STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LII**, Number 4, December 2007

MEAN CONVERGENCE OF FOURIER SUMS ON UNBOUNDED INTERVALS

G. MASTROIANNI AND D. OCCORSIO

Dedicated to Professor D. D. Stancu on his 80^{th} *birthday*

Abstract. In this paper we consider the approximation of functions by suitable "truncated" Fourier Sums in the generalized Freud and Laguerre systems. We prove necessary and sufficient conditions for the uniform boundedness in L_p weighted spaces.

1. **Introduction**

Let be $W_{\alpha,\beta}(x) =: W_\alpha(x) = |x|^\alpha e^{-|x|^\beta}, x \in \mathbb{R}, \alpha > -1, \beta > 1$ a generalized Freud weight and denote by $\{p_m(W_\alpha)\}_m$ the corresponding sequence of orthonormal polynomials with positive leading coefficients, i.e.

$$p_m(W_\alpha, x) = \gamma_m(W_\alpha)x^m + \dots, \quad \gamma_m(W_\alpha) > 0, \quad m = 0, 1, \dots.$$

These polynomials introduced and studied in [3](see also [4], [5]) are a generalization of Sonin-Markov polynomials. Let be $S_m(W_\alpha, f)$ the $m-$th partial Fourier sum of a measurable function f in the system $\{p_m(W_\alpha)\}_m$, i.e.

$$S_m(W_\alpha, f, x) = \sum_{k=0}^{m} c_k p_k(W_\alpha, x), \quad c_k = \int_{\mathbb{R}} f(t) p_k(W_\alpha, t) W_\alpha(t) dt.$$

For $\alpha = 0$, the boundedness in weighted L_p spaces of $S_m(W_\alpha, f, x)$ holds only for a "small" range of p (see [2]). To be more precise, in [2] the authors proved the bound

$$\|S_m(W_0(x), f,)\sqrt{W_0}\|_p \le \mathcal{C}\|f\sqrt{W_0}\|_p \tag{1}$$

Received by the editors: 18.04.2007.

2000 *Mathematics Subject Classification.* 41A10, 42C10.

Key words and phrases. Fourier series, othogonal polynomials, approximation by polynomials.

Research was supported by University of Basilicata.

for $\frac{4}{3} < p < 4$ and $\beta = 2, 4, 6...$, while for $p \geq 4$ and $p \leq \frac{4}{3}$ estimate of kind (1) cannot always hold. In the same paper [2] the authors, in order to extend the range of p, modify the weight in the norm obtaining, under suitable assumptions on b, B, β, not homogenous estimates of the kind

$$\|S_m(W_0(x), f)\sqrt{W_0}(1 + |x|)^b\|_p \leq \mathcal{C}\|f\sqrt{W_0}(1 + |x|)^B\|_p, \quad 1 < p < \infty. \tag{2}$$

In the case $\alpha = 0$ and $\beta = 2$ (Hermite polynomials) estimates of types (1) and (2) were already proved in [12] (see also [1]).
Let be $U_\gamma(x) = |x|^\gamma e^{-\frac{|x|^\beta}{2}}, x \in \mathbb{R}, \gamma > -\frac{1}{p}$. Denote by $a_m = a_m(W_\alpha)$ the Mhaskar-Rahmanov-Saff number (M-R-S number) with respect to W_α and by $\Delta_{m,\theta}$ the characteristic function of the segment $A_m = [-\theta a_m, \theta a_m]$, with $0 < \theta < 1$. In this paper, we will prove inequalities of kind

$$\|S_m(W_\alpha, f\Delta_{m,\theta})U_\gamma\Delta_{m,\theta}\|_p \leq \mathcal{C}\|fU_\gamma\Delta_{m,\theta}\|_p, \tag{3}$$

with $1 < p < \infty$, under certain conditions on α and γ which are necessary and sufficient. Since we prove also that, for $m \to \infty$, the norm $\|[f - \Delta_{m,\theta}S_m(W_\alpha, \Delta_{m,\theta}f)]U_\gamma\|_p$ converges to zero essentially like the error of the best approximation in $L^p_{U_\gamma}$, then in order to approximate a function $f \in L^p_{U_\gamma}$ (see (7) for the definition) the sequence $\{\Delta_{m,\theta}S_m(W_\alpha, f\Delta_{m,\theta})\}_m$ is simpler and more convenient than the ordinary Fourier sum.

An inequality of type (3) has been proved in [12], in the special case of the Hermite weight. The proof in [12] requires a precise estimate of the difference $|p_{m+1}(x) - p_{m-1}(x)|$ where $p_m(x)$ is the $m-$th Hermite polynomial. This estimate for weights W_α is not available in the literature and, on the other hand, it isn't required in our proof. The case $p = 1$ is also considered when the functions are in the Calderon-Zygmund spaces.

As consequence of estimate (3) we derive the analogous one for Fourier sums in the system of orthogonal polynomials w.r.t generalized Laguerre weights $w_\alpha(x) = x^\alpha e^{-x^\beta/2}$, $x \geq 0, \alpha > -1, \beta > \frac{1}{2}$.

The plan of the paper is the following: next section contains some basic facts necessary to introduce the main results given in section 3. Section 4 contains all the proofs.

2. **Preliminary**

In the sequel $\mathcal{C}$ denotes a positive constant which can be different in different formulas. Moreover we write $\mathcal{C} \neq \mathcal{C}(a, b, ..)$ when the constant $\mathcal{C}$ is independent of $a, b, ..$

Let be $U_\gamma(x) = |x|^\gamma e^{-\frac{|x|^\beta}{2}}, \gamma > -\frac{1}{p}, \beta > 1$ and denote by $\overline{a}_m = \overline{a}_m(U_\gamma)$ the M-R-S number w.r.t. U_γ. The following "infinite-finite range inequality" holds [3]

$$\|P_m U_\gamma\|_{L_p(\mathbb{R})} \leq \mathcal{C} \|P_m U_\gamma\|_{L_p(|x| \leq \overline{a}_m(1-\mathcal{C}m^{-2/3})}.$$

We remark that $\overline{a}_m = \overline{a}_m(U_\gamma)$ can be expressed as [7]

$$\overline{a}_m = m^{\frac{1}{\beta}} \mathcal{C}(\beta, \gamma), \tag{4}$$

where the positive constant $\mathcal{C}(\beta, \gamma)$ will not be used in the sequel (analogously for $a_m = a_m(W_\alpha)$). Moreover we recall the following inequalities [7]:

$$\|P_m U_\gamma\|_{L_p(|x| \geq \overline{a}_m(1+\delta))} \leq \mathcal{C}_1 e^{-\mathcal{C}_2 m} \|P_m U_\gamma\|_{L_p(-\overline{a}_m, \overline{a}_m)} \tag{5}$$

and

$$\|P_m U_\gamma\|_{L_p(\mathbb{R})} \leq \mathcal{C} \|P_m U_\gamma\|_{L_p(\frac{\overline{a}_m}{m} \leq |x| \leq \overline{a}_m)}, \tag{6}$$

where $\delta > 0$ is fixed and the constants $\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2$ are independent of m and P_m.
For $1 \leq p < \infty$ define the space

$$L^p_{U_\gamma} = \left\{ f : \left(\int_{-\infty}^{\infty} |f(x) U_\gamma(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\} \tag{7}$$

and denote by

$$E_m(f)_{U_\gamma, p} = \inf_{P \in \mathbb{P}_m} \|(f - P) U_\gamma\|_p \tag{8}$$

the error of the best approximation in $L^p_{U_\gamma}$.

For a fixed real θ with $0 < \theta < 1$ we shall denote by $\overline{\Delta}_{m,\theta}$ the characteristic function of $D_m = (-\theta\overline{a}_m, \theta\overline{a}_m), \overline{a}_m = \overline{a}_m(U_\gamma)$. Next Proposition is useful for our goals.

Proposition 2.1. *Let $f \in L^p_{U_\gamma}$ and $1 \le p \le \infty$. For m sufficiently large (say $m > m_0$) we have*

$$\|f(1-\overline{\Delta}_{m,\theta})U_\gamma\|_p \le \mathcal{C}_1\left(E_M(f)_{U_\gamma,p} + e^{-\mathcal{C}_2 m}\|fU_\gamma\|_p\right), \tag{9}$$

where $M = \left[m\left(\frac{\theta}{1+\theta}\right)^\beta\right]$ [1] *and the constants $\mathcal{C}, \mathcal{C}_1, C_2$ are independent on m and f.*
By (9) we get

$$\|fU_\gamma\|_p \le \mathcal{C}\left(E_M(f)_{U_\gamma,p} + \|f\overline{\Delta}_{m,\theta}U_\gamma\|_p\right). \tag{10}$$

Then, by virtue of Proposition 2.1 we will go to consider the behaviour of the sequence $\{\Delta_{m,\theta}S_m(W_\alpha, \Delta_{m,\theta}f)\}_m$ instead of $\{S_m(W_\alpha, f)\}_m$, where here and in the sequel $\Delta_{m,\theta}$ is the characteristic function of $[-\theta a_m, \theta a_m]$, with $a_m = a_m(W_\alpha) < \overline{a}_m(U_\gamma)$.

3. **Main results**

Now we are able to state the next two Theorems.

Theorem 3.1. *Let be $U_\gamma(x) = |x|^\gamma e^{-|x|^\beta/2}, \gamma > -\frac{1}{p}, \beta > 1$, $1 < p < \infty$ and $f \in L^p_{U_\gamma}$. Then, there exists a constant $\mathcal{C} \ne \mathcal{C}(m,f)$ such that*

$$\|S_m(W_\alpha, f\Delta_{m,\theta})U_\gamma\Delta_{m,\theta}\|_p \le \mathcal{C}\|fU_\gamma\Delta_{m,\theta}\|_p, \tag{11}$$

if and only if

$$-\frac{1}{p} < \gamma - \frac{\alpha}{2} < \frac{1}{q}, \quad q = \frac{p}{p-1}. \tag{12}$$

Moreover, if (12) holds, it results also

$$\|[f - \Delta_{m,\theta}S_m(W_\alpha, \Delta_{m,\theta}f)]U_\gamma\|_p \le \mathcal{C}\left(E_M(f)_{U_\gamma,p} + e^{-\mathcal{C}_1 m}\|fU_\gamma\|_p\right) \tag{13}$$

with $\mathcal{C} \ne \mathcal{C}(m,f)$, $\mathcal{C}_1 \ne \mathcal{C}_1(m,f)$.

Setting

$$\log^+ f(x) = \log\left(max(1, f(x))\right),$$

we prove

[1] $[a]$ denotes the largest integer smaller than or equal to $a \in \mathbb{R}^+$

Theorem 3.2. *Let be* $U_\gamma(x) = |x|^\gamma e^{-|x|^\beta/2}, \gamma > -1, \beta > 1$, *and let be* f *such that* $\int_{I\!R} |f(x)U_\gamma(x)| \log^+ |f(x)| dx < \infty$. *If it results*

$$-1 < \gamma - \frac{\alpha}{2} < 0 \tag{14}$$

then

$$\|S_m(W_\alpha, f\Delta_{m,\theta})U_\gamma\Delta_{m,\theta}\|_1 \leq \mathcal{C} + \mathcal{C}\int_{I\!R} |f(x)U_\gamma(x)| \left[1 + \log^+|f(x)| + \log^+|x|\right] dx, \tag{15}$$

where $\mathcal{C} \neq \mathcal{C}(m, f)$.

Theorem 3.2 can be useful to prove the convergence of some product integration rules. We state now some inequalities that can be useful in different contests. Assuming (12) true with p belonging to the right hand mentioned intervals, the following inequalities hold

$$\|S_m(W_\alpha, f)U_\gamma\Delta_{m,\theta}\|_p \leq \mathcal{C}\|fU_\gamma\|_p, \quad 1 < p < 4 \tag{16}$$

$$\|S_m(W_\alpha, f\Delta_{m,\theta})U_\gamma\|_p \leq \mathcal{C}\|fU_\gamma\Delta_{m,\theta}\|_p, \quad p > \frac{4}{3} \tag{17}$$

$$\|S_m(W_\alpha, f)U_\gamma\|_p \leq \mathcal{C}\|fU_\gamma\|_p, \quad \frac{4}{3} < p < 4 \tag{18}$$

$$\|S_m(W_\alpha, f)U_\gamma\|_p \leq \mathcal{C}m^{\frac{1}{3}}\|fU_\gamma\|_p, \quad p \in \left(1, \frac{4}{3}\right) \cup (4, \infty) \tag{19}$$

with $\mathcal{C} \neq \mathcal{C}(m, f)$.

For $\beta = 2$ Theorem 3.1 and inequalities (16)-(19) were proved in [6]. Estimates of $E_m(f)_{U_\gamma,p}$ can be found in [7] and [8].

Now we want to show an useful consequence of the previous results. Let $w_\alpha(x) = x^\alpha e^{-x^\beta}, x > 0, \alpha > -1, \beta > \frac{1}{2}$ be a generalized Laguerre weight and let $\{p_m(w_\alpha)\}_m$ be the corresponding sequence of orthonormal polynomials with positive leading coefficients. With $u_\gamma(x) = x^\gamma e^{-x^\beta/2}, \gamma > -\frac{1}{p}, \beta > \frac{1}{2}$, let $L^p_{u_\gamma}$, $1 < p < \infty$, be the set of measurable functions with norm

$$\|fu_\gamma\|_p = \left(\int_0^\infty |f(x)u_\gamma(x)|^p dx\right)^{\frac{1}{p}} < \infty$$

and denote by $S_m(w_\alpha, f)$ the $m-$th Fourier sum of $f \in L^p_{u_\gamma}$, i.e.

$$S_m(w_\alpha, f, x) = \sum_{k=0}^{m} c_k p_k(w_\alpha, x), \quad c_k = \int_0^\infty f(t)p_k(w_\alpha, t)w_\alpha(t)dt.$$

The theorems that we are going to establish are a direct consequence of Theorems 3.1-3.2. To introduce these results, let $a_m = a_m(w_\alpha)$ the M-R-S number with respect to w_α and for $\theta \in (0,1)$ let be $\chi_{m,\theta}$ the characteristic function of $[0, \theta a_m]$. We have

Theorem 3.3. *Let $u_\gamma(x) = x^\gamma e^{-x^\beta/2}, \gamma > -\frac{1}{p}, \beta > \frac{1}{2}$, $f \in L^p_{u_\gamma}$ and $1 < p < \infty$. Then there exists a constant $\mathcal{C} \neq \mathcal{C}(m, f)$ such that*

$$\|S_m(w_\alpha, f\chi_{m,\theta})u_\gamma\chi_{m,\theta}\|_p \leq \mathcal{C}\|fu_\gamma\chi_{m,\theta}\|_p, \tag{20}$$

if and only if

$$\frac{v^\gamma}{\sqrt{v^\alpha\varphi}} \in L_p(0,1) \quad \text{and} \quad \sqrt{\frac{v^\alpha}{\varphi}}\frac{1}{v^\gamma} \in L_q(0,1), \tag{21}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $v^r = x^r$, and $\varphi(x) = \sqrt{x}$.
Moreover, if (21) holds, it results also

$$\|[f - \chi_{m,\theta}S_m(w_\alpha, \chi_{m,\theta}f)]u_\gamma\|_p \leq \mathcal{C}\left(E_M(f)_{u_\gamma,p} + e^{-\mathcal{C}_1 m}\|fu_\gamma\|_p\right) \tag{22}$$

with $\mathcal{C} \neq \mathcal{C}(m, f)$, $\mathcal{C}_1 \neq \mathcal{C}_1(m, f)$.

Theorem 3.4. *Let $u_\gamma(x) = x^\gamma e^{-x^\beta/2}, \gamma > -1, \beta > \frac{1}{2}$, and let be $\int_0^\infty |f(x)u_\gamma(x)| \log^+ |f(x)| dx < \infty$. If it results*

$$\frac{v^\gamma}{\sqrt{v^\alpha\varphi}} \in L_1(0,1) \quad \frac{\sqrt{v^\alpha}}{v^\gamma\sqrt{\varphi}} \in L_\infty(0,1), \tag{23}$$

then

$$\|S_m(w_\alpha, f\chi_{m,\theta})u_\gamma\chi_{m,\theta}\|_1 \leq \mathcal{C} + \mathcal{C}\int_0^\infty |f(x)u_\gamma(x)|\left[1 + \log^+|f(x)| + \log^+ x\right] dx, \tag{24}$$

where $\mathcal{C} \neq \mathcal{C}(m, f)$, $v^r = x^r$, and $\varphi(x) = \sqrt{x}$.

The case $\beta = 1$ in the Theorem 3.3 was proved in [9].

The following inequalities

$$\|S_m(w_\alpha, f)u_\gamma\chi_{m,\theta}\|_p \leq \mathcal{C}\|fu_\gamma\|_p, \quad 1 < p < 4 \tag{25}$$

$$\|S_m(w_\alpha, f\chi_{m,\theta})u_\gamma\|_p \leq \mathcal{C}\|fu_\gamma\chi_{m,\theta}\|_p, \quad p > \frac{4}{3} \tag{26}$$

$$\|S_m(w_\alpha, f)u_\gamma\|_p \leq \mathcal{C}\|fu_\gamma\|_p, \quad \frac{4}{3} < p < 4 \tag{27}$$

$$\|S_m(w_\alpha, f)u_\gamma\|_p \leq \mathcal{C}m^{\frac{1}{3}}\|fu_\gamma\|_p, \quad p \in (1,\infty)\backslash(\frac{4}{3}, 4) \tag{28}$$

are true with $\mathcal{C} \neq \mathcal{C}(m, f)$, and assuming (21) true with p belonging to the indicated intervals.
The case $\beta = 1$ in Theorem 3.3 and in the inequalities (25)-(28) was just proved in [9].

4. **Proofs**

4.1. **Proof of Proposition 2.1.** We have:

$$\begin{aligned}\|f(1-\overline{\Delta}_{m,\theta})U_\gamma\|_p &= \|fU_\gamma\|_{L_p(|x|\geq\theta\overline{a}_m)} \\ &\leq \|[f-P_M]U_\gamma\|_p + \|P_M U_\gamma\|_{L_p(|x|\geq\theta\overline{a}_m)},\end{aligned}$$

$$M = \left[m\left(\frac{\theta}{1+\theta}\right)^\beta\right],$$

where P_M is the best approximation polynomial of $f \in L^p_{U_\gamma}$. Since (5)

$$\begin{aligned}\|f(1-\overline{\Delta}_{m,\theta})U_\gamma\|_p &\leq E_M(f)_{U_\gamma,p} + \mathcal{C}e^{-\mathcal{C}_2 m}\|P_M U_\gamma\|_p \\ &\leq \mathcal{C}_1(E_M(f)_{U_\gamma,p} + e^{-\mathcal{C}_2 m}\|fU_\gamma\|_p),\end{aligned}$$

i.e. the Proposition is proved. □

In the sequel we need some inequalities about the polynomials $p_m(W_\alpha)$.
In [3, Th. 1.8, p. 16] the authors proved

$$|p_m(W_\alpha, x)|\sqrt{W_\alpha(x)} \leq \frac{\mathcal{C}}{\sqrt{a_m}\sqrt[4]{\left|1-\frac{|x|}{a_m}\right| + m^{-\frac{2}{3}}}}, \quad \frac{a_m}{m} \leq |x| \leq a_m.$$

from which, for a fixed θ, with $0 < \theta < 1$, we can deduce

$$|p_m(w_\alpha, x)|\sqrt{w_\alpha(x)} \leq \mathcal{C}\frac{1}{\sqrt{a_m}}, \quad \frac{a_m}{m} \leq |x| \leq \theta a_m. \tag{29}$$

Denote by x_d a zero of $p_m(W_\alpha)$ closest to x, by $l_{m,d}$ the $d-$th fundamental Lagrange polynomial based on the zeros of $p_m(W_\alpha)$, and recall the following Erdös-Turán estimate [4]

$$\frac{l^2_{m,d}(x)W_\alpha(x)}{W_\alpha(x_d)} + \frac{l^2_{m,d+1}(x)W_\alpha(x)}{W_\alpha(x_d)} > 1. \tag{30}$$

Denoted by $\lambda_m(W_\alpha, x)$ the m-th Christoffel function $\quad m = 1, 2, \dots,$

$$\lambda_m(W_\alpha; x) = \left[\sum_{k=0}^{m-1} p_k^2(W_\alpha; x)\right]^{-1},$$

in [3] the authors proved

$$\frac{1}{\mathcal{C}}\varphi_m(x) \le \frac{\lambda_m(W_\alpha, x)}{\left(|x| + \frac{a_m}{m}\right)^\alpha e^{-|x|^\beta}} \le \mathcal{C}\varphi_m(x), \tag{31}$$

where

$$\varphi_m(x) = \frac{a_m}{m}\frac{1}{\sqrt{\left|1 - \frac{|x|}{a_m}\right| + m^{-\frac{1}{3}}}}, \quad |x| \le a_m.$$

Combining (30) and (31) we deduce

$$\frac{l_{m,d}^2(x)W_\alpha(x)}{W_\alpha(x_d)} \sim 1. \tag{32}$$

Since from [3, p.16-17], for $|x_d| \le \theta a_m$,

$$W_\alpha(x_d){p_m'}^2(W_\alpha, x_d) \sim \frac{1}{\Delta^2 x_d}, \quad |\Delta x_d| = |x_{d\pm1} - x_d|,$$

we deduce

$$|p_m(w_\alpha, x)|\sqrt{W_\alpha(x)}\sqrt{a_m} \sim \left|\frac{x - x_d}{x_d - x_{d\pm1}}\right|, \quad \frac{a_m}{m} \le |x| \le \theta a_m. \quad \square \tag{33}$$

The following proposition will be useful in the sequel.

Proposition 4.1. *Let be* $W_\alpha(x) = v^\alpha(x)e^{-|x|^\beta}$, $v^\alpha(x) = |x|^\alpha$ *and* $U_\rho(x) = v^\rho(x)e^{-\frac{|x|^\beta}{2}}$, $v^\rho(x) = |x|^\rho$. *For a fixed* $0 < \theta < 1$, $1 \le p < \infty$ *and* $\rho - \frac{\alpha}{2} > -\frac{1}{p}$, *we have*

$$\|p_m(W_\alpha)U_\rho\|_{L_p[-\theta a_m, \theta a_m]} \ge \frac{C}{\sqrt{a_m}}\left\|\frac{v^\rho}{\sqrt{v^\alpha}}\right\|_{L^p(-1,1)}, \tag{34}$$

where $\mathcal{C}$ *is independent of* m.

Proof. Let $\delta > 0$ be "small". Define $\delta_k = \frac{\delta}{4}\Delta x_k = \frac{\delta}{4}(x_{k+1} - x_k)$, and $I_m = \bigcup_{1 \le k \le m}([x_k - \delta_k, x_k + \delta_k])$. To prove (34), set $CI_m = [-1, 1]\backslash I_m$. By (33) we get

$$|p_m(W_\alpha, x)|U_\rho(x) \ge C\frac{|x|^{\rho - \frac{\alpha}{2}}}{\sqrt{a_m}}, \quad x \in CI_m,$$

and consequently

$$\|p_m(W_\alpha)\sigma\|_{L^p[-a_m\theta,a_m\theta]} \geq \frac{C}{\sqrt{a_m}} \left\| \frac{v^\rho}{\sqrt{v^\alpha}} \right\|_{L^p(CI_m)}.$$

Since the measure of I_m is bounded by δ, for a suitable δ, we conclude

$$\|p_m(W_\alpha)U_\rho\|_{L^p[-a_m\theta,a_m\theta]} \geq \frac{C}{\sqrt{a_m}} \left\| \frac{v^\rho}{\sqrt{v^\alpha}} \right\|_{L^p([-1,1])}. \qquad \square$$

In order to prove next theorem, we recall the following expression for $S_m(W_\alpha, f)$

$$\begin{aligned} S_m(W_\alpha, f, x) \quad &= \quad \frac{\gamma_{m-1}(W_\alpha)}{\gamma_m(W_\alpha)} \{p_m(W_\alpha, x)H(f\Delta_{m,\theta}p_{m-1}(W_\alpha)W_\alpha; x) \\ &+ \quad p_{m-1}(W_\alpha, x)H(f\Delta_{m,\theta}p_m(W_\alpha)W_\alpha; x)\}, \end{aligned} \tag{35}$$

where

$$H(g,t) = \int_{\mathbb{R}} \frac{g(x)}{x-t} dx$$

is the Hilbert transform of g in $\mathbb{R}$, and [3]

$$\frac{\gamma_{m-1}(W_\alpha)}{\gamma_m(W_\alpha)} \sim a_m(W_\alpha). \tag{36}$$

4.2. **Proof of Theorem 3.1.** By (6) we have

$$\|S_m(W_\alpha, f\Delta_{m,\theta})U_\gamma\Delta_{m,\theta}\|_p \leq \mathcal{C}\|S_m(W_\alpha, f\Delta_{m,\theta})U_\gamma\Delta_{m,\theta}\|_{L_p(C_m)},$$

$$C_m = \{x : \mathcal{C}\frac{a_m}{m} \leq |x| \leq \theta a_m\},$$

Taking into account (35) and (36)

$$\begin{aligned} &\|S_m(W_\alpha, f\Delta_{m,\theta})U_\gamma\Delta_{m,\theta}\|_p \\ &\leq a_m \left(\int_{C_m} |p_m(W_\alpha, t)H(p_{m-1}(W_\alpha)W_\alpha f\Delta_{m,\theta}; t)U_\gamma(t)|^p \, dt \right)^{\frac{1}{p}} \\ &+ a_m \left(\int_{C_m} |p_{m-1}(W_\alpha, t)H(p_m(W_\alpha)W_\alpha f\Delta_{m,\theta}; t)U_\gamma(t)|^p \, dt \right)^{\frac{1}{p}} = B_1 + B_2 \end{aligned} \tag{37}$$

Using (29)

$$B_1 \leq \mathcal{C}\sqrt{a_m} \left(\int_{C_m} |t|^{\gamma-\frac{\alpha}{2}} \left| \int_{C_m} \frac{p_{m-1}(W_\alpha, x)f(x)\Delta_{m,\theta}(x)W_\alpha(x)}{x-t} dx \right|^p dt \right)^{\frac{1}{p}}$$

By the changes of variables $x = a_m y$, $t = a_m z$, we get

$$B_1 \leq \mathcal{C}a_m^{\frac{1}{2}+\gamma-\frac{\alpha}{2}+\frac{1}{p}} \left(\int_{\tilde{C}_m} |z|^{\gamma-\frac{\alpha}{2}} \left| \int_{\tilde{C}_m} \frac{(p_{m-1}(W_\alpha)f\Delta_{m,\theta}W_\alpha)(a_m y)}{y-z} dy \right|^p dz \right)^{\frac{1}{p}}$$

where

$$\tilde{C}_m = [-1,1] \backslash \left[-\frac{\mathcal{C}}{m}, \frac{\mathcal{C}}{m} \right]$$

Under the assumptions (12), $|z|^{\gamma-\frac{\alpha}{2}}$ is an A_p weight and therefore, recalling a result in [13] (see also [11, p.57 and 313-314]) about the boundedness of the Hilbert transform in $[-1,1]$, we have

$$B_1 \le \mathcal{C} a_m^{\frac{1}{2}+\gamma-\frac{\alpha}{2}+\frac{1}{p}} \left(\int_{-1}^{1} |z|^{\gamma-\frac{\alpha}{2}} \left| (p_{m-1}(W_\alpha) f \Delta_{m,\theta} W_\alpha)(a_m z) \right|^p dz \right)^{\frac{1}{p}}.$$

So, by the change of variable $a_m z = x$, we have

$$B_1 \le \mathcal{C} a_m^{\frac{1}{2}} \left(\int_{-a_m}^{a_m} |x|^{\gamma-\frac{\alpha}{2}} \left| (p_{m-1}(W_\alpha, x) f(x) \Delta_{m,\theta}(x) W_\alpha(x) \right|^p dx \right)^{\frac{1}{p}}$$

and using again (29)

$$B_1 \le \mathcal{C} \left(\int_{\mathbb{R}} |f(x) \Delta_{m,\theta}(x) U_\gamma(x)|^p \, dx \right)^{\frac{1}{p}}. \tag{38}$$

By similar arguments used to bound B_1, we get

$$B_2 \le \mathcal{C} \left(\int_{\mathbb{R}} |f(x) \Delta_{m,\theta}(x) U_\gamma(x)|^p \, dx \right)^{\frac{1}{p}}. \tag{39}$$

Combining (38),(39) with (37), (11) follows.

Now we prove (11) implies (12). Let be

$$C_m = \left\{ x : \mathcal{C} \frac{a_m}{m} \le |x| \le \theta a_m \right\}, \quad C_{m-1} = \left\{ x : \mathcal{C} \frac{a_{m-1}}{m} \le |x| \le \theta a_{m-1} \right\},$$

and let $\Delta_{m,\theta}$, $\Delta_{m-1,\theta}$ the corresponding characteristic functions. Setting $\tilde{f} = f \Delta_{m-1,\theta}$, we have

$$\begin{aligned} & \|[S_m(W_\alpha, \tilde{f}\Delta_{m,\theta}) - S_{m-1}(W_\alpha, \tilde{f}\Delta_{m,\theta})] U_\gamma \Delta_{m,\theta}\|_p \\ &= \left| \int_{\mathbb{R}} \tilde{f}(x) \Delta_{m,\theta}(x) p_m(W_\alpha, x) W_\alpha(x) dx \right| \|\Delta_{m,\theta} p_m(W_\alpha) U_\gamma\|_p. \end{aligned}$$

In view of (11) for $1 < p < \infty$

$$\left| \int_{\mathbb{R}} \tilde{f}(x) \Delta_{m,\theta}(x) p_m(W_\alpha, x) W_\alpha(x) dx \right| \|\Delta_{m,\theta} p_m(W_\alpha) U_\gamma\|_p \le 2 \|f U_\gamma\|_p.$$

Then

$$\|\Delta_{m,\theta} p_m(W_\alpha) U_\gamma\|_p \sup_{||h||_q=1} \left| \int_{\mathbb{R}} \tilde{h}(x) \Delta_{m,\theta}(x) p_m(W_\alpha, x) \frac{W_\alpha(x)}{U_\gamma(x)} dx \right| \le 2\mathcal{C}$$

and also

$$\|\Delta_{m,\theta} p_m(W_\alpha) U_\gamma\|_p \cdot \|\Delta_{m,\theta} p_m(W_\alpha) \frac{W_\alpha}{U_\gamma}\|_q \leq 2\mathcal{C}.$$

Using then Proposition 4.1

$$\frac{1}{a_m}\left(\int_{-1}^{1} |x|^{(\gamma-\frac{\alpha}{2})p} dx\right)^{\frac{1}{p}} \left(\int_{-1}^{1} |x|^{(\frac{\alpha}{2}-\gamma)q} dx\right)^{\frac{1}{q}} \leq 2\mathcal{C},$$

by which conditions in (12) follow.

Now we prove (13). Let $P \in \mathbb{P}_M$, with $M = \left[m\left(\frac{\theta}{1+\theta}\right)^{\beta}\right]$, the polynomial of best approximation of f in $L^p_{U_\gamma}$. By

$$\begin{aligned}
&\|[f - \Delta_{m,\theta} S_m(W_\alpha, f\Delta_{m,\theta})]U_\gamma\|_p \\
&\leq \|(1-\Delta_{m,\theta}) f U_\gamma\|_p + \|[f - S_m(W_\alpha, f\Delta_{m,\theta})]U_\gamma \Delta_{m,\theta}\|_p \\
&\leq \|(1-\Delta_{m,\theta}) f U_\gamma\|_p + \|(f-P)\Delta_{m,\theta} U_\gamma\|_p \qquad (40)\\
&+\|S_m(W_\alpha, (f-P)\Delta_{m,\theta})\Delta_{m,\theta} U_\gamma\|_p \\
&+\|S_m(W_\alpha, P(1-\Delta_{m,\theta})\Delta_{m,\theta} U_\gamma\|_p \\
&=: I_1 + I_2 + I_3 + I_4. \qquad (41)
\end{aligned}$$

Using Proposition 2.1,

$$I_1 + I_2 \leq \mathcal{C}_1 \left(E_M(f)_{U_\gamma,p} + e^{-\mathcal{C}_2 m}\|fU_\gamma\|_p\right)$$

and by (11)

$$I_3 \leq \mathcal{C}\|(f-P)\Delta_{m,\theta} U_\gamma\|_p \leq \mathcal{C} E_M(f)_{U_\gamma,p}.$$

To estimate I_4 we use (19)

$$I_4 \leq \mathcal{C} m^{\frac{1}{3}} |P(1-\Delta_{m,\theta})U_\gamma\|_p,$$

and by (5), we have

$$I_4 \leq \mathcal{C} m^{\frac{1}{3}} e^{-\mathcal{C}_1 m}\|P\Delta_{m,\theta} U_\gamma\|_p.$$

Therefore

$$\|[f - \Delta_{m,\theta} S_m(W_\alpha, f\Delta_{m,\theta})]U_\gamma\|_p \leq \mathcal{C}[E_M(f)_{U_\gamma,p} + e^{-Am}\|fU_\gamma\|_p],$$

that is (13) follows. □

4.3. **Proof of Theorem 3.2.** Using (6)

$$\|S_m(W_\alpha, f\Delta_{m,\theta})U_\gamma\Delta_{m,\theta}\|_1 \le \mathcal{C}\|S_m(W_\alpha, f\Delta_{m,\theta})U_\gamma\Delta_{m,\theta}\|_{L_1(C_m)},$$

$$C_m = \{x : \mathcal{C}\frac{a_m}{m} \le |x| \le \theta a_m\}, \tag{42}$$

and setting $g = sgn(S_m(W_\alpha, f\Delta_{m,\theta}))$,

$$\|S_m(W_\alpha, f\Delta_{m,\theta})U_\gamma\Delta_{m,\theta}\|_1 \le \mathcal{C}\int_{C_m} S_m(W_\alpha, f\Delta_{m,\theta}, x)g(x)U_\gamma(x)dx. \tag{43}$$

By (35) and (36)

$$\begin{aligned} &\|S_m(W_\alpha, f\Delta_{m,\theta})U\Delta_{m,\theta}\|_1 \\ &\le \mathcal{C}\left[a_m\int_{C_m}|p_m(W_\alpha,x)H(f\Delta_{m,\theta}p_{m-1}(W_\alpha)W_\alpha;x)|\,U_\gamma(x)dx\right. \\ &\left.+\, a_m\int_{C_m}|p_{m-1}(W_\alpha,x)H(f\Delta_{m,\theta}p_m(W_\alpha)W_\alpha;x)|\,U_\gamma(x)dx\right] \\ &=: A_1 + A_2. \end{aligned} \tag{44}$$

First we bound A_1. By (29)

$$A_1 \le \mathcal{C}\sqrt{a_m}\int_{C_m}|x|^{\gamma-\frac{\alpha}{2}}\,|H(f\Delta_{m,\theta}p_m(W_\alpha)W_\alpha;x)|\,dx \le \mathcal{C}\int_{\mathbb{R}}|x|^{\gamma-\frac{\alpha}{2}}\,|H(G_m;x)|\,dx$$

where $G_m = \sqrt{a_m}f\Delta_{m,\theta}p_m(W_\alpha)W_\alpha$. Here we recall the following inequality due to Muckenhoupt in [12, Lemma 9, p.440]:

$$\begin{aligned} &\int_{\mathbb{R}}\left(\frac{|x|}{1+|x|}\right)^r(1+|x|)^s\left|\int_{\mathbb{R}}\frac{g(y)}{x-y}dy\right|dx \\ &\le \mathcal{C}+\mathcal{C}\int_{\mathbb{R}}|g(x)\left(\frac{|x|}{1+|x|}\right)^R(1+|x|)^S(1+\log^+|g(x)|+\log^+|x|)dx \end{aligned}$$

under the assumptions $r > -1, s < 0, R \le 0, S \ge -1, r \ge R, s \le S$ and $f\log^+ f \in L_1$.

Using previous result with $r = R = \gamma - \frac{\alpha}{2} = s = S$, under the assumption $0 < \gamma - \frac{\alpha}{2} < 1$ and taking into account $|G_m(x)| \le \mathcal{C}|f(x)|\sqrt{W_\alpha(x)}$, we have

$$A_1 \le \mathcal{C}+\mathcal{C}\int_{\mathbb{R}}|f(x)U_\gamma(x)|\left\{1+\log^+|f(x)|+\log^+|x|\right\}dx. \tag{45}$$

Similarly we obtain

$$A_2 \le \mathcal{C}+\mathcal{C}\int_{\mathbb{R}}|f(x)U_\gamma(x)|\left\{1+\log^+|f(x)|+\log^+|x|\right\}dx. \tag{46}$$

Combining (45), (46) with (44), the Theorem follows. □

To prove Theorems 3.3 and 3.4 we need some relations between generalized Freud and generalized Laguerre polynomials and then we apply the previous estimates about Fourier Sums with respect to generalized Freud weights.

Setting $\tilde{W}_\alpha(x) = |x|^{2\alpha+1}e^{-|x|^{2\beta}}$ and $\tilde{U}_\gamma(x) = |x|^{2\gamma+\frac{1}{p}}e^{-|x|^\beta}$, for the orthogonal polynomials we have

$$p_{2m}(\tilde{W}_\alpha, x) = p_m(w_\alpha, x^2) \tag{47}$$

Moreover, assuming F be an even extension in $\mathbb{R}$ of f defined on $(0,\infty)$, the following relation holds

$$S_{2m}(\tilde{W}_\alpha, F, x) = S_m(w_\alpha, f, x^2). \tag{48}$$

Denoted by $\tilde{\chi}_{2m,\theta}$ the characteristic function of $\tilde{C}_{2m} = [-\theta a_{2m}(\tilde{W}_\alpha)^2, \theta a_{2m}(\tilde{W}_\alpha)^2]$, from (48) easily follows

$$\|S_{2m}(\tilde{W}_\alpha, F\tilde{\chi}_{2m,\theta})\tilde{\Delta}_{2m,\theta}\tilde{U}_\gamma\|_p = \|S_m(w_\alpha, f\chi_{m,\theta})u_\gamma\chi_{m,\theta}\|_p \tag{49}$$

4.4. **Proof of Theorem 3.3.** Let F be an even extension in $\mathbb{R}$ of f defined on $(0,\infty)$. Using Theorem 3.1 we have

$$\|S_{2m}(\tilde{W}_\alpha, F\tilde{\Delta}_{2m,\theta})\tilde{\Delta}_{2m,\theta}\tilde{U}_\gamma\|_p \le \mathcal{C}\|F\tilde{U}_\gamma\tilde{\Delta}_{2m,\theta}\|_p, \tag{50}$$

if and only if

$$\gamma - \frac{\alpha}{2} + \frac{1}{4} < \frac{1}{q} \quad \textit{and} \quad \gamma - \frac{\alpha}{2} - \frac{1}{4} > -\frac{1}{p},$$

which are equivalent to (21).

By (49), and using $\|F\tilde{U}_\gamma\tilde{\Delta}_{2m,\theta}\|_p = \|fu_\gamma\Delta_{m,\theta}\|_p$, with $a_m(w_\alpha) = a_{2m}^2(\tilde{W}_\alpha$, the first part of the Theorem follows.

To prove (22), we premit a Proposition which is the equivalent in $\mathbb{R}^+$ of the Proposition 2.1.

Proposition 4.2. *Let $f \in L^p_{u_\gamma}$ and $1 \le p < \infty$. For m sufficiently large (say $m > m_0$) we have*

$$\|f(1-\chi_{m,\theta})u_\gamma\|_p \le \mathcal{C}_1\left(E_M(f)_{u_\gamma,p} + e^{-\mathcal{C}_2 m}\|fu_\gamma\|_p\right), \tag{51}$$

where $M = \left[m\left(\frac{\theta}{1+\theta}\right)^\beta\right]$ and the constants $\mathcal{C}, \mathcal{C}_1, C_2$ are independent on m and f.

Now we prove (22). Let $P \in \mathbb{P}_M$, with $M = \left[m \left(\frac{\theta}{1+\theta} \right)^{\beta} \right]$, the polynomial of best approximation of f in $L^p_{u_\gamma}$. By

$$\begin{aligned}
\|[f - \chi_{m,\theta} S_m(w_\alpha, f\chi_{m,\theta})]u_\gamma\|_p &\le \|(1- \chi_{m,\theta}) f u_\gamma\|_p + \|[f - S_m(w_\alpha, f\chi_{m,\theta})]u_\gamma \chi_{m,\theta}\|_p \\
&\le \|(1 - \chi_{m,\theta}) f u_\gamma\|_p + \|(f - P)\chi_{m,\theta} u_\gamma\|_p \\
&+ \|S_m(w_\alpha, (f - P)\chi_{m,\theta})\chi_{m,\theta} u_\gamma\|_p \\
&+ \|S_m(w_\alpha, P(1 - \chi_{m,\theta})\chi_{m,\theta} u_\gamma\|_p \\
&=: I'_1 + I'_2 + I'_3 + I'_4.
\end{aligned}$$

Estimate (22) follows using Proposition 4.2,(20) and (28).□

We omit the proof of Theorem 3.4 since it follows by arguments similar to those used in the proof of Theorem 3.3.

References

[1] Askey, R., and Wainger, S., *Mean convergence of expansions in Laguerre and Hermite series,* Amer. J. Math. **87** (1965), 695-708.

[2] Jha, S.W., and Lubinsky, D.S., *Necessary and Sufficient Conditions for Mean convergence of Orthogonal Expansions for Freud Weights*, Constr. Approx. (1995) **11**, p. 331-363.

[3] Kasuga, T., and Sakai, R., *Orthonormal polynomials with generalized Freud-type weights,* J. Approx. Theory **121**(2003), 13-53.

[4] Levin, E., and Lubinsky, D., *Orthogonal polynomials for exponential weights* $x^{2\rho}e^{-2Q(x)}$ *on* $[0, d)$, J. Approx. Theory, **134**(2005), no. 2, 199-256.

[5] Levin, E., and Lubinsky, D., *Orthogonal polynomials for exponential weights* $x^{2\rho}e^{-2Q(x)}$ *on* $[0, d)$*. II*, J. Approx. Theory, **139**(2006), no. 1-2, 107-143.

[6] Mastroianni, G., and Occorsio, D., *Fourier sums in Sonin-Markov polynomials*, Rendiconti del Circolo Matematico di Palermo, Proceedings of the Fifth FAAT, serie II n. **76**(2005), 469-485.

[7] Mastroianni, G., and Szabados, J., *Polynomial approximation on infinite intervals with weights having inner zeros* , Acta Math. Hungar. **96**(2002), no. 3, 221-258.

[8] Mastroianni, G., and Szabados, J., *Direct and converse polynomial approximation theorems on the real line with weights having zeros*, Frontiers in Interpolation and Approximation Dedicated to the memory of A. Sharma, (Eds. N.K. Govil, H.N. Mhaskar, R.N.

Mohapatra, Z. Nashed and J. Szabados), 2006 Taylor & Francis Books, Boca Raton, Florida, 287-306.

[9] Mastroianni, G., and Vertesi, J., *Fourier Sums and Lagrange Interpolation on* $(0, +\infty)$ *and* $(-\infty, +\infty)$, Frontiers in Interpolation and Approximation Dedicated to the memory of A. Sharma, (Eds. N.K.Govil, H.N. Mhaskar, R.N. Mohapatra, Z. Nashed and J. Szabados), 2006 Taylor & Francis Books, Boca Raton, Florida, 307-344.

[10] Mhaskar, H.N., and Saff, E.B., *Extremal Problems for Polynomials with Laguerre Weights*, Approx. Theory IV, (College Station, Tex., 1983), Academic Press, New York, 1983, 619-624.

[11] Michlin, S.G., and Prössdorf, S., *Singular Integral Operators*, Mathematical Textbooks and Monographs, Part II: Mathematical Monographs, **52** Akademie-Verlag, Berlin, (1980).

[12] Muckenhoupt, B., *Mean convergence of Hermite and Laguerre series II*, Trans. Amer. Math. Soc. **147**(1970), 433-460.

[13] Muckenhoupt, B., *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc., **165**(1972), 207-226.

Dipartimento di Matematica ed Informatica,
Università della Basilicata,
Via dell'Ateneo Lucano 10,
85100 Potenza, Italy
E-mail address: giuseppe.mastroianni@unibas.it

Dipartimento di Matematica ed Informatica,
Università della Basilicata,
Via dell'Ateneo Lucano 10,
85100 Potenza, Italy
E-mail address: donatella.occorsio@unibas.it