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$\mathcal{A}$-SUMMABILITY AND APPROXIMATION OF CONTINUOUS PERIODIC FUNCTIONS

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Dedicated to Professor D. D. Stancu on his 80^{th} birthday

Abstract. The aim of this paper is to present a generalization of the classical Korovkin approximation theorem by using a matrix summability method, for sequences of positive linear operators defined on the space of all real-valued continuous and 2π-periodic functions. This approach is motivated by the works of O. Duman [4] and C. Orhan, Ö.G. Atlihan [1].

1. Introduction

One of the most recently studied subject in approximation theory is the approximation of continuous function by linear positive operators using A-statistical convergence or a matrix summability method ([1], [3], [5], [7]).

In this paper, following [1], we will give a Korovkin type approximation theorem for a sequence of positive linear operators defined on the space of all real-valued continuous and 2π-periodic functions via $\mathcal{A}$-summability. Particular cases are also punctuated.

First of all, we recall some notation and definitions used in this paper.

Let $\mathcal{A} := (A^n)_{n\geq 1}$, $A^n = (a^n_{kj})_{k,j\in\mathbb{N}}$ be a sequence of infinite non-negative real matrices.

For a sequence of real numbers, $x = (x_j)_{j\in\mathbb{N}}$, the double sequence

$$\mathcal{A}x := \{(Ax)^n_k : \ k, n \in \mathbb{N}\}$$

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defined by $(Ax)_k^n := \sum_{j=1}^{\infty} a_{kj}^n x_j$ is called the $\mathcal{A}$-transform of x whenever the series converges for all k and n. A sequence x is said to be $\mathcal{A}$-summable to a real number L if $\mathcal{A}x$ converges to L as k tends to infinity uniformly in n (see [2]).

We denote by $C_{2\pi}(\mathbb{R})$ the space of all 2π-periodic and continuous functions on $\mathbb{R}$. Endowed with the norm $\|\cdot\|_{2\pi}$ this space is a Banach space, where

$$\|f\|_{2\pi} := \sup_{t\in\mathbb{R}} |f(t)|, \quad f \in C_{2\pi}(\mathbb{R}).$$

We also have to recall the classical Bohman-Korovkin theorem.

Theorem A. *If $\{L_j\}$ is a sequence of positive linear operators acting from $C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$ such that*

$$\lim_{j\to\infty} \|L_j f_i - f_i\|_{2\pi} = 0 \quad (i = 1, 2, 3),$$

where $f_1(t) = 1,\ f_2(t) = \cos t,\ f_3(t) = \sin t$ for all $t \in \mathbb{R}$, then, for all $f \in C_{2\pi}(\mathbb{R})$ we have

$$\lim_{j\to\infty} \|L_j f - f\|_{2\pi} = 0.$$

Recently, the statistical analog of Theorem A has been studied by O. Duman [4]. It will be read as follows.

Theorem B. *Let $A = (a_{kj})$ be a non-negative regular summability matrix, and let $\{L_j\}$ be a sequence of positive linear operators mapping $C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$. Then, for all $f \in C_{2\pi}(\mathbb{R})$,*

$$st_A - \lim_{j\to\infty} \|L_j f - f\|_{2\pi} = 0$$

if and only if

$$st_A - \lim_{j\to\infty} \|L_j f_i - f_i\|_{2\pi} = 0 \quad (i = 1, 2, 3),$$

where $f_1(t) = 1,\ f_2(t) = \cos t,\ f_3(t) = \sin t$ for all $t \in \mathbb{R}$.

2. **A Korovkin type theorem**

Theorem 2.1. *Let $\mathcal{A} = (A^n)_{n\geq 1}$ be a sequence of infinite non-negative real matrices such that*

$$\sup_{n,k} \sum_{j=1}^{\infty} a_{kj}^n < \infty \tag{2.1}$$

and let $\{L_j\}$ be a sequence of positive linear operators mapping $C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$.

Then, for all $f \in C_{2\pi}(\mathbb{R})$ we have

$$\lim_{k\to\infty} \sum_{j=1}^{\infty} a_{kj}^n \|L_j f - f\|_{2\pi} = 0, \tag{2.2}$$

uniformly in n if and only if

$$\lim_{k\to\infty} \sum_{j=1}^{\infty} a_{kj}^n \|L_j f_i - f_i\|_{2\pi} = 0 \quad (i = 1, 2, 3), \tag{2.3}$$

uniformly in n, where $f_1(t) = 1,\ f_2(t) = \cos t,\ f_3(t) = \sin t$ for all $t \in \mathbb{R}$.

Proof. Since f_i $(i = 1, 2, 3)$ belong to $C_{2\pi}(\mathbb{R})$, the implication (2.2) $\Rightarrow$ (2.3) is obvious.

Now, assume that (2.3) holds. Let $f \in C_{2\pi}(\mathbb{R})$ and let I be a closed subinterval of length 2π of $\mathbb{R}$. Fix $x \in I$. By the continuity of f at x, it follows that for any $\varepsilon > 0$ there exists a number $\delta > 0$ such that

$$|f(t) - f(x)| < \varepsilon \text{ for all } t \text{ satisfying } |t - x| < \delta. \tag{2.4}$$

By the boundedness of f follows

$$|f(t) - f(x)| \leq 2\|f\|_{2\pi} \text{ for all } t \in \mathbb{R}. \tag{2.5}$$

Further on, we consider the subinterval $(x - \delta, 2\pi + x - \delta]$ of length 2π. We show that

$$|f(t) - f(x)| < \varepsilon + \frac{2\|f\|_{2\pi}}{\sin^2 \dfrac{\delta}{2}} \psi(t) \text{ holds for all } t \in (x - \delta, 2\pi + x - \delta], \tag{2.6}$$

where $\psi(t) := \sin^2 \left(\dfrac{t - x}{2} \right)$.

To prove (2.6) we examine two cases.

Case 1. Let $t \in (x-\delta, x+\delta)$. In this case we get $|t-x| < \delta$ and the relation (2.6) follows by (2.4).

Case 2. Let $t \in [x+\delta, 2\pi + x - \delta]$. In this case we have $\delta \leq t - x \leq 2\pi - \delta$ and $\delta \in (0, \pi]$. We get

$$\sin^2 \frac{\delta}{2} \leq \sin^2 \left(\frac{t-x}{2} \right) \leq \sin^2 \left(\pi - \frac{\delta}{2} \right), \tag{2.7}$$

for all $\delta \in (0, \pi]$ and $t \in [x+\delta, 2\pi + x - \delta]$.

Then, from (2.5) and (2.7) we obtain

$$|f(t) - f(x)| \leq \frac{2\|f\|_{2\pi}}{\sin^2 \frac{\delta}{2}} \psi(t) \text{ for all } t \in [x+\delta, 2\pi + x - \delta].$$

Since the function $f \in C_{2\pi}(\mathbb{R})$ is 2π-periodic, the inequality (2.6) holds for all $t \in \mathbb{R}$.

Now, applying the operator L_j, we get

$$\begin{aligned} |L_j(f;x) - f(x)| &\leq L_j(|f - f(x)|; x) + |f(x)||L_j(f_1;x) - f_1(x)| \\ &< L_j \left(\varepsilon + \frac{2\|f\|_{2\pi}}{\sin^2 \frac{\delta}{2}} \psi; x \right) + \|f\|_{2\pi} |L_j(f_1;x) - f_1(x)| \\ &= \varepsilon L_j(f_1;x) + \frac{2\|f\|_{2\pi}}{\sin^2 \frac{\delta}{2}} L_j(\psi;x) + \|f\|_{2\pi}|L_j(f_1;x) - f_1(x)| \\ &\leq \varepsilon + (\varepsilon + \|f\|_{2\pi})|L_j(f_1;x) - f_1(x)| + \frac{2\|f\|_{2\pi}}{\sin^2 \frac{\delta}{2}} L_j(\psi;x). \end{aligned}$$

Since

$$L_j(\psi;x) \leq \frac{1}{2}\{|L_j(f_1;x) - f_1(x)| + |\cos x||L_j(f_2;x) - f_2(x)| + |\sin x||L_j(f_3;x) - f_3(x)|\}, \tag{2.8}$$

(see [8], Theorem 4) we obtain

$$|L_j(f;x) - f(x)| < \varepsilon + \left(\varepsilon + \|f\|_{2\pi} + \frac{\|f\|_{2\pi}}{\sin^2 \dfrac{\delta}{2}}\right) \Big\{ |L_j(f_1;x) - f_1(x)|$$

$$+ |L_j(f_2;x) - f_2(x)| + |L_j(f_3;x) - f_3(x)| \Big\}$$

$$\leq \varepsilon + K\{\|L_j f_1 - f_1\|_{2\pi} + \|L_j f_2 - f_2\|_{2\pi} + \|L_j f_3 - f_3\|_{2\pi}\},$$

where

$$K := \varepsilon + \|f\|_{2\pi} + \frac{\|f\|_{2\pi}}{\sin^2 \dfrac{\delta}{2}}.$$

Taking supremum over x, for all $j \in \mathbb{N}$ we obtain

$$\|L_j f - f\|_{2\pi} \leq \varepsilon + K\{\|L_j f_1 - f_1\|_{2\pi} + \|L_j f_2 - f_2\|_{2\pi} + \|L_j f_3 - f_3\|_{2\pi}\}.$$

Consequently, we get

$$\sum_{j=1}^{\infty} a_{kj}^n \|L_j f - f\|_{2\pi} \leq \varepsilon \sum_{j=1}^{\infty} a_{kj}^n + K \sum_{j=1}^{\infty} a_{kj}^n \|L_j f_1 - f_1\|_{2\pi}$$

$$+ K \sum_{j=1}^{\infty} a_{kj}^n \|L_j f_2 - f_2\|_{2\pi} + K \sum_{j=1}^{\infty} a_{kj}^n \|L_j f_3 - f_3\|_{2\pi}.$$

By taking limit as $k \to \infty$ and by using (2.1), (2.3) we obtain the desired result. $\square$

Using the concept of A-statistical convergence, O. Duman and E. Erkuş [6] obtained a Korovkin type approximation theorem by positive linear operators defined on $C_{2\pi}(\mathbb{R}^m)$, the space of all real-valued continuous and 2π-periodic functions on $\mathbb{R}^m$ ($m \in \mathbb{N}$) endowed with the norm $\|\cdot\|_{2\pi}$ of the uniform convergence. The same result stands for $\mathcal{A}$-summability.

Theorem 2.2. *Let $\mathcal{A} = (A^n)_{n \geq 1}$ be a sequence of infinite non-negative real matrices such that*

$$\sup_{n,k} \sum_{j=1}^{\infty} a_{kj}^n < \infty$$

and let $\{L_j\}$ be a sequence of positive linear operators mapping $C_{2\pi}(\mathbb{R}^m)$ into $C_{2\pi}(\mathbb{R}^m)$.

Then, for all $f \in C_{2\pi}(\mathbb{R}^m)$ *we have*

$$\lim_{k\to\infty}\sum_{j=1}^{\infty} a_{kj}^n \|L_j f - f\|_{2\pi} = 0,$$

uniformly in n*, if and only if*

$$\lim_{k\to\infty}\sum_{j=1}^{\infty} a_{kj}^n \|L_j f_p - f_p\|_{2\pi} = 0 \quad (p = 1, 2, \ldots, (2m+1)),$$

uniformly in n*, where* $f_1(t_1, t_2, \ldots, t_m) = 1$*,* $f_p(t_1, t_2, \ldots, t_m) = \cos t_{p-1}$ $(p = 2, 3, \ldots, m+1)$*,* $f_q(t_1, t_2, \ldots, t_m) = \sin t_{q-m-1}$ $(q = m+2, \ldots, 2m+1)$*.*

3. **Particular cases**

Taking $A^n = I$, I being the identity matrix, Theorem 2.1 reduces to Theorem A.

If $A^n = A$, for some matrix A, then $\mathcal{A}$-summability is the ordinary matrix summability by A.

Note that statistical convergence is a regular summability method. Considering Theorem B and our Theorem 2.1 we obtain the next result.

Corollary 3.1. *Let* $\mathcal{A} = (A^n)_{n\in\mathbb{N}}$ *be a sequence of non-negative regular summability matrices and let* $\{L_j\}$ *be a sequence of positive linear operators mapping* $C_{2\pi}(\mathbb{R})$ *into* $C_{2\pi}(\mathbb{R})$*.*

Then, for all $f \in C_{2\pi}(\mathbb{R})$ *we have*

$$st_{A_n} - \lim_{j\to\infty} \|L_j f - f\|_{2\pi} = 0, \text{ uniformly in } n$$

if and only if

$$st_{A_n} - \lim_{j\to\infty} \|L_j f_i - f_i\|_{2\pi} = 0 \quad (i = 1, 2, 3), \text{ uniformly in } n,$$

where $f_1(t) = 1$*,* $f_2(t) = \cos t$*,* $f_3(t) = \sin t$ *for all* $t \in \mathbb{R}$*.*

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