$(\underset{1}{A}{}^i{}_j, \underset{2}{A}{}^i{}_j,\ \underset{(00)}{B}{}^i{}_{jk},\ \underset{(10)}{B}{}^i{}_{jk},\ \underset{(20)}{B}{}^i{}_{jk},\ \underset{(01)}{D}{}^i{}_{jk},\ \underset{(11)}{D}{}^i{}_{jk},\ \underset{(21)}{D}{}^i{}_{jk},\ \underset{(02)}{D}{}^i{}_{jk},\ \underset{(12)}{D}{}^i{}_{jk},\ \underset{(22)}{D}{}^i{}_{jk}\,)$$

are called the **difference** tensor fields of $D\Gamma(N)$ to $D\bar{\Gamma}(\bar{N})$ and the mapping $D\Gamma(N) \to D\bar{\Gamma}(\bar{N})$ given by (2.3) is called **the transformation** of N-linear connection to $\bar{N}$-linear connection, and it is noted by:

$$t(\underset{1}{A}{}^i{}_j, \underset{2}{A}{}^i{}_j,\ \underset{(00)}{B}{}^i{}_{jk},\ \underset{(10)}{B}{}^i{}_{jk},\ \underset{(20)}{B}{}^i{}_{jk},\ \underset{(01)}{D}{}^i{}_{jk},\ \underset{(11)}{D}{}^i{}_{jk},\ \underset{(21)}{D}{}^i{}_{jk},\ \underset{(02)}{D}{}^i{}_{jk},\ \underset{(12)}{D}{}^i{}_{jk},\ \underset{(22)}{D}{}^i{}_{jk}\,).$$

Theorem 2.3. *The set $\mathcal{T}$ of the transformations of N-linear connections to $\bar{N}$- linear connections, together with the composition of mappings isn't a group.*

Proof. Let

$$t(\underset{1}{A}{}^i{}_j, \underset{2}{A}{}^i{}_j,\ \underset{(00)}{B}{}^i{}_{jk},\ \underset{(10)}{B}{}^i{}_{jk},\ \underset{(20)}{B}{}^i{}_{jk},\ \underset{(01)}{D}{}^i{}_{jk},\ \underset{(11)}{D}{}^i{}_{jk},\ \underset{(21)}{D}{}^i{}_{jk},\ \underset{(02)}{D}{}^i{}_{jk},\ \underset{(12)}{D}{}^i{}_{jk},\ \underset{(22)}{D}{}^i{}_{jk}\,) : D\Gamma(N) \to D\bar{\Gamma}(\bar{N})$$

and

$$t(\underset{1}{\bar{A}^i{}_j}, \underset{2}{\bar{A}^i{}_j}, \underset{(00)}{\bar{B}^i{}_{jk}}, \underset{(10)}{\bar{B}^i{}_{jk}}, \underset{(20)}{\bar{B}^i{}_{jk}}, \underset{(01)}{\bar{D}^i{}_{jk}}, \underset{(11)}{\bar{D}^i{}_{jk}}, \underset{(21)}{\bar{D}^i{}_{jk}}, \underset{(02)}{\bar{D}^i{}_{jk}}, \underset{(12)}{\bar{D}^i{}_{jk}}, \underset{(22)}{\bar{D}^i{}_{jk}}) : D\bar{\Gamma}(\bar{N}) \to D\bar{\bar{\Gamma}}(\bar{\bar{N}})$$

be two transformations from $\mathcal{T}$, given by (2.3).

From (2.3) we have:

$$\underset{\beta}{\bar{\bar{N}}^i{}_j} = \underset{\beta}{N^i{}_j} - (\underset{\beta}{A^i{}_j} + \underset{\beta}{\bar{A}^i{}_j}), (\beta = 1, 2).$$

We obtain:

$$\begin{cases} \underset{(\alpha 0)}{\bar{\bar{L}}^i{}_{jk}} = \underset{(\alpha 0)}{L^i{}_{jk}} + \underset{(\alpha 1)}{C^i{}_{jl}} (\underset{1}{A^l{}_k} + \underset{1}{\bar{A}^l{}_k}) + \underset{(\alpha 2)}{C^i{}_{jm}} \underset{1}{N^m{}_l}(\underset{1}{A^l{}_k} + \underset{1}{\bar{A}^l{}_k}) + \\ \qquad + \underset{(\alpha 2)}{C^i{}_{jl}} (\underset{2}{A^l{}_k} + \underset{2}{\bar{A}^l{}_k}) + (\underset{(\alpha 2)}{C^i{}_{jm}} + \underset{(\alpha 2)}{D^i{}_{jm}}) \underset{1}{A^m{}_l} \underset{1}{\bar{A}^l{}_k} - \\ \qquad - (\underset{(\alpha 1)}{D^i{}_{jl}} \underset{1}{\bar{A}^l{}_k} + \underset{(\alpha 2)}{D^i{}_{jm}} \underset{1}{N^m{}_l} \underset{1}{A^l{}_k} + \underset{(\alpha 2)}{C^i{}_{jm}} \underset{(1)}{A^m{}_l} \underset{1}{\bar{A}^l{}_k} + \\ \qquad + \underset{(\alpha 2)}{D^i{}_{jl}} \underset{(2)}{\bar{A}^l{}_k} - (\underset{(\alpha 0)}{B^i{}_{jk}} + \underset{(\alpha 0)}{\bar{B}^i{}_{jk}}), \\ \underset{(\alpha 1)}{\bar{\bar{C}}^i{}_{jk}} = \underset{(\alpha 1)}{C^i{}_{jk}} + \underset{(\alpha 2)}{C^i{}_{jl}} (\underset{1}{A^l{}_k} + \underset{1}{\bar{A}^l{}_k}) - (\underset{(\alpha 2)}{D^i{}_{jl}} \underset{1}{\bar{A}^l{}_k} + \underset{(\alpha 1)}{D^i{}_{jk}} + \underset{(\alpha 1)}{\bar{D}^i{}_{jk}}), \\ \underset{(\alpha 2)}{\bar{\bar{C}}^i{}_{jk}} = \underset{(\alpha 2)}{C^i{}_{jk}} - (\underset{(\alpha 2)}{D^i{}_{jk}} + \underset{(\alpha 2)}{\bar{D}^i{}_{jk}}), (\alpha = 0, 1, 2). \end{cases} \tag{2.4}$$

So $\underset{(\alpha 0)}{\bar{\bar{L}}^i{}_{jk}}$, $(\alpha = 0, 1, 2)$ hasn't the form (2.3). Result that the mapping of two transformations from $\mathcal{T}$, isn't a transformation from $\mathcal{T}$, so $\mathcal{T}$, together with the composition of mappings isn't a group.

Remark 2.1. If we consider $\underset{\beta}{A^i{}_j} = 0, (\beta = 1, 2)$, in (2.3) we obtain the set $\mathcal{T}_N$ of transformations of N-linear connections, having the same nonlinear connection N:

$$\mathcal{T}_N = \{t(0, 0, \underset{(00)}{B^i{}_{jk}}, \underset{(10)}{B^i{}_{jk}}, \underset{(20)}{B^i{}_{jk}}, \underset{(01)}{D^i{}_{jk}}, \underset{(11)}{D^i{}_{jk}}, \underset{(21)}{D^i{}_{jk}}, \underset{(02)}{D^i{}_{jk}}, \underset{(12)}{D^i{}_{jk}}, \underset{(22)}{D^i{}_{jk}}) \in \mathcal{T}\}.$$

We have:

Theorem 2.4. *The set $\mathcal{T}_N$ of the transformations of N-linear connections to N-linear connections, together with the composition of mappings is a group. This group, $\mathcal{T}_N$, acts effectively and transitivelly on the set of N-linear connections.*

Proof. Let

$$t(0,0,\underset{(00)}{B^i{}_{jk}},\underset{(10)}{B^i{}_{jk}},\underset{(20)}{B^i{}_{jk}},\underset{(01)}{D^i{}_{jk}},\underset{(11)}{D^i{}_{jk}},\underset{(21)}{D^i{}_{jk}},\underset{(02)}{D^i{}_{jk}},\underset{(12)}{D^i{}_{jk}},\underset{(22)}{D^i{}_{jk}}):D\Gamma(N)\to D\bar{\Gamma}(N)$$

be a transformation from $\mathcal{T}_N$ given by (2.5):

$$\begin{cases} \underset{\beta}{\bar{N}^i{}_j} = \underset{\beta}{N^i{}_j}, \\ \underset{(\alpha 0)}{\bar{L}^i{}_{jk}} = \underset{(\alpha 0)}{L^i{}_{jk}} - \underset{(\alpha 0)}{B^i{}_{jk}}, \\ \underset{(\alpha 1)}{\bar{C}^i{}_{jk}} = \underset{(\alpha 1)}{C^i{}_{jk}} - \underset{(\alpha 1)}{D^i{}_{jk}}, \\ \underset{(\alpha 2)}{\bar{C}^i{}_{jk}} = \underset{(\alpha 2)}{C^i{}_{jk}} - \underset{(\alpha 2)}{D^i{}_{jk}}, (\alpha = 0,1,2;\ \beta = 1,2). \end{cases} \tag{2.5}$$

The composition of two transformations from $\mathcal{T}_N$ is a transformation from $\mathcal{T}_N$, given by:

$$t(0,0,\underset{(00)}{\bar{B}^i{}_{jk}},\underset{(10)}{\bar{B}^i{}_{jk}},\underset{(20)}{\bar{B}^i{}_{jk}},\underset{(01)}{\bar{D}^i{}_{jk}},\underset{(11)}{\bar{D}^i{}_{jk}},\underset{(21)}{\bar{D}^i{}_{jk}},\underset{(02)}{\bar{D}^i{}_{jk}},\underset{(12)}{\bar{D}^i{}_{jk}},\underset{(22)}{\bar{D}^i{}_{jk}})$$

$$\circ t(0,0,\underset{(00)}{B^i{}_{jk}},\underset{(10)}{B^i{}_{jk}},\underset{(20)}{B^i{}_{jk}},\underset{(01)}{D^i{}_{jk}},\underset{(11)}{D^i{}_{jk}},\underset{(21)}{D^i{}_{jk}},\underset{(02)}{D^i{}_{jk}},\underset{(12)}{D^i{}_{jk}},\underset{(22)}{D^i{}_{jk}})$$

$$= t(0,0,\underset{(00)}{B^i{}_{jk}} + \underset{(00)}{\bar{B}^i{}_{jk}},\underset{(10)}{B^i{}_{jk}} + \underset{(10)}{\bar{B}^i{}_{jk}},\underset{(20)}{B^i{}_{jk}} + \underset{(20)}{\bar{B}^i{}_{jk}},\underset{(01)}{D^i{}_{jk}} + \underset{(01)}{\bar{D}^i{}_{jk}},\underset{(11)}{D^i{}_{jk}} + \underset{(11)}{\bar{D}^i{}_{jk}},$$

$$\underset{(21)}{D^i{}_{jk}} + \underset{(21)}{\bar{D}^i{}_{jk}},\underset{(02)}{D^i{}_{jk}} + \underset{(02)}{\bar{D}^i{}_{jk}},\underset{(12)}{D^i{}_{jk}} + \underset{(12)}{\bar{D}^i{}_{jk}},\underset{(22)}{D^i{}_{jk}} + \underset{(22)}{\bar{D}^i{}_{jk}}).$$

The inverse of a transformation from $\mathcal{T}_N$ is the transformation:

$$t(0,0,-\underset{(00)}{B^i{}_{jk}},-\underset{(10)}{B^i{}_{jk}},-\underset{(20)}{B^i{}_{jk}},-\underset{(01)}{D^i{}_{jk}},-\underset{(11)}{D^i{}_{jk}},-\underset{(21)}{D^i{}_{jk}},-\underset{(02)}{D^i{}_{jk}},-\underset{(12)}{D^i{}_{jk}},-\underset{(22)}{D^i{}_{jk}}):$$

$$D\Gamma(N)\to D\bar{\Gamma}(N).$$

The transformation (2.5) preserves all the N-linear connections D if

$$\underset{(\alpha 0)}{B^i{}_{jk}} = \underset{(\alpha 0)}{D^i{}_{jk}} = 0, (\alpha = 0,1,2).$$

Therefore $\mathcal{T}_N$ acts effectively on the set of N-linear connections. From the Theorem 2.1. results that $\mathcal{T}_N$ acts transitively on this set. □

Let us consider:

$$\mathcal{T}_{NL}=\{t(0,0,0,0,0,\underset{(01)}{D}{}^i{}_{jk},\underset{(11)}{D}{}^i{}_{jk},\underset{(21)}{D}{}^i{}_{jk},\underset{(02)}{D}{}^i{}_{jk},\underset{(12)}{D}{}^i{}_{jk},\underset{(22)}{D}{}^i{}_{jk})\in\mathcal{T}_N\},$$

$$\mathcal{T}_{N\underset{(1)}{C}}=\{t(0,0,\underset{(00)}{B}{}^i{}_{jk},\underset{(10)}{B}{}^i{}_{jk},\underset{(20)}{B}{}^i{}_{jk},0,0,0,\underset{(02)}{D}{}^i{}_{jk},\underset{(12)}{D}{}^i{}_{jk},\underset{(22)}{D}{}^i{}_{jk})\in\mathcal{T}_N\},$$

$$\mathcal{T}_{N\underset{(2)}{C}}=\{t(0,0,\underset{(00)}{B}{}^i{}_{jk},\underset{(10)}{B}{}^i{}_{jk},\underset{(20)}{B}{}^i{}_{jk},\underset{(01)}{D}{}^i{}_{jk},\underset{(11)}{D}{}^i{}_{jk},\underset{(21)}{D}{}^i{}_{jk},0,0,0)\in\mathcal{T}_N\},$$

$$\mathcal{T}_{N\underset{(1)}{C}\underset{(2)}{C}}=\{t(0,0,\underset{(00)}{B}{}^i{}_{jk},\underset{(10)}{B}{}^i{}_{jk},\underset{(20)}{B}{}^i{}_{jk},0,0,0,0,0,0)\in\mathcal{T}_N\}.$$

Proposition 2.2. *$\mathcal{T}_{NL},\mathcal{T}_{N\underset{(1)}{C}},\mathcal{T}_{N\underset{(2)}{C}}$ and $\mathcal{T}_{N\underset{(1)}{C}\underset{(2)}{C}}$ are abelian subgroups of $\mathcal{T}_N$.*

Proposition 2.3. *The group $\mathcal{T}_N$ preserves the nonlinear connection N; $\mathcal{T}_{NL}$ preserves the nonlinear connection N and the components $\underset{(\alpha0)}{L}$, $(\alpha=0,1,2)$ of the local coefficients $D\Gamma(N)$; $\mathcal{T}_{N\underset{(1)}{C}}$ preserves the nonlinear connection N and the components $\underset{(\alpha1)}{C}$, $(\alpha=0,1,2)$ of the local coefficients $D\Gamma(N)$; $\mathcal{T}_{N\underset{(2)}{C}}$ preserves the nonlinear connection N and the components $\underset{(\alpha2)}{C}$, $(\alpha=0,1,2)$ of the local coefficients $D\Gamma(N)$ and $\mathcal{T}_{N\underset{(1)}{C}\underset{(2)}{C}}$ preserves the nonlinear connection N and the components $\underset{(\alpha1)}{C}$ and $\underset{(\alpha2)}{C}$, $(\alpha=0,1,2)$ of the local coefficients $D\Gamma(N)$.*

3. The transformations of the d-tensors of torsion and curvature in $\mathcal{T}_N$

In the following, we shall study the Abelian group $\mathcal{T}_N$. Its elements are the transformations $t:D\Gamma(N)\to D\bar{\Gamma}(N)$ given by

$$\begin{cases}\underset{\beta}{\bar{N}}{}^i{}_j=\underset{\beta}{N}{}^i{}_j,\\ \underset{(\alpha0)}{\bar{L}}{}^i{}_{jk}=\underset{(\alpha0)}{L}{}^i{}_{jk}-\underset{(\alpha0)}{B}{}^i{}_{jk},\\ \underset{(\alpha1)}{\bar{C}}{}^i{}_{jk}=\underset{(\alpha1)}{C}{}^i{}_{jk}-\underset{(\alpha1)}{D}{}^i{}_{jk},\\ \underset{(\alpha2)}{\bar{C}}{}^i{}_{jk}=\underset{(\alpha2)}{C}{}^i{}_{jk}-\underset{(\alpha2)}{D}{}^i{}_{jk},(\alpha=0,1,2;\ \beta=1,2).\end{cases}\tag{3.1}$$

Firstly, we shall study the transformations of the d-tensors of torsion of $D\Gamma(N)$ (see, (7.2) and (7.5), [1]). We obtain:

Proposition 3.1. *The transformations of the Abelian group $\mathcal{T}_N$, given by (3.1) lead to the transformations of the d-tensors of torsion in the following way:*

$$\underset{(0\beta)}{\bar{R}^i{}_{jk}} = \underset{(0\beta)}{R^i{}_{jk}}\ , \tag{3.2}$$

$$\underset{(0)}{\overset{\alpha}{\bar{T}}{}^i{}_{jk}} = \underset{(0)}{\overset{\alpha}{T}{}^i{}_{jk}} + (\underset{(\alpha 0)}{B^i{}_{kj}} - \underset{(\alpha 0)}{B^i{}_{jk}}), \tag{3.3}$$

$$\underset{(\beta)}{\overset{\alpha}{\bar{S}}{}^i{}_{jk}} = \underset{(\beta)}{\overset{\alpha}{S}{}^i{}_{jk}} + (\underset{(\alpha\beta)}{D^i{}_{kj}} - \underset{(\alpha\beta)}{D^i{}_{jk}}), \tag{3.4}$$

$$\underset{(21)}{\bar{Q}^i{}_{jk}} = \underset{(21)}{Q^i{}_{jk}} - \underset{(12)}{D^i{}_{jk}}\ , \tag{3.5}$$

$$\underset{(22)}{\overset{\alpha}{\bar{Q}}{}^i{}_{jk}} = \underset{(22)}{\overset{\alpha}{Q}{}^i{}_{jk}} + \underset{(\alpha 1)}{D^i{}_{kj}}\ , \tag{3.6}$$

$$\underset{(12)}{\bar{S}^i{}_{jk}} = \underset{(12)}{S^i{}_{jk}}\ , \tag{3.7}$$

$$\underset{(\beta\beta)}{\overset{\alpha}{\bar{P}}{}^i{}_{jk}} = \underset{(\beta\beta)}{\overset{\alpha}{P}{}^i{}_{jk}} + \underset{(\alpha 0)}{B^i{}_{kj}}\ , \tag{3.8}$$

$$\underset{(\beta 0)}{\bar{P}^i{}_{jk}} = \underset{(\beta 0)}{P^i{}_{jk}} - \underset{(0\beta)}{D^i{}_{jk}}\ , \tag{3.9}$$

$$\underset{(12)}{\bar{P}^i{}_{jk}} = \underset{(12)}{P^i{}_{jk}}\ , \tag{3.10}$$

$$\underset{(21)}{\bar{P}^i{}_{jk}} = \underset{(21)}{P^i{}_{jk}}\ , \quad (\alpha = 0,1,2;\ \beta = 1,2). \tag{3.11}$$

Now, we shall study the transformations of the d-tensors of curvature of $D\Gamma(N)$ (see, (7.11),[1]). We get:

Proposition 3.2. *The transformations of the Abelian group $\mathcal{T}_N$, given by (3.1) lead to the transformations of the d-tensors of curvature in the following way:*

$$\underset{(0\alpha)}{\bar{R}_h{}^i{}_{jk}} = \underset{(0\alpha)}{R_h{}^i{}_{jk}} - \underset{(\alpha 1)}{D^i{}_{hs}}\ \underset{(01)}{R^s{}_{jk}} - \underset{(\alpha 2)}{D^i{}_{hs}}\ \underset{(02)}{R^s{}_{jk}} - \underset{(\alpha 0)}{B^i{}_{hs}}\ \underset{(0)}{\overset{\alpha}{T}{}^s{}_{jk}} + \tag{3.12}$$

$$+\mathcal{A}_{jk}\{-\underset{(\alpha 0)}{B^i{}_{hj|\alpha k}} + \underset{(\alpha 0)}{B^s{}_{hj}}\ \underset{(\alpha 0)}{B^i{}_{sk}}\},$$

$$\underset{(1\alpha)}{\bar{P}_h{}^i{}_{jk}} = \underset{(1\alpha)}{P_h{}^i{}_{jk}} - \underset{(\alpha 1)}{D^i{}_{hs}}\ \underset{(11)}{\overset{\alpha}{P}{}^s{}_{jk}} - \underset{(\alpha 2)}{D^i{}_{hs}}\ \underset{(12)}{P^s{}_{jk}} - \underset{(\alpha 0)}{B^i{}_{hs}}\ \underset{(\alpha 1)}{C^s{}_{jk}} + \tag{3.13}$$

$$+ \underset{(\alpha 0)}{L^s{}_{kj}} \underset{(\alpha 1)}{D^i{}_{hs}} - \underset{(\alpha 0)}{B^i{}_{hj}} \overset{(1)}{|}_{\alpha k} + \underset{(\alpha 1)}{D^i{}_{hk|\alpha j}} + \underset{(\alpha 0)}{B^s{}_{hj}} \underset{(\alpha 1)}{D^i{}_{sk}} -$$

$$- \underset{(\alpha 1)}{D^s{}_{hk}} \underset{(\alpha 0)}{B^i{}_{sj}} + \underset{(\alpha 1)}{C^i{}_{hs}} \underset{(\alpha 0)}{B^s{}_{kj}} - \underset{(\alpha 1)}{D^i{}_{hs}} \underset{(\alpha 0)}{B^s{}_{kj}} ,$$

$$\underset{(2\alpha)}{\bar{P}{}_h{}^i{}_{jk}} = \underset{(2\alpha)}{P{}_h{}^i{}_{jk}} - \underset{(\alpha 1)}{D^i{}_{hs}} \underset{(21)}{P^s{}_{jk}} - \underset{(\alpha 2)}{D^i{}_{hs}} \underset{(22)}{\overset{\alpha}{P}{}^s{}_{jk}} - \underset{(\alpha 0)}{B^i{}_{hs}} \underset{(\alpha 2)}{C^s{}_{jk}} + \tag{3.14}$$

$$+ \underset{(\alpha 0)}{L^s{}_{kj}} \underset{(\alpha 2)}{D^i{}_{hs}} - \underset{(\alpha 0)}{B^i{}_{hj}} \overset{(2)}{|}_{\alpha k} + \underset{(\alpha 2)}{D^i{}_{hk|\alpha j}} + \underset{(\alpha 0)}{B^s{}_{hj}} \underset{(\alpha 2)}{D^i{}_{sk}} -$$

$$- \underset{(\alpha 2)}{D^s{}_{hk}} \underset{(\alpha 0)}{B^i{}_{sj}} + \underset{(\alpha 2)}{C^i{}_{hs}} \underset{(\alpha 0)}{B^s{}_{kj}} - \underset{(\alpha 2)}{D^i{}_{hs}} \underset{(\alpha 0)}{B^s{}_{kj}} ,$$

$$\underset{(2\alpha)}{\bar{Q}{}_h{}^i{}_{jk}} = \underset{(2\alpha)}{Q{}_h{}^i{}_{jk}} - \underset{(\alpha 2)}{C^s{}_{jk}} \underset{(\alpha 1)}{D^i{}_{hs}} + \underset{(\alpha 1)}{C^s{}_{kj}} \underset{(\alpha 2)}{D^i{}_{hs}} - \underset{(\alpha 1)}{D^i{}_{hj}} \overset{(2)}{|}_{\alpha k} + \tag{3.15}$$

$$+ \underset{(\alpha 2)}{D^i{}_{hk}} \overset{(1)}{|}_{\alpha j} + \underset{(\alpha 1)}{D^s{}_{hj}} \underset{(\alpha 2)}{D^i{}_{sk}} - \underset{(\alpha 2)}{D^s{}_{hk}} \underset{(\alpha 1)}{D^i{}_{sj}} - \underset{(\alpha 2)}{D^i{}_{hs}} \underset{(21)}{P^s{}_{jk}} ,$$

$$\underset{(\beta\alpha)}{\bar{S}{}_h{}^i{}_{jk}} = \underset{(\beta\alpha)}{S{}_h{}^i{}_{jk}} - \underset{(\alpha\beta)}{D^i{}_{hs}} \underset{(\beta)}{\overset{\alpha}{S}{}^s{}_{jk}} + \mathcal{A}_{jk}\{- \underset{(\alpha\beta)}{D^i{}_{hj}} \overset{(\beta)}{|}_{\alpha k} + \tag{3.16}$$

$$+ \underset{(\alpha\beta)}{D^s{}_{hj}} \underset{(\alpha\beta)}{D^i{}_{sk}} \} - \underset{(\alpha 2)}{D^i{}_{hs}} \underset{(\beta 2)}{R^s{}_{jk}} , \quad (\alpha = 0,1,2;\ \beta = 1,2),$$

where $\mathcal{A}_{ij}$ denotes the alternate summation.

We shall consider the tensor fields:

$$\underset{(0\alpha)}{\mathbb{K}{}_h{}^i{}_{jk}} = \underset{(0\alpha)}{R{}_h{}^i{}_{jk}} - \underset{(\alpha 1)}{C^i{}_{hs}} \underset{(01)}{R^s{}_{jk}} - \underset{(\alpha 2)}{C^i{}_{hs}} \underset{(02)}{R^s{}_{jk}} , \tag{3.17}$$

$$\underset{(1\alpha)}{\mathbb{P}{}_h{}^i{}_{jk}} = \mathcal{A}_{jk} \{ \underset{(1\alpha)}{P{}_h{}^i{}_{jk}} - \underset{(\alpha 1)}{C^i{}_{hs}} \frac{\delta \underset{1}{N^s{}_j}}{\delta y^{(1)k}} - \underset{(\alpha 2)}{C^i{}_{hs}} (\underset{1}{N^s{}_m} \frac{\delta \underset{1}{N^m{}_j}}{\delta y^{(1)k}} + \tag{3.18}$$

$$+ \frac{\delta \underset{2}{N^s{}_j}}{\delta y^{(1)k}} - \frac{\delta \underset{2}{N^s{}_k}}{\delta y^{(1)j}})\},$$

$$\underset{(2\alpha)}{\mathbb{P}{}_h{}^i{}_{jk}} = \mathcal{A}_{jk} \{ \underset{(2\alpha)}{P{}_h{}^i{}_{jk}} - \underset{(\alpha 1)}{C^i{}_{hs}} \frac{\partial \underset{1}{N^s{}_j}}{\delta y^{(2)k}} - \underset{(\alpha 2)}{C^i{}_{hs}} (\underset{1}{N^s{}_m} \frac{\partial \underset{1}{N^m{}_j}}{\delta y^{(2)k}} + + \frac{\partial \underset{2}{N^s{}_j}}{\delta y^{(2)k}})\}, \tag{3.19}$$

$$\underset{(2\alpha)}{\mathbb{Q}{}_h{}^i{}_{jk}} = \mathcal{A}_{jk} \{ \underset{(2\alpha)}{Q{}_h{}^i{}_{jk}} + \underset{(\alpha 2)}{C^i{}_{hs}} \frac{\partial \underset{1}{N^s{}_j}}{\delta y^{(2)k}} \}, \tag{3.20}$$

$$\underset{(\beta\alpha)}{\mathbb{S}{}_h{}^i{}_{jk}} = \underset{(\beta\alpha)}{S{}_h{}^i{}_{jk}} - \underset{(\alpha 2)}{C^i{}_{hs}} \underset{(\beta 2)}{R^s{}_{jk}} , (\alpha = 0,1,2;\ \beta = 1,2). \tag{3.21}$$

Proposition 3.3. *By a transformation of the Abelian group $\mathcal{T}_N$, given by (3.1), the tensor fields:* $\underset{(0\alpha)}{\mathbb{K}}{}_{h\ jk}^{\ i}$, $\underset{(1\alpha)}{\mathbb{P}}{}_{h\ jk}^{\ i}$, $\underset{(2\alpha)}{\mathbb{P}}{}_{h\ jk}^{\ i}$, $\underset{(2\alpha)}{\mathbb{Q}}{}_{h\ jk}^{\ i}$ *and* $\underset{(\beta\alpha)}{\mathbb{S}}{}_{h\ jk}^{\ i}$, $(\alpha = 0, 1, 2;\ \beta = 1, 2)$ *are transformed according to the following laws:*

$$\underset{(0\alpha)}{\bar{\mathbb{K}}}{}_{h\ jk}^{\ i} = \underset{(0\alpha)}{\mathbb{K}}{}_{h\ jk}^{\ i} - \underset{(\alpha0)}{B}{}^i_{hs}\ \underset{(0)}{\overset{\alpha}{T}}{}^{s}_{\ jk} + \mathcal{A}_{jk}\{-\underset{(\alpha0)}{B}{}^i_{hj|\alpha k} + \underset{(\alpha0)}{B}{}^s_{kj}\ \underset{(\alpha0)}{B}{}^i_{sk}\}, \tag{3.22}$$

$$\underset{(\beta\alpha)}{\bar{\mathbb{P}}}{}_{h\ jk}^{\ i} = \underset{(\beta\alpha)}{\mathbb{P}}{}_{h\ jk}^{\ i} - 2\underset{(\alpha\beta)}{D}{}^i_{hs}\ \underset{(0)}{\overset{\alpha}{T}}{}^{s}_{\ jk} - \underset{(\alpha0)}{B}{}^i_{hs}\ \underset{(\beta)}{\overset{\alpha}{S}}{}^{s}_{\ jk} + \mathcal{A}_{jk}\{-\underset{(\alpha0)}{B}{}^i_{hj}\overset{(\beta)}{|}_{\alpha k} - \tag{3.23}$$

$$-\underset{(\alpha\beta)}{D}{}^i_{hj|\alpha k} + \underset{(\alpha0)}{B}{}^s_{hj}\ \underset{(\alpha\beta)}{D}{}^i_{sk} + \underset{(\alpha\beta)}{D}{}^s_{hj}\ \underset{(\alpha0)}{B}{}^i_{sk} + \underset{(\alpha\beta)}{D}{}^i_{hs}\ \underset{(\alpha0)}{B}{}^s_{jk} - \underset{(\alpha\beta)}{C}{}^i_{hs}\ \underset{(\alpha0)}{B}{}^s_{jk}\},$$

$$\underset{(2\alpha)}{\bar{\mathbb{Q}}}{}_{h\ jk}^{\ i} = \underset{(2\alpha)}{\mathbb{Q}}{}_{h\ jk}^{\ i} + \underset{(2)}{\overset{\alpha}{S}}{}^{s}_{\ jk}\ \underset{(\alpha1)}{D}{}^i_{hs} - \underset{(1)}{\overset{\alpha}{S}}{}^{s}_{\ jk}\ \underset{(\alpha2)}{D}{}^i_{hs} \tag{3.24}$$

$$+\mathcal{A}_{jk}\{\underset{(\alpha1)}{D}{}^i_{hj}\overset{(2)}{|}_{\alpha k} + \underset{(\alpha2)}{D}{}^i_{hk}\overset{(1)}{|}_{\alpha j} + + \underset{(\alpha1)}{D}{}^s_{hj}\ \underset{(\alpha2)}{D}{}^i_{sk} - \underset{(\alpha2)}{D}{}^s_{hk}\ \underset{(\alpha1)}{D}{}^i_{sj}\},$$

$$\underset{(\beta\alpha)}{\bar{\mathbb{S}}}{}_{h\ jk}^{\ i} = \underset{(\beta\alpha)}{\mathbb{S}}{}_{h\ jk}^{\ i} - \underset{(\alpha\beta)}{D}{}^i_{hs}\ \underset{(\beta)}{\overset{\alpha}{S}}{}^{s}_{\ jk} + \mathcal{A}_{jk}\{-\underset{(\alpha\beta)}{D}{}^i_{hj}\overset{(\beta)}{|}_{\alpha k} + \underset{(\alpha\beta)}{D}{}^s_{hj}\ \underset{(\alpha\beta)}{D}{}^i_{sk}\}, \tag{3.25}$$

$(\alpha = 0, 1, 2;\ \beta = 1, 2)$.

The transformations for the coefficients of an N-linear connection on the tangent bundle of order two, T^2M, by a transformation of a nonlinear connection in T^2M, together with the transformation laws of the torsion and curvature tensor fields, with respect to the transformations of the group $\mathcal{T}_N$, given in the present paper are necessary for the study of a important subgroup of the group $\mathcal{T}_N$: the group of transformations of the metric semi-symmetric N-linear connections in T^2M, $\overset{ms}{\mathcal{T}}_N$. This study is in our attention.

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