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POLYNOMIAL APPROXIMATION ON THE REAL SEMIAXIS WITH GENERALIZED LAGUERRE WEIGHTS

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Dedicated to Professor D. D. Stancu on his 80^{th} birthday

Abstract. We present a complete collection of results dealing with the polynomial approximation of functions on $(0, +\infty)$.

1. Introduction

This paper is dedicated to the approximation of functions which are defined on $(0, +\infty)$, have singularities in the origin and increase exponentially for $x \to +\infty$. Therefore, it is natural to consider weighted approximation with the generalized Laguerre weight $w_\alpha(x) = x^\alpha e^{-x^\beta}$. We first prove the main polynomial inequalities: "infinite-finite" range inequalities, Remez-type inequalities, Markov-Bernstein and Nikolski inequalities. In Section 2 we introduce a new modulus of continuity, the equivalent $K-$functional and some function spaces. With these tools we prove the Jackson theorem, the Stechkin inequality and estimate the derivatives of the polynomial of best approximation (or "near best approximant" polynomial). We will also prove an embedding theorem between functional spaces. In Section 5, generalizing analogous results proved in [10], we will study the behaviour of Fourier sums and Lagrange polynomials. This paper can be considered as a survey on the topic. However, all the results are new and cover the ones available in the literature.

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2. **Polynomial inequalities**

In this context the main idea is to prove polynomial inequalities with exponential weights on unbounded intervals by using well known polynomial inequalities (eventually with weight) on bounded intervals. To this end the main gradients are the "infinite-finite range inequality" and the approximation of weight by polynomials on a finite interval.

In our case, the weight $w_\alpha(x) = w_{\alpha\beta}(x) = x^\alpha e^{-x^\beta}$ is related, by a quadratic transformation, to the generalized Freud weight $u(x) = |x|^{2\alpha+1}e^{-x^{2\beta}}$.

The Mhaskar-Rakhmanov-Saff number $\bar{a}_m(u)$, related to the weight u, is [9]: $\bar{a}_m(u) \sim m^{1/2\beta}$ where the constant in "$\sim$" depends on α and β and does not depend on m. Then for the weight w_α we have

$$a_m(w) = \bar{a}_{2m}(u)^2 \sim m^{1/\beta} \tag{2.1}$$

and, for an arbitrary polynomial P_m, the following inequalities easily follow:

$$\left(\int_0^\infty |P_m(x)w_{\alpha\beta}(x)|^p dx\right)^{1/p} \le C\left(\int_{\Gamma_m} |P_m(x)w_{\alpha\beta}(x)|^p dx\right)^{1/p}, \tag{2.2}$$

$$\left(\int_{a_m(1+\delta)}^{+\infty} |P_m(x)w_{\alpha\beta}(x)|^p dx\right)^{1/p} \le Ce^{-Am}\left(\int_0^{+\infty} |P_m(x)w_{\alpha\beta}(x)|^p dx\right)^{1/p} \tag{2.3}$$

where $\Gamma_m = [0, a_m(1-k/m^{2/3})]$ (k =const), $p \in (0, +\infty]$, $\beta > \frac{1}{2}$, $\alpha > -\frac{1}{p}$ if $p < +\infty$ and $\alpha \ge 0$ if $p = +\infty$; the constants A and C are independent of m and p and A depend on $\delta > 0$. Then, as a consequence of some results in [5], [11], with $x \in [0, Aa_m]$, $A \ge 1$ fixed, there exist polynomials Q_m such that $Q_m(x) \sim e^{-x^\beta}$ and

$$\frac{\sqrt{a_m}}{m}|\sqrt{x}Q_m'(x)| \le Ce^{-x^\beta}, \tag{2.4}$$

where C and the constants in "$\sim$" are independent of x. Therefore, by using (2.2) and (2.4) and a linear transformation in $[0, 1)$, polynomial inequalities of Bernstein-type, Remez and Shur can be deduced by analogous inequalities in $[0, 1]$ with Jacobi weights x^α.

The next theorems can be proved by using the previous considerations.

With $A>0$ $0<t_1<\ldots<t_r<a_m$ fixed, we put

$$A_m=\left[A\frac{a_m}{m^2},a_m\left(1-\frac{A}{m^{2/3}}\right)\right]\setminus\left(\bigcup_{i=1}^{r}\left[t_i-A\frac{\sqrt{a_m}}{m},t_i+A\frac{\sqrt{a_m}}{m}\right]\right)$$

where m is sufficiently large ($m>m_0$), $r\geq 0$. Let us specify that if $r=0$ then $A_m=\left[A\frac{a_m}{m^2},a_m\left(1-\frac{A}{m^{2/3}}\right)\right]$.

Theorem 2.1. *Let $A,t_1,\ldots,t_r$ be as in the previous definition and $0<p\leq+\infty$. Then, for each polynomial P_m, there exists a constant $C=C(A)$, independent of m, p and P_m, such that*

$$\left(\int_0^{+\infty}|(P_mw_{\alpha\beta})(x)|^pdx\right)^{1/p}\leq C\left(\int_{A_m}|(P_mw_{\alpha\beta})(x)|^pdx\right)^{1/p}. \qquad (2.5)$$

Theorem 2.2. *For each polynomial P_m and $0<p\leq+\infty$ we have*

$$\left(\int_0^{+\infty}|P_m'(x)\sqrt{x}w_{\alpha\beta}(x)|^pdx\right)^{1/p}\leq C\frac{m}{\sqrt{a_m}}\left(\int_0^{+\infty}|P_m(x)w_{\alpha\beta}(x)|^pdx\right)^{1/p} \qquad (2.6)$$

and

$$\left(\int_0^{+\infty}|P_m'(x)w_{\alpha\beta}(x)|^pdx\right)^{1/p}\leq C\left(\frac{m}{\sqrt{a_m}}\right)^2\left(\int_0^{+\infty}|P_m(x)w_{\alpha\beta}(x)|^pdx\right)^{1/p} \qquad (2.7)$$

with $C\neq C(m,p,P_m)$.

As in the Markoff-Bernstein inequalities, we have two versions of Nikolski inequality.

Theorem 2.3. *Let $P_m\in\mathbb{P}_m$ be an arbitrary polynomial and $1\leq q<p\leq+\infty$. Then there exists a constant K, independent of m,p,q and P_m such that, for $\alpha\geq 0$ if $p=+\infty$ and $\alpha>-\frac{1}{p}$ if $p<+\infty$, we have*

$$\|P_mw_{\alpha\beta}\varphi^{\frac{1}{q}}\|_p\leq K\left(\frac{m}{\sqrt{a_m}}\right)^{\frac{1}{q}-\frac{1}{p}}\|P_mw_{\alpha\beta}\|_q, \qquad (2.8)$$

$$\|P_mw_{\alpha\beta}\|_p\leq K\left(\frac{m}{\sqrt{a_m}}\right)^{\frac{2}{q}-\frac{2}{p}}\|P_mw_{\alpha\beta}\|_q, \qquad (2.9)$$

where $\varphi(x)=\sqrt{x}$.

Proof. We first suppose $\alpha \geq 0$ and prove (2.8) with $p = +\infty$ and $1 \leq q < +\infty$. Set $I_x = [x, x + \Delta_m(x)]$, where $x \geq 0$, $\Delta_m(x) = \frac{\sqrt{a_m}}{m}\sqrt{x}$.
From the relation

$$\int_{I_x} P_m(t)dt = P_m(x)\Delta_m(x) + \int_{I_x} P'_m(t)\,(x + \Delta_m(x) - t)\,dt,$$

(by using Hölder inequality for $q > 1$) we get for $q \geq 1$:

$$|P_m(x)\varphi(x)^{\frac{1}{q}}| \leq \left(\frac{m}{\sqrt{a_m}}\right)^{1/q}\left[\left(\int_{I_x} |P_m(t)|^q\,dt\right)^{1/q} + \frac{\sqrt{a_m}}{m}\left(\int_{I_x} |P'_m(t)\varphi(t)|^q\,dt\right)^{1/q}\right]. \tag{2.10}$$

Since $w_{\alpha\beta}(x) \sim w_{\alpha\beta}(t)$ for $t \in I_x$, $\alpha \geq 0$ it also holds

$$\begin{aligned} \left|P_m(x)w_{\alpha\beta}(x)\varphi(x)^{1/q}\right| &\leq C\left(\frac{m}{\sqrt{a_m}}\right)^{1/q}\left[\left(\int_{I_x} |P_m(t)w_{\alpha\beta}(t)|^q\,dt\right)^{1/q}\right. \qquad (2.11)\\ &+ \left.\frac{\sqrt{a_m}}{m}\left(\int_{I_x} |P'_m(t)\varphi(t)w_{\alpha\beta}(t)|^q\,dt\right)^{1/q}\right]. \end{aligned}$$

By extending the integrals to $(0, +\infty)$ and by using Bernstein inequality we deduce:

$$\left\|P_m w_{\alpha\beta}\varphi^{\frac{1}{q}}\right\|_\infty \leq K\left(\frac{m}{\sqrt{a_m}}\right)^{1/q} \|P_m w_{\alpha\beta}\|_q\,. \tag{2.12}$$

Moreover, using (2.5) with $r = 0$ and $A = 1$, one has

$$\begin{aligned} \|P_m w_{\alpha\beta}\|_\infty &\leq C\left\|P_m w_{\alpha\beta}\varphi^{1/q}\varphi^{-1/q}\right\|_{L^\infty([\frac{a_m}{m^2},\infty))}\\ &\leq C\left(\frac{m}{\sqrt{a_m}}\right)^{1/q}\left\|P_m w_{\alpha\beta}\varphi^{1/q}\right\|_\infty. \end{aligned}$$

Then from (2.12) it follows

$$\|P_m w_{\alpha\beta}\|_\infty \leq K\left(\frac{m}{\sqrt{a_m}}\right)^{2/q} \|P_m w_{\alpha\beta}\|_q\,. \tag{2.13}$$

Then (2.8) and (2.9) are true with $\alpha \geq 0$, $p = +\infty$, $1 \leq q < +\infty$.
When $\alpha \geq 0$ and $1 \leq q < p < +\infty$, then to prove (2.9), we write

$$\begin{aligned} \|P_m w_{\alpha\beta}\|_p^p &= \left\| |P_m w_{\alpha\beta}|^{p-q} \, |P_m w_{\alpha\beta}|^q \right\|_1 \\ &\leq \|P_m w_{\alpha\beta}\|_\infty^{p-q} \int_0^{+\infty} |P_m w_{\alpha\beta}|^q \, (x) dx \leq \\ &\leq K^{p-q} \left(\frac{m}{\sqrt{a_m}} \right)^{(p-q)\frac{2}{q}} \|P_m w_{\alpha\beta}\|_q^{p-q} \, \|P_m w_{\alpha\beta}\|_q^q \, , \end{aligned}$$

from which

$$\|P_m w_{\alpha\beta}\|_p \leq K \left(\frac{m}{\sqrt{a_m}} \right)^{2\left(\frac{1}{q} - \frac{1}{p}\right)} \|P_m w_{\alpha\beta}\|_q$$

i.e. (2.9) with $\alpha \geq 0$. In an analogous way we can prove(2.8).
Let us suppose now $1 \leq q < p < +\infty$ and $-\frac{1}{p} < \alpha < 0$. From Theorem 2.1 we get

$$\|P_m w_{\alpha\beta}\|_p \sim \|P_m w_{\alpha\beta}\|_{L^p\left(\frac{a_m}{m^2}, a_m\right)} .$$

In the interval $\left[\frac{a_m}{m^2}, a_m\right]$ we can construct a polynomial Q_{lm} (with l a fixed integer) for which it holds $Q_{lm} \sim x^\alpha$ (see [8] in $[-1, 1]$) and

$$\|P_m w_{0\beta}\|_p \sim \|(P_m Q_{lm}) w_{0\beta}\|_{L^p\left(\frac{a_m}{m^2}, a_m\right)} \leq C \, \|(P_m Q_{lm}) w_{0\beta}\|_p \, .$$

Then we can use (2.9) with $\alpha = 0$, $P_m Q_{lm}$ instead of P_m and, finally, Theorem 2.1 to replace Q_{lm} by x^α.
Relation (2.8) can be proved in a similar way and the proof is complete. □

3. **Function spaces, modulus of continuity and K-functionals**

With $w_{\alpha\beta}(x) = x^\alpha e^{-x^\beta}$ and $1 \leq p < +\infty$ we denote by $L^p_{w_{\alpha\beta}}$ the set of all measurable functions such that

$$\|f w_{\alpha\beta}\|_p^p = \int_0^{+\infty} |f w_{\alpha\beta}|^p (x) dx < +\infty, \quad \alpha > -\frac{1}{p}.$$

If $p = +\infty$ we define

$$L^\infty_{w_{\alpha\beta}} = \{ f \in C^0[(0, +\infty)] : \lim_{x \to 0, x \to +\infty} (f w_{\alpha\beta})(x) = 0 \}, \quad \alpha > 0$$

and

$$L^\infty_{w_{0\beta}} = \{ f \in C^0[[0, +\infty)] : \lim_{x \to +\infty} (f w_{\alpha\beta})(x) = 0 \},$$

where $C^0(A)$ is the set of all continuous functions in $A \subseteq [0,+\infty)$.

For more regular functions we introduce the Sobolev-type space

$$W_r^p = W_r^p(w_{\alpha\beta}) = \{f \in L^p_{w_{\alpha\beta}} : f^{(r-1)} \in AC((0,+\infty)) \text{ and } \|f^{(r)}\varphi^r w_{\alpha\beta}\|_p < +\infty\}$$

where $r \geq 1$, $1 \leq p \leq +\infty$, $\varphi(x) = \sqrt{x}$ and $AC(A)$ is the set of absolutely continuous functions in $A \subseteq [0,+\infty)$.

In order to define in $L^p_{w_{\alpha\beta}}$ a modulus of smoothness, for every $h > 0$ we introduce the quantity $h^* = \frac{1}{h^{\frac{2}{2\beta-1}}}$, $\beta > \frac{1}{2}$ and the segment $I_{rh} = [8r^2h^2, Ah^*]$ where A is a fixed positive constant.

Then, following [3] (see also [1]), we define

$$\Omega^r_\varphi(f,t)_{w_{\alpha\beta},p} = \sup_{0<h\leq t} \|(\Delta^r_{h\varphi} f) w_{\alpha\beta}\|_{L^p(I_{rh})} \tag{3.1}$$

as the main part of the modulus of continuity, where $r \geq 1$, $1 \leq p \leq +\infty$, $\Delta^r_{h\varphi} f(x) = \sum_{k=0}^{r} (-1)^k \begin{pmatrix} r \\ k \end{pmatrix} f(x+(r-k)h\sqrt{x})$. The complete modulus of continuity ω^r_φ is defined by

$$\begin{aligned} \omega^r_\varphi(f,t)_{w_{\alpha\beta},p} \quad = \quad & \inf_{P\in\mathbb{P}_{r-1}} \|(f-P)w_{\alpha\beta}\|_{L^p([0,8r^2t^2])} + \Omega^r_\varphi(f,t)_{w_{\alpha\beta},p} + \\ + \quad & \inf_{P\in\mathbb{P}_{r-1}} \|(f-P)w_{\alpha\beta}\|_{L^p(At^*,\infty)}. \end{aligned} \tag{3.2}$$

Connected with the modulus of continuity ω^r_φ is the K-functional

$$K(f,t^r)_{\omega_{\alpha\beta},p} = \inf_{g\in W^p_r} \{\|(f-g)w_{\alpha\beta}\|_p + t^r \|g^{(r)}\varphi^r w_{\alpha\beta}\|_p\} \tag{3.3}$$

where $r \geq 1$ and $1 \leq p \leq +\infty$, $0 < t < 1$.

In some contexts it is useful to define the main part of the previous K- functional

$$\tilde{K}(f,t^r)_{\omega_{\alpha\beta},p} = \sup_{0<h\leq t} \inf_{g\in W^p_r} \{\|(f-g)w_{\alpha\beta}\|_{L^p(I_{rh})} + h^r \|g^{(r)}\varphi^r w_{\alpha\beta}\|_{L^p(I_{rh})}\} \tag{3.4}$$

In fact the following theorem holds

Theorem 3.1. *Let $f \in L^p_{w_{\alpha\beta}}$ and $1 \leq p \leq +\infty$. Then, as $t \to 0$, we have*

$$\omega^r_\varphi(f,t)_{w_{\alpha\beta},p} \sim K(f,t^r)_{w_{\alpha\beta},p} \tag{3.5}$$

and

$$\Omega^r_\varphi(f,t)_{w_{\alpha\beta},p} \sim \tilde{K}(f,t^r)_{w_{\alpha\beta},p} \tag{3.6}$$

where the constants in "$\sim$" are independent of f and t.

The proof of this theorem is similar to the proof in [1] and later we will prove some crucial steps.

It is useful to observe that, by (3.6) and (3.4) with $g = f$, it follows

$$\Omega_\varphi^r(f,t)_{w_{\alpha\beta},p} \le C \inf_{0<h\le t} h^r \|f^{(r)}\varphi^r w_{\alpha\beta}\|_{L^p(I_{rh})};$$

this last relation allows us to evaluate the main part of the modulus of continuity of differentiable functions in $(0,+\infty)$. For example, for $f(x) = |\log x|$ we have $\Omega_\varphi^r(f,t)_{w_{\alpha\beta},1} \sim t^{2+2\alpha}$.

Now, as in the case of periodic functions or of functions defined on finite intervals, we can define the Besov-type spaces $B_{sq}^p(w_{\alpha\beta})$ by means of modulus of continuity. To this end, with $1 \le p \le +\infty$, we introduce the seminorms

$$\|f\|_{p,q,s} = \begin{cases} \left(\displaystyle\int_0^{1/k} \left[\frac{\omega_\varphi^k(f,t)_{w_{\alpha\beta},p}}{t^{s+1/q}}\right]^q dt\right)^{1/q}, & 1 \le q < +\infty, \quad k > s \\ \displaystyle\sup_{t>0} \frac{\omega_\varphi^k(f,t)_{w_{\alpha\beta},p}}{t^s}, & q = +\infty, \quad k > s \end{cases} \tag{3.7}$$

and define

$$B_{sq}^p = B_{sq}^p(w_{\alpha\beta}) = \{f \in L_{w_{\alpha\beta}}^p : \|f\|_{p,q,s} < +\infty\}$$

equipped with the norm $\|f\|_{B_{sq}^p(w_{\alpha\beta})} = \|fw_{\alpha\beta}\|_p + \|f\|_{p,q,s}$. Here we cannot study these spaces in details. In the next section we will prove some embedding theorems and will characterize the Besov spaces by the error of the best approximation.

4. **Polynomial approximation**

For each function $f \in L_{w_{\alpha\beta}}^p$ with $1 \le p \le +\infty$, $\beta > \frac{1}{2}$, $\alpha > -\frac{1}{p}$ if $p < +\infty$ and $\alpha \ge 0$ if $p = +\infty$, we define, as usual,the error of best approximation

$$E_m(f)_{w_{\alpha\beta},p} = \inf_{P \in \mathbb{P}_{m-1}} \|(f-P)w_{\alpha\beta}\|_p.$$

In this section we will estimate $E_m(f)_{w_{\alpha\beta},p}$ by means of the modulus of continuity and will characterize the classes functions defined in the previous section.

In order to establish a Jakson theorem it is necessary the following

Proposition 4.1. *For each function $f \in W_1^p(w_{\alpha\beta})$, $1 \le p \le +\infty$, we have*

$$E_m(f)_{w_{\alpha\beta},p} \le C\frac{\sqrt{a_m}}{m}\|f'\varphi w_{\alpha\beta}\|_p, \tag{4.1}$$

where $\varphi(x) = \sqrt{x}$, $C \ne C(m,f)$ and $a_m \sim m^{1/\beta}$.

Proof. We first prove that the condition

$$\left(\int_0^{+\infty} |f'(x)e^{-x^\beta}|^p dx\right)^{1/p} < +\infty \tag{4.2}$$

implies the estimate

$$E_m(f)_{w_{\alpha\beta},p} \le C\frac{\sqrt{a_m}}{m}\left(\int_0^{+\infty}\left|f'(x)\left(x+\frac{a_m}{m^2}\right)^{\alpha+\frac{1}{2}}e^{-x^\beta}\right|^p dx\right)^{1/p}. \tag{4.3}$$

To this end, let $1 \le p < +\infty$, $u(x) = |x|^{2\alpha+1/p}e^{-x^{2\beta}}$, $g(x) = f(x^2)$, $x \in \mathbb{R}$ and p_{2m} the best approximation of g. By using Theorem 2.1 in [9] we have

$$\begin{aligned} A \ &:= \left(\int_{-\infty}^{+\infty}|(g(x)-p_{2m}(x))u(x)|^p dx\right)^{1/p} \le \\ &\le C\frac{\bar a_{2m}}{2m}\left(\int_{-\infty}^{+\infty}\left|g'(x)\left(|x|+\frac{\bar a_{2m}}{2m}\right)^{2\alpha+\frac{1}{p}}e^{-x^{2\beta}}\right|^p dx\right)^{1/p} =: B \end{aligned}$$

where $\bar a_{2m} = \bar a_{2m}(u) \sim m^{\frac{1}{2\beta}}$ is the M-R-S number related to the weight u and as we first observed $\bar a_{2m} \sim \sqrt{a_m(w_{\alpha\beta})}$. Then a change of variables in A and B leads to (4.3).

Now we suppose $f \in W_1^p(w_{\alpha\beta})$ and we introduce the function

$$f_m(x) = \begin{cases} f\left(\frac{a_m}{m^2}\right) & x \in \left[0, \frac{a_m}{m^2}\right] \\ f(x) & x \ge \frac{a_m}{m^2} \end{cases}.$$

Obviously the condition $\|f_m' e^{-x^\beta}\|_p < +\infty$ is satisfied, (4.3) can be used and we easily deduce

$$E_m(f_m)_{w_{\alpha\beta},p} \le C\frac{\sqrt{a_m}}{m}\|f'\varphi w_{\alpha\beta}\|_{L^p\left(\left[\frac{a_m}{m^2},\infty\right)\right)}. \tag{4.4}$$

Then, since $E_m(f)_{w_{\alpha\beta},p} \le \|(f-f_m)w_{\alpha\beta}\|_p + E_m(f_m)_{w_{\alpha\beta},p}$, we have to estimate only the $L^p_{w_{\alpha\beta}}$-norm of $f - f_m$.

To this end, we put $x_0 = \frac{a_m}{m^2}$ and get

$$\|(f-f_m)w_{\alpha\beta}\|_p = \left(\int_0^{x_0}|[f(x)-f(x_0)]w_{\alpha\beta}(x)|^p dx\right)^{1/p}$$

$$= \left(\int_0^{x_0} \left| \int_0^{x_0} (t-x)_+^0 f'(t) w_{\alpha\beta}(x) dt \right|^p dx \right)^{1/p}$$

$$\leq \int_0^{x_0} |f'(t)| \left(\int_0^{x_0} (t-x)_+^0 w_{\alpha\beta}^p(x) dx \right)^{1/p} dt$$

$$= \int_0^{x_0} |f'(t)| \left(\int_0^{t} w_{\alpha\beta}^p(x) dx \right)^{1/p} dt \sim \int_0^{x_0} |f'(t)| t^{\alpha+\frac{1}{p}} e^{-t^\beta} dt \leq$$

$$\leq C \|f' \varphi w_{\alpha\beta}\|_{L^p((0,x_0))} \left(\int_0^{x_0} t^{q(1/p-1/2)} dt \right)^{1/q} \sim \frac{\sqrt{a_m}}{m} \|f' \varphi w_{\alpha\beta}\|_{L^p\left(0,\frac{a_m}{m^2}\right)},$$

which, with (4.4), proves (4.1) when $1 \leq p < +\infty$. The case $p = +\infty$ is similar and (4.1) is proved.

By iterating (4.1) we have, for each $g \in W_r^p(w_{\alpha\beta})$, the estimate

$$E_m(g)_{w_{\alpha\beta},p} \leq C \left(\frac{\sqrt{a_m}}{m} \right)^r \|g^{(r)} \varphi^r w_{\alpha\beta}\|_p, \quad C \neq C(m,f),$$

from which, by using the K-functional and its equivalence with ω_φ^r, the Jackson theorem follows. □

Theorem 4.2. *For all $f \in L^p_{w_{\alpha\beta}}$, $1 \leq p \leq +\infty$ and $r < m$ we have*

$$E_m(f)_{w_{\alpha\beta}} \leq C \omega_\varphi^r \left(f, \frac{\sqrt{a_m}}{m} \right)_{w_{\alpha\beta},p}, \quad C \neq C(f,m). \tag{4.5}$$

By using the K-functional and the Bernstein inequality, in a usual way we obtain the Stechkin inequality formulated in the following theorem

Theorem 4.3. *For each $f \in L^p_{w_{\alpha\beta}}$, $1 \leq p \leq +\infty$, and an arbitrary integer $r \geq 1$ we have:*

$$\omega_\varphi^r \left(f, \frac{\sqrt{a_m}}{m} \right)_{w_{\alpha\beta},p} \leq C \left(\frac{\sqrt{a_m}}{m} \right)^r \sum_{k=0}^{m} \left(\frac{1+k}{\sqrt{a_k}} \right)^r \frac{E_k(f)_{w_{\alpha\beta},p}}{1+k} \tag{4.6}$$

with $C = C(r)$ independent of m and f.

By proceeding as in [1], Lemma 3.5 (see also [3], p. 94-95) it is not difficult to show that, setting

$$\widetilde{E}_m(f)_{w_{\alpha\beta},p} = \inf_{P_m} \|(f - P_m) w_{\alpha\beta}\|_{L^p\left(\frac{a_m}{m^2}, a_m\right)}, \quad 1 \leq p \leq +\infty,$$

if $t^{-1} \Omega_\varphi^r(f,t)_{w_{\alpha\beta},p} \in L^1$, it results

$$\widetilde{E}_m(f)_{w_{\alpha\beta},p} \leq C \Omega_\varphi^r \left(f, \frac{\sqrt{a_m}}{m} \right)_{w_{\alpha\beta},p}. \tag{4.7}$$

From this last result the next theorem easily follows.

Theorem 4.4. *For each function $f \in L^p_{w_{\alpha\beta}}$, $1 \le p \le +\infty$, we have*

$$E_m(f)_{w_{\alpha\beta},p} \le C \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi^k(f,t)_{w_{\alpha\beta},p}}{t} dt \tag{4.8}$$

where $C \ne C(m,f)$ and $k < m$.

Recall that the main part of the modulus Ω_φ^k is smaller than ω_φ^k and generally the two moduli are not equivalent. Moreover if, for some p, $\Omega_\varphi^k(f,t)_{w_{\alpha\beta},p} \sim t^\lambda$, $0 < \lambda < k$, then by (4.8), we have $E_m(f)_{w_{\alpha\beta},p} \sim \left(\frac{\sqrt{a_m}}{m}\right)^\lambda$ and, by using (4.6), also $\omega_\varphi^k(f,t)_{w_{\alpha\beta},p} \sim t^\lambda$. Then for these classes of functions the two moduli are equivalent. By using Jackson and Stechkin inequalities we can represent the seminorms of the Besov spaces in (3.7) by means of the error of best approximation (see, for instance, [3]). In fact, for $1 \le p \le +\infty$, the following equivalences hold:

$$\|f\|_{pqs} \sim \left(\sum_{k=1}^{+\infty} k^{\left(1-\frac{1}{2\beta}\right)sq-1} E_k(f)^q_{w_{\alpha\beta},p} \right)^{1/q}, \quad 1 \le q < +\infty$$

$$\|f\|_{pqs} \sim \sup_{m\ge 1} m^{\left(1-\frac{1}{2\beta}\right)s} E_m(f)_{w_{\alpha\beta},p}, \quad q = +\infty.$$

The next theorem is useful in more contexts.

Theorem 4.5. *For each $f \in L^p_{w_{\alpha\beta}}, 1 \le p \le +\infty$, we have*

$$\omega_\varphi^r \left(f, \frac{\sqrt{a_m}}{m}\right)_{w_{\alpha\beta},p} \sim \inf_{P \in I\!P_m} \left\{ \|(f-P)w_{\alpha\beta}\|_p + \left(\frac{\sqrt{a_m}}{m}\right)^r \|P^{(r)}\varphi^r w_{\alpha\beta}\|_p \right\} \tag{4.9}$$

where the constants in "$\sim$" are independent of m and f.

A consequence of formula (4.9) is the useful inequality

$$\left(\frac{\sqrt{a_m}}{m}\right)^r \left\| P_m^{(r)} \varphi^r w_{\alpha\beta} \right\|_p \le C \omega_\varphi^r \left(f, \frac{\sqrt{a_m}}{m}\right)_{w_{\alpha\beta},p}, \tag{4.10}$$

being P_m the polynomial of quasi best approximation, i.e.

$$\|(f-P_m)\,w_{\alpha\beta}\|_p \le C E_m(f)_{w_{\alpha\beta},p}.$$

For the proof of Theorem 4.5 the reader can use the same tool in [1] with some small changes.

Now we will show some embedding theorems which connect different function norms and moduli of smoothness. For different classes of functions the reader can consult [2].

In the sequel, to simplify the notations, we will set $w = w_{\alpha\beta}$ with $\alpha \geq 0$.

Theorem 4.6. *Let $f \in L^p_w$, $1 \leq p < +\infty$ and let us assume that the condition*

$$\int_0^1 \frac{\Omega^r_\varphi(f,t)_{w,p}}{t^{1+1/p}} dt < +\infty \tag{4.11}$$

is satisfied. Then f is a continuous function in any interval $[a, +\infty)$, $a > 0$. Moreover, if, with $\widetilde{w} = w/\varphi^{1/p}$, and

$$\int_0^1 \frac{\Omega^r_\varphi(f,t)_{\widetilde{w},p}}{t^{1+1/p}} dt < +\infty \tag{4.12}$$

then we have

$$\left.\begin{array}{c} E_m(f)_{w,\infty} \\ \Omega^r_\varphi\left(f, \frac{\sqrt{a_m}}{m}\right)_{w,\infty} \end{array}\right\} \leq C \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega^r_\varphi(f,t)_{\widetilde{w},p}}{t^{1+1/p}} dt \tag{4.13}$$

and

$$\|fw\|_\infty \leq C \left(\|f\widetilde{w}\|_p + \int_0^1 \frac{\Omega^r_\varphi(f,t)_{\widetilde{w},p}}{t^{1+1/p}} dt \right). \tag{4.14}$$

Finally (4.12) implies (4.13) and (4.14) with w in place of $\widetilde{w}$ and $\frac{2}{p}$ in place of $\frac{1}{p}$. Here the positive constants C are independent of m, t and f.

Proof. In virtue of (4.8), (4.11) implies, for $1 \leq p < +\infty$, $\lim_m E_m(f)_{w,p} = 0$. Therefore, if P_m denotes the polynomial of best approximation (or quasi best approximation) in L^p_w, the equality

$$w(f - P_m) = \sum_{k=0}^{+\infty} (P_{2^{k+1}m} - P_{2^k m})\, w \tag{4.15}$$

is true a.e. in $(0, +\infty)$. If we prove that the series uniformly converges on each half-line $[a, +\infty)$, $a > 0$, then the equality holds everywhere in $[a, +\infty)$ and f is continuous.

Now, by using (2.8), with $p = +\infty$ and $q = p$, one has

$$\|(P_{2^{k+1}m} - P_{2^k m})\, w\|_{L^\infty([a,+\infty))} \leq a^{-\frac{1}{2p}} \left\|(P_{2^{k+1}m} - P_{2^k m})\, w\varphi^{\frac{1}{p}}\right\|_{L^\infty((a,+\infty))}$$

$$\leq a^{-\frac{1}{2p}} K \left(\frac{2^{k+1}m}{\sqrt{a_{2^{k+1}m}}} \right)^{1/p} \|(P_{2^{k+1}m} - P_{2^k m})\, w\|_p$$

$$\leq a^{-\frac{1}{2p}} KC \left(\frac{2^{k+1}m}{\sqrt{a_{2^{k+1}m}}} \right)^{1/p} \|(P_{2^{k+1}m} - P_{2^k m})\, w\|_{L^p(I_{mk})}$$

having used (2.5) in the last inequality and setting $I_{mk} = \left[\frac{a_{2^{k+1}m}}{(2^{k+1}m)^2}, a_{2^{k+1}m} \right]$. Consequently one has, for (4.7),

$$\begin{aligned} \|(P_{2^{k+1}m} - P_{2^k m})\, w\|_{L^\infty([a,+\infty))} &\leq C \left(\frac{2^k m}{\sqrt{a_{2^k m}}} \right)^{1/p} \widetilde{E}_{2^k m}(f)_{w,p} \\ &\leq C \left(\frac{2^k m}{\sqrt{a_{2^k m}}} \right)^{1/p} \Omega_\varphi^r \left(f, \frac{\sqrt{a_{2^k m}}}{2^k m} \right)_{w,p} \end{aligned}$$

and

$$\begin{aligned} \sum_{k=0}^{+\infty} \|(P_{2^{k+1}m} - P_{2^k m})\, w\|_{L^\infty([a,+\infty))} &\leq C \sum_{k=0}^{+\infty} \left(\frac{2^k m}{\sqrt{a_{2^k m}}} \right)^{1/p} \Omega_\varphi^r \left(f, \frac{\sqrt{a_{2^k m}}}{2^k m} \right)_{w,p} \\ &\leq C \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi^r(f,t)_{w,p}}{t^{1+1/p}} dt < +\infty. \end{aligned}$$

Then the series in (4.15) absolutely and uniformly converges and the equality in (4.15) is true everywhere in $[a, +\infty)$.

To prove the first relation of (4.13) we use (2.8) in an equivalent form and with the previous notations we obtain

$$\begin{aligned} \|(P_{2^{k+1}m} - P_{2^k m})\, w\|_\infty &\leq K \left(\frac{2^{k+1}m}{\sqrt{a_{2^{k+1}m}}} \right)^{1/p} \left\| (P_{2^{k+1}m} - P_{2^k m})\, w/\varphi^{1/p} \right\|_p \\ &\leq K \left(\frac{2^{k+1}m}{\sqrt{a_{2^{k+1}m}}} \right)^{1/p} \widetilde{E}_{2^k m}(f)_{\widetilde{w},p} \\ &\leq C \left(\frac{2^{k+1}m}{\sqrt{a_{2^{k+1}m}}} \right)^{1/p} \Omega_\varphi^r \left(f, \frac{\sqrt{a_{2^k m}}}{2^k m} \right)_{\widetilde{w},p}. \end{aligned}$$

It follows

$$\begin{aligned} \|(f - P_m)w\|_\infty &\leq \lim_k \|(P_{2^{k+1}m} - P_{2^k m})\, w\|_\infty = \lim_k \left\| \sum_{i=0}^{k} (P_{2^{i+1}m} - P_{2^i m})\, w \right\|_\infty \\ &\leq \sum_{i=0}^{+\infty} \|(P_{2^{i+1}m} - P_{2^i m})\, w\|_\infty \leq C \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi^r(f,t)_{\widetilde{w},p}}{t^{1+1/p}} dt. \end{aligned}$$

To prove the second estimate in (4.13) we observe that, with P_m as the polynomial of best approximation in $L^p_{\widetilde{w}}$, we have

$$\begin{aligned}\Omega^r_\varphi\left(f,\frac{\sqrt{a_m}}{m}\right)_{w,\infty} &\leq C\left[\|(f-P_m)w\|_\infty+\left(\frac{\sqrt{a_m}}{m}\right)^r\left\|P_m^{(r)}\varphi^r w\right\|_\infty\right]\\ &\leq C\left[E_m(f)_{w,\infty}+\left(\frac{\sqrt{a_m}}{m}\right)^r\left\|P_m^{(r)}\varphi^r\widetilde{w}\right\|_p\left(\frac{m}{\sqrt{a_m}}\right)^{1/p}\right].\end{aligned}$$

Now for the first term let us use the first estimate of (4.13). The second term, by proceeding as in [3], p.99-100 (see also [1]) is dominated by

$$C\left(\frac{m}{\sqrt{a_m}}\right)^{1/p}\int_0^{\frac{\sqrt{a_m}}{m}}\frac{\Omega^r_\varphi(f,t)_{\widetilde{w},p}}{t}dt.\leq C\int_0^{\frac{\sqrt{a_m}}{m}}\frac{\Omega^r_\varphi(f,t)_{\widetilde{w},p}}{t^{1+1/p}}dt.$$

Then the second estimate in (4.13) follows.
Finally to prove (4.14) we write

$$\|fw\|_\infty\leq\|(f-P_1)w\|_\infty+\|P_1w\|_\infty$$

with P_1 as best approximation in $L^p_{\widetilde{w}}$. Since

$$\|P_1w\|_\infty\leq KP_1\widetilde{w}\|_p\leq 2K\|f\widetilde{w}\|_p,$$

for the first term we use the first estimate of (4.13) with $m=1$.
To show the last part of the theorem we proceed as in the proof of (4.13), using inequality (2.9) in place of (2.8). □

5. **Fourier Sum and Lagrange Polynomial**

The approximation of functions by means of their Fourier sums in the system $\{p_m(w_\alpha)\}_m$, where $p_m(w_\alpha,x)=\gamma_m x^m+\cdots,\gamma_m>0$, and

$$\int_0^{+\infty}p_m(w_\alpha,x)p_n(w_\alpha,x)w_\alpha(x)dx=\delta_{mn},$$

is useful in different contexts. Moreover, the weighted Lagrange interpolation based on the zeros of $p_m(w_\alpha,x)$ is useful in numerous problems of numerical analysis, too. We will consider these two approximation processes in the space L^p_u, where $u(x)=x^\gamma e^{-\frac{x^\beta}{2}}$ and $1\leq p\leq+\infty$.

5.1. **Fourier Sums.** For $f \in L^p_u$, the m-th Fourier sum $S_m(w_\alpha, f)$ is defined as follows

$$S_m(w_\alpha, f) = \sum_{k=0}^{m-1} c_k p_k(w_\alpha),$$

where

$$c_k = \int_0^{+\infty} f(t) p_k(w_\alpha, t) w_\alpha(t) dt.$$

Analogously to the cases of Laguerre, Hermite and Freud polynomials (see [10]) the uniform boundedness of $S_m(w_\alpha)$ in L^p_u holds true for $p \in \left(\frac{4}{3}, 4\right)$ and then for a restricted class of functions. This fact leads to modify the polynomial $S_m(w_\alpha, f)$ following a procedure used in [7][6][10] that we will briefly illustrate. Let $a_m := a_m(u)$ be the M-R-S number related to the weight u. Let $\theta \in (0,1), M = \left\lfloor \frac{m\theta}{1+\theta} \right\rfloor \sim m$ and let $\bar{\Delta}_{\theta m}$ be the characteristic function of the segment $[0, \theta a_m]$. Then, using (2.3) with u in place of $w_{\alpha\beta}$, for every $f \in L^p_u$, we get

$$\|f(1 - \bar{\Delta}_{\theta m})u\|_p \le \mathcal{C} \left(E_M(f)_{u,p} + e^{-Am} \|fu\|_p \right) \tag{5.1}$$

and

$$\|fu\|_p \le \mathcal{C} \left(\|f \bar{\Delta}_{\theta m} u\|_p + E_M(f)_{u,p} \right), \tag{5.2}$$

where $1 \le p \le +\infty$ and $E_M(f)_{u,p}$ is the error of best approximation of f in $\mathbb{P}_M$. Therefore, by (5.2), it is sufficient to approximate the function f in the more restricted interval $[0, \theta a_m]$ or, equivalently, to replace $\{S_m(w_\alpha, f)\}_m$ with the sequence $\{\Delta_{\theta m} S_m(w_\alpha, f\Delta_{\theta m})\}_m$, where $a_m = a_m(w_\alpha)$ and $\Delta_{\theta m}$ is the characteristic function of $[0, \theta a_m]$ with $\theta \in (0,1)$ arbitrary. The theorems that follow show that this procedure is convenient.

Theorem 5.1. *Let $u \in L^p$ with $1 < p < +\infty$. Then, for every $f \in L^p_u$ there exists a constant $\mathcal{C} \ne \mathcal{C}(m, f)$ such that*

$$\|S_m(w_\alpha, \Delta_{\theta m} f) \Delta_{\theta m} u\|_p \le \mathcal{C} \|f \Delta_{\theta m} u\|_p \tag{5.3}$$

if and only if

$$\frac{v^\gamma}{\sqrt{v^\alpha \varphi}} \in L^p(0,1) \quad \textit{and} \quad \sqrt{\frac{v^\alpha}{\varphi}} \frac{1}{v^\gamma} \in L^q(0,1), \tag{5.4}$$

where $v^\rho(x) = x^\rho, \varphi(x) = \sqrt{x}$ *and* $p^{-1} + q^{-1} = 1$. *Moreover, under the conditions* (5.4), (5.3) *is equivalent to*

$$\|[f - \Delta_{\theta m} S_m(w_\alpha, \Delta_{\theta m} f)]u\|_p \leq \mathcal{C}\left(E_M(f)_{u,p} + e^{-Am}\|fu\|_p\right), \tag{5.5}$$

where A *and* $\mathcal{C}$ *are positive constant independent of* m *and* f.

As an example, if $f \in W_r^p(u), r \geq 1$, and (5.4) holds true, we have

$$\|[f - \Delta_{\theta m} S_m(w_\alpha, \Delta_{\theta m} f)]u\|_p \leq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r \|f\|_{W_r^p(u)},$$

i.e. the error of best approximation of functions belonging to $W_r^p(u)$. If $w_\alpha(x) = x^\alpha e^{-x}$ and $u(x) = x^\gamma e^{-\frac{x}{2}}$ (Laguerre case), then Theorem 5.1 is equivalent to Theorem 2.2 in [10]. Moreover, as in the Laguerre case, if (5.4) holds true with $1 < p < 4$ then we get the estimate

$$\|S_m(w_\alpha, \Delta_{\theta m} f)\Delta_{\theta m} u\|_p \leq \mathcal{C}\|f\Delta_{\theta m} u\|_p \tag{5.6}$$

and if (5.4) holds true with $p > \frac{4}{3}$ then it results

$$\|S_m(w_\alpha, f)\Delta_{\theta m} u\|_p \leq \mathcal{C}\|fu\|_p. \tag{5.7}$$

Moreover, we have

$$\|S_m(w_\alpha, f)u\|_p \leq \mathcal{C}\|fu\|_p, \tag{5.8}$$

$$\|S_m(w_\alpha, f)u\|_p \leq \mathcal{C}\begin{cases} m^{\frac{1}{3}}\|fu\|_p \\ \\ \|fu(1+\cdot^3)\|_p \end{cases} \tag{5.9}$$

if (5.4) is satisfied with $p \in \left(\frac{4}{3}, 4\right)$ or $p \in (1, +\infty) \setminus \left[\frac{4}{3}, 4\right]$ respectively. The cases $p = 1$ or $p = +\infty$ are considered in the following theorems.

Theorem 5.2. *Let* f *be such that*

$$\int_0^{+\infty} |f(x)u(x)| \log^+ |f(x)| < +\infty,$$

with

$$\log^+ |z| = \begin{cases} 0 & \text{if } |z| \leq 1 \\ \\ \log|z| & \text{if } z > 1. \end{cases}$$

If

$$\frac{v^\gamma}{\sqrt{v^\alpha \varphi}} \in L^1 \quad \text{and} \quad \sqrt{\frac{v^\alpha}{\varphi}}\frac{1}{v^\gamma} \in L^\infty, \quad v^\rho(x) = x^\rho, \quad \varphi(x) = \sqrt{x}, \tag{5.10}$$

then we have

$$\|S_m(w_\alpha, \Delta_{\theta m} f) u \Delta_{\theta m}\|_1 \le \mathcal{C} \left[1 + \int_0^{+\infty} |fu|(x)(1 + \log^+ |f(x)| + \log^+ x) dx \right],$$

with $\mathcal{C} \neq \mathcal{C}(m, f)$.

Theorem 5.3. *Let* $f \in L_u^\infty, u(x) = x^\gamma e^{-\frac{x^\beta}{2}}, \beta > \frac{1}{2}, \gamma \ge 0$. *If* $\frac{\alpha}{2} + \frac{1}{4} \le \gamma \le \frac{\alpha}{2} + \frac{3}{4}$, *then we have*

$$\|S_m(w_\alpha, \Delta_{\theta m} f) u \Delta_{\theta m}\|_\infty \le \mathcal{C} \|f \Delta_{\theta m} u\|_\infty (\log m),$$

where $\mathcal{C} \neq \mathcal{C}(m, f)$.

Theorems 5.1 and 5.2 and estimates (5.6)-(5.9) have been proved in [10]. Theorem 5.3 has been proved in [6].

5.2. **Lagrange interpolation.** If f is a continuous function in $(0, +\infty)$ then the Lagrange polynomial interpolating f on the zeros $x_1 < x_2 < \cdots < x_m$ of $p_m(w_\alpha)$ is defined as

$$L_m(w_\alpha, f, x) = \sum_{i=1}^m l_i(x) f(x_i), \quad l_i(x) = \frac{p_m(w_\alpha, x)}{p_m'(w_\alpha, x_k)(x - x_k)}.$$

In the sequel we will consider the behaviour of $L_m(w_\alpha, f)$ in L_u^p with $u(x) = x^\gamma e^{-\frac{x^\beta}{2}}$. Analogously to the Fourier sums, the behaviour of $L_m(w_\alpha, f)$ in L_u^p is "poor", i.e. it can be used with good results only for a restricted class of functions. For example, if $p = +\infty$ and $f \in L_u^\infty$ with $\gamma \ge 0$, then for every choice of α and γ,

$$\|L_m(w_\alpha)\| := \sup_{\|fu\|_\infty = 1} \|L_m(w_\alpha, f) u\|_\infty > \mathcal{C} m^\rho,$$

with $\rho > 0$ and $\mathcal{C} \neq \mathcal{C}(f, m)$. Then, as for the Fourier sums, we modify the Lagrange polynomial. To this end, we introduce the following notations. Let

$$x_j = \min_{k=1,\dots,m} \{x_k \ : \ x_k \ge \theta a_m\},$$

where $\theta \in (0, 1)$ and $a_m = a_m(w_\alpha), m$ sufficiently large. With

$$\Psi(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x \ge 1 \end{cases} \quad \text{and} \quad \Psi_j(x) = \Psi\left(\frac{x - x_j}{x_j - x_{j-1}}\right),$$

define the truncated function $f_j := \Phi_j f$, where $\Phi_j = 1 - \Psi_j$. By definition, we deduce that f_j has the same smoothness as f and

$$f_j(x) = \begin{cases} f(x) & \text{if } x \in [0, x_j] \\ 0 & \text{if } x \in [x_{j+1}, +\infty). \end{cases}$$

Now, letting $\theta_1 \in (\theta, 1)$ and denoting by $\Delta_{\theta_1} := \Delta_{\theta_1 m}$ the characteristic function of $[0, \theta_1 a_m]$, we consider the behaviour of the sequence $\{\Delta_{\theta_1} L_m(w_\alpha, f_j)\}_m$ in $L^p_u, u(x) = x^\gamma e^{-\frac{x^\beta}{2}}, 1 < p \le +\infty$.

Theorem 5.4. *If the parameters α and γ of the weights w_α and u satisfy*

$$\frac{\alpha}{2} + \frac{1}{4} \le \gamma \le \frac{\alpha}{2} + \frac{5}{4}, \quad \gamma \ge 0,$$

then

$$\|\Delta_{\theta_1} L_m(w_\alpha, f_j) u\|_\infty \le \mathcal{C} \|f_j u\|_\infty (\log m),$$

with $\mathcal{C} \ne \mathcal{C}(m, f)$.

The following lemma will be useful in the sequel, but it can be used in more contexts too.

Lemma 5.5. *Let $0 < \theta < \theta_1 < 1, 1 \le p < +\infty$ and $\Delta x_k = x_{k+1} - x_k$. Then, for an arbitrary polynomial $P \in P_{ml}$ (l fixed integer), we have*

$$\sum_{k=1}^{j} \Delta x_k |Pu|^p(x_k) \le \mathcal{C} \int_{x_1}^{\theta_1 a_m} |Pu|^p(x) dx,$$

with $\mathcal{C} \ne \mathcal{C}(m, p, P)$.

In order to simplify the notations, from now on we let $v^\rho(x) = x^\rho$.

Theorem 5.6. *Let $1 < p < +\infty$ and assume that*

$$\frac{v^\gamma}{\sqrt{v^\alpha \varphi}} \in L^p \quad \text{and} \quad \frac{\sqrt{v^\alpha \varphi}}{v^\gamma} \in L^q, \quad \varphi(x) = \sqrt{x}, \quad q = \frac{p}{p-1}. \tag{5.11}$$

Then, for every $f \in C^0(0, +\infty)$, we have

$$\|L_m(w_\alpha, f_j) u \Delta_{\theta_1}\|_p \le \mathcal{C} \sum_{k=1}^{j} \Delta x_k |fu|^p(x_k), \tag{5.12}$$

with $\mathcal{C} \ne \mathcal{C}(m, f)$.

The following lemma estimates the right-hand side of (5.12) in terms of the main part of the modulus of smoothness.

Lemma 5.7. *For every function f belonging to $C^0(0,+\infty)$ we have*

$$\left(\sum_{k=1}^{j}\Delta x_k|fu|^p(x_k)\right)^{\frac{1}{p}} \le \mathcal{C}\left[\|fu\|_{L^p(0,x_j)} + \left(\frac{\sqrt{a_m}}{m}\right)^{\frac{1}{p}}\int_0^{\frac{\sqrt{a_m}}{m}}\frac{\Omega_\varphi^r(f,t)_{u,p}}{t^{1+\frac{1}{p}}}dt\right],$$

with $r < m$ and $\mathcal{C} \ne \mathcal{C}(m,f)$.

Now we can state the following

Theorem 5.8. *Under the assumptions of Theorem* 5.6*, for every continuous function in* $(0,+\infty)$*, we have*

$$\|[f - \Delta_{\theta_1}L_m(w_\alpha, f_j)]u\|_p \le \mathcal{C}\left[\left(\frac{\sqrt{a_m}}{m}\right)^{\frac{1}{p}}\int_0^{\frac{\sqrt{a_m}}{m}}\frac{\Omega_\varphi^r(f,t)_{u,p}}{t^{1+\frac{1}{p}}}dt + e^{-Am}\|fu\|_{L^p}\right],$$

where the constants A and $\mathcal{C}$ are independent of m and f.

As an example, for every $f \in W_r^p(u)$, we have

$$\|[f - \Delta_{\theta_1}L_m(w_\alpha, f_j)]u\|_p \le \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r \|f\|_{W_r^p(u)},$$

that is the error of best approximation in $W_r^p(u)$.

6. **Proofs**

We first state two propositions whose proofs are easy.

Proposition 6.1. *Let $x \in [(2rh)^2, h^*]$, with $h^* = \frac{1}{h^{\frac{2}{2\beta-1}}}, \beta > \frac{1}{2}$, and $y \in [x - rh\sqrt{x}, x + rh\sqrt{x}]$. Then it results:*

$$w_{\alpha\beta}(x) \sim w_{\alpha\beta}(y),$$

where the constant in " $\sim$ " are independent of x and h.

Proposition 6.2. *Let $z > 0$ be such that $w_{\alpha\beta}(x) = x^\alpha e^{-x^\beta}, \beta > \frac{1}{2}$ is a non-decreasing function in $[z,+\infty]$. Then, for every $f \in W_r^p(w_{\alpha\beta})$, with $r \ge 1$ and $1 \le p \le +\infty$,*

$$\left(\int_z^{+\infty}\left|w_{\alpha\beta}(x)\int_z^x(x-u)^{r-1}f^{(r)}(u)du\right|^p dx\right)^{\frac{1}{p}} \le \frac{\mathcal{C}}{(z^{\beta-\frac{1}{2}})^r}\|f^{(r)}\varphi^r w_{\alpha\beta}\|_p,$$

with $\mathcal{C} \ne \mathcal{C}(f,z,p)$.

Proof of Theorem 3.1. We first point out the main steps of the proof. In order to prove (3.6), constructing a suitable function $G_h \in W_r^p(w_{\alpha\beta})$, we state the inequality

$$\tilde{K}(f,t^r)_{w_{\alpha\beta},p} \le \mathcal{C}\Omega_\varphi^r(f,t)_{w_{\alpha\beta},p}. \tag{6.13}$$

Let $t_0 < 8r^2h^2 \le t_1 < t_2 < \cdots < t_j \le h^* < t_{j+1}, h > 0$, be a system of knots such that $t_{i+1} - t_i \sim h\sqrt{t_i}, i = 0, \dots, j$. With $\Psi \in C^\infty(\mathbb{R})$ a non-decreasing function such that

$$\Psi_(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x \ge 1, \end{cases}$$

and with $y_k = \frac{t_k + t_{k+1}}{2}$, define the functions $\Psi_k(x) = \Psi\left(\frac{x - y_k}{t_{k+1} - y_k}\right)$, where $k = 1, 2, \dots, j$ and $\Psi_0(x) = 0 = \Psi_{j+1}(x)$. With

$$f_\tau(x) = r^r \int_0^{\frac{1}{r}} \cdots \int_0^{\frac{1}{r}} \left(\sum_{l=1}^{r} (-1)^{l+1} \binom{r}{l} f(x + l\tau(u_1 + \cdots + u_r)) \right) du_1 \dots du_r$$

and

$$F_{hk}(x) = \frac{2}{h} \int_{\frac{h}{2}} h f_{\tau\varphi(t_k)}(x) d\tau,$$

we introduce the function

$$G_h(x) = \sum_{k=1}^{j} F_{hk}(x) \Psi_{k-1}(x)(1 - \Psi_k(x)). \tag{6.14}$$

After that, in order to prove the inequalities

$$\left.\begin{array}{r} \|(f - G_h)w_{\alpha\beta}\|_{L^p(8r^2h^2, h^*)} \\ h^r \|G_h^{(r)} \varphi^r w_{\alpha\beta}\|_{L^p(8r^2h^2, h^*)} \end{array}\right\} \le \mathcal{C} \|w_{\alpha\beta} \overrightarrow{\Delta}_{h\varphi} f\|_{L^p(8r^2h^2, Ah^*)},$$

for some constant A, it is sufficient to repeat word for word [3], p. 194-197, with some simplifications due to the forward difference $\overrightarrow{\Delta}_{h\varphi}$ appearing in the definition of the modulus Ω_φ^r. Thus (3.6) follows. In order to prove the inverse inequality of (3.6), we now prove that for every $g \in W_r^p(w_{\alpha\beta})$

$$\|w_{\alpha\beta} \overrightarrow{\Delta}_{h\varphi} f\|_{L^p(8r^2h^2, h^*)}$$

$$\le \mathcal{C} \left\{ \|(f - g)w_{\alpha\beta}\|_{L^p(8r^2h^2, Ah^*)} + h^r \|g^{(r)} \varphi^r w_{\alpha\beta}\|_{L^p(8r^2h^2, Ah^*)} \right\},$$

with $A = 1 + rh^{\frac{2\beta}{2\beta - 1}}$. In fact, we have

$$|w_{\alpha\beta}(x)(\overrightarrow{\Delta}_{h\varphi} f)(x)| \le \sum_{k=0}^{r} \binom{r}{k} |f - g|(x + (r - k)h\sqrt{x}) w_{\alpha\beta}(x) + |w_{\alpha\beta}(x)(\overrightarrow{\Delta}_{h\varphi} g)(x)|.$$

Now, x and $x+(r-k)h\sqrt{x}$ belong to $[8r^2h^2, Ah^*]$ and $|x-(x+(r-k)h\sqrt{x})| \le rh\sqrt{x}$. Thus, by Proposition 6.1, $w_{\alpha\beta}(x) \le \mathcal{C} w_{\alpha\beta}(x+(r-k)h\sqrt{x})$ and

$$\begin{aligned}\|w_{\alpha\beta}\overrightarrow{\Delta}_{h\varphi}(f-g)\|_{L^p(8r^2h^2,h^*)} &\le \mathcal{C}\sum_{k=0}^{r}\binom{r}{k}\|(f-g)w_{\alpha\beta}(\cdot+(r-k)h\sqrt{\cdot})\|_{L^p(8r^2h^2,h^*)}\\ &\le \mathcal{C}2^r\|(f-g)w_{\alpha\beta}\|_{L^p(8r^2h^2,Ah^*)},\end{aligned}$$

making the change of variable $u = x+(r-k)h\sqrt{x}$ and using $\left|\frac{du}{dx}\right| \le 2$. Moreover, since

$$\begin{aligned}\overrightarrow{\Delta}_h^r g(x) &= r!h^r\int_0^1\int_0^{t_1}\cdots\int_0^{t_{r-1}} g^{(r)}(x+h(t_1+\cdots+t_r))dt_1\dots dt_r\\ &=: r!h^r\int_{T_r} g^{(r)}(x+h\tau)dT_r,\end{aligned}$$

with $\tau = t_1+\cdots+t_r < r$ and $T_r = [0,1]\times[0,t_1]\times\cdots\times[0,t_r]$, we can write

$$w_{\alpha\beta}(x)\overrightarrow{\Delta}_{h\varphi}^r g(x) = r!(h\varphi)^r\int_{T_r} g^{(r)}(x+h\tau\sqrt{x})w_{\alpha\beta}(x)dT_r.$$

Consequently, by Proposition 6.1, we have

$$\begin{aligned}\|w_{\alpha\beta}\overrightarrow{\Delta}_{h\varphi}^r g\|_{L^p(8r^2h^2,h^*)} &\le \mathcal{C}r!h^r\left(\int_{8r^2h^2}^{h^*}\left|\int_{T_r} g^{(r)}(x+h\tau\sqrt{x})w_{\alpha\beta}(x)dT_r\right|^p\right)^{\frac{1}{p}}\\ &\le \mathcal{C}r!h^r\int_{T_r}\left(\int_{8r^2h^2}^{h^*}\left|g^{(r)}\varphi^r w_{\alpha\beta}\right|^p(x+h\tau\sqrt{x})dx\right)^{\frac{1}{p}}dT_r\\ &\le \mathcal{C}h^r\left(\int_{8r^2h^2}^{Ah^*}\left|g^{(r)}\varphi^r w_{\alpha\beta}\right|^p(u)du\right)^{\frac{1}{p}},\end{aligned}$$

being $\int_{T_r} dT_r = \frac{1}{r!}$. Then the equivalence (3.6) easily follows. Now we prove equivalence (3.5), i.e.

$$\omega_\varphi^r(f,t)_{w_{\alpha\beta},p} \sim K(f,t^r)_{w_{\alpha\beta},p}.$$

In order to prove

$$\omega_\varphi^r(f,t)_{w_{\alpha\beta},p} \le \mathcal{C}K(f,t^r)_{w_{\alpha\beta},p},$$

since

$$\Omega_\varphi^r(f,t)_{w_{\alpha\beta},p} \le \mathcal{C}K(f,t^r)_{w_{\alpha\beta},p}, \quad 1\le p\le+\infty,$$

holds true, it remains to prove that the first and third terms in the definition of ω_φ^r are dominated by the $K-$functional. About the first term, in [1], p. 200, we proved that, with $u_\alpha = x^\alpha e^{-x}$,

$$\inf_{q_r\in\mathbb{P}_r} \|(f-q_r)u_\alpha\|_{L^p(0,8r^2t^2)} \le \mathcal{C}\|(f-g)u_\alpha\|_{L^p(0,8r^2t^2)} + t^r\|g^{(r)}\varphi^r u_\alpha\|_{L^p(0,8r^2t^2)}$$

and then, since $e^{-x}\sim e^{-x^\beta}\sim 1$ for $x\in[0,(2rh)^2]$, we can replace u_α with $w_{\alpha\beta}$ in the above norms. About the third term, we have

$$\inf_{q_{r-1}\in\mathbb{P}_{r-1}} \|(f-q_{r-1})w_{\alpha\beta}\|_{L^p(t^*,+\infty)} \le \|(f-g)w_{\alpha\beta}\|_{L^p(t^*,+\infty)} + \|(g-T_{r-1})w_{\alpha\beta}\|_{L^p(t^*,+\infty)},$$

where $g\in W_r^p(w_{\alpha\beta})$ is arbitrary and T_{r-1} is the Taylor polynomial of g with initial point t^*. Consequently

$$\|(g-T_{r-1})w_{\alpha\beta}\|_{L^p(t^*,+\infty)} = \left(\int_{t^*}^{+\infty}\left|w_{\alpha\beta}(x)\int_{t^*}^x (x-u)^{r-1}g^{(r-1)}(u)du\right|^p dx\right)^{\frac{1}{p}}.$$

Then, using Proposition 6.2 with $z=t^*$ and $f=g$, the right-hand side of the above equality is dominated by $\frac{\mathcal{C}}{\left[(t^*)^{\frac{2\beta-1}{2}}\right]^r}\|g^{(r)}\varphi^r w_{\alpha\beta}\|_{L^p(t^*,+\infty)}$. By definition $t^*=\frac{1}{t^{\frac{2}{2\beta-1}}}$, i.e. $\frac{1}{\left[(t^*)^{\frac{2\beta-1}{2}}\right]^r}=t^r$, and the inequality

$$\omega_\varphi^r(f,t)_{w_{\alpha\beta},p} \le \mathcal{C}K(f,t^*)_{w_{\alpha\beta},p}$$

follows. In order to prove the inverse inequality, recall that for two suitable polynomials p_1 and p_2 belonging to $\mathbb{P}_{r-1}$,

$$\begin{aligned}\|(f-p_1)w_{\alpha\beta}\|_{L^p(0,8r^2t^2)} + t^r\|p_1^{(r)}\varphi^{r-1}w_{\alpha\beta}\|_{L^p(0,8r^2t^2)} &\le \omega_\varphi^r(f,t)_{w_{\alpha\beta},p}\\ \|(f-p_2)w_{\alpha\beta}\|_{L^p(t^*-1,+\infty)} + t^r\|p_2^{(r)}\varphi^{r-1}w_{\alpha\beta}\|_{L^p(t^*-1,+\infty)} &\le \omega_\varphi^r(f,t)_{w_{\alpha\beta},p}\end{aligned}$$

as previously proved. Moreover, for the function $G_t(x)$ defined in (6.14), the inequality

$$\begin{aligned}\|(f-G_t)w_{\alpha\beta}\|_{L^p(8r^2t^2,h^*)} + t^r\|G_t^{(r)}\varphi^r w_{\alpha\beta}\|_{L^p(8r^2t^2,h^*)} &\le \mathcal{C}\Omega_\varphi^r(f,t)_{w_{\alpha\beta},p}\\ &\le \mathcal{C}\omega_\varphi^r(f,t)_{w_{\alpha\beta},p}\end{aligned}$$

holds. Now, with $x_1 = 4r^2t^2, x_2 = 8r^2t^2, x_3 = t^* - 1, x_4 = t^*$, consider the function

$$\begin{aligned}\Gamma_t(x) &= \left(1 - \Psi\left(\frac{x - x_1}{x_2 - x_1}\right)\right) p_1(x) + \Psi\left(\frac{x - x_1}{x_2 - x_1}\right)\left(1 - \Psi\left(\frac{x - x_3}{x_4 - x_3}\right)\right) G_t(x) \\ &+ \Psi\left(\frac{x - x_3}{x_4 - x_3}\right) p_2(x).\end{aligned}$$

Obviously $\Gamma_t \in W_r^p$ and it is not difficult to verify the inequality

$$\|(f - \Gamma_t) w_{\alpha\beta}\|_p + t^r \|\Gamma_t^{(r)} \varphi^r w_{\alpha\beta}\|_{L^p(8r^2t^2, h^*)} \leq \mathcal{C}\omega_\varphi^r(f, t)_{w_{\alpha\beta}, p}.$$

Thus the proof of the theorem is complete. □

In order to prove the theorems on interpolation, we recall some basic facts on the orthonormal polynomials $\{p_m(w_\alpha)\}_m$. The zeros of $p_m(w_\alpha)$ are located as follows:

$$\mathcal{C}\frac{a_m}{m^2} \leq x_1 < \cdots < x_m \leq a_m \left(1 - \frac{\mathcal{C}}{m^{\frac{2}{3}}}\right).$$

Moreover,

$$\Delta x_k = x_{k+1} - x_k \sim \frac{\sqrt{a_m}}{m} \sqrt{x_k} \frac{1}{\sqrt{1 - \frac{x_k}{a_m} + \frac{1}{m^{\frac{2}{3}}}}},$$

where $a_m = a_m(w_\alpha)$ and $\mathcal{C}$ is a positive constant independent of m. The following estimates are useful:

$$|p_m(w_\alpha, x)\sqrt{w_\alpha(x)}| \leq \frac{\mathcal{C}}{\sqrt[4]{a_m x}\sqrt[4]{\left|1 - \frac{x}{a_m}\right| + \frac{1}{m^{\frac{2}{3}}}}},$$

where $\mathcal{C}\frac{a_m}{m^2} \leq x \leq \mathcal{C}a_m(1 + m^{-\frac{2}{3}})$ and $\mathcal{C} \neq \mathcal{C}(m, x)$, and

$$\frac{1}{|p_m'(w_\alpha, x_k)\sqrt{w_\alpha(x_k)}|} \sim \sqrt[4]{x_k a_m}\Delta x_k \sqrt{1 - \frac{x_k}{a_m} + \frac{1}{m^{\frac{2}{3}}}}, \quad k = 1, \ldots, m,$$

where the constants in " $\sim$ " are independent of m and k. The above estimates can be found in [5] or can be directly obtained by [4].

Proof of Theorem 5.4. Since

$$u(x)L_m(w_\alpha, f_j, x) = \sum_{i=1}^{j} u(x)\frac{l_i(x)}{u(x_i)}(fu)(x_i)$$

and, denoting by x_d a knot closest to x, it results $\left|u(x)\frac{l_d(x)}{u(x_d)}\right| \sim 1$, for $x \in [0, x_j]$. Then we have

$$|u(x)L_m(w_\alpha, f_j, x)| \leq \mathcal{C}\|fu\|_{L^\infty([0,x_j])}\left(1 + \sum_{\substack{i=1\\ i\neq d}}^{j} \frac{u(x)}{u(x_i)}|l_i(x)|\right). \tag{6.15}$$

Using the previous estimates and a Remez-type inequality, we get

$$\frac{|u(x)p_m(w_\alpha, x)|}{|p'_m(w_\alpha, x_i)u(x_i)|} \leq \mathcal{C}\left(\frac{x}{x_i}\right)^{\gamma - \frac{\alpha}{2} - \frac{1}{4}} \frac{\Delta x_i}{|x - x_i|},$$

where $i = 1, 2, \ldots, j, i \neq d$, and $x \in \left[\frac{a_m}{m^2}, x_j\right]$. Then, under the assumptions of α and γ, the sum in (6.15) is dominated by $\log m$ and the theorem follows. □

Here we omit the proofs of Lemmas 5.5 and 5.7 and the proofs of Theorems 5.6 and 5.8, being completely similar to the proofs of Lemmas 2.5 and 2.7 and Theorems 2.6 and 2.8 in [10] respectively.

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