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ON BERNSTEIN-STANCU TYPE OPERATORS

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Dedicated to Professor D. D. Stancu on his 80^{th} *birthday*

Abstract. D.D. Stancu defined in [5] a class of approximation operators depending of two non-negative parameters α and β, $0 \leq \alpha \leq \beta$. We consider here another class of Bernstein-Stancu type operators.

1. Introduction

Let f be a continuous functions, $f : [0,1] \to \mathbb{R}$. For every natural number n we denote by $B_n f$ Bernstein's polynomial of degree n,

$$(B_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right),$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0, 1, \ldots n.$$

In 1968 D.D. Stancu introduced in [5] a linear positive operator depending on two non-negative parameters α and β satisfying the condition $0 \leq \alpha \leq \beta$.

For every continuous function f and for every $n \in \mathbb{N}$ the polynomial $P_n^{(\alpha,\beta)} f$ defined in [5] is given by

$$(P_n^{(\alpha,\beta)} f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k+\alpha}{n+\beta}\right).$$

Note that for $\alpha = \beta = 0$ the Bernstein-Stancu operators become the classical Bernstein operators B_n. In [2] were introduced the following linear operators A_n :

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$C[0,1] \to \Pi_n$, defined as

$$A_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) T_{n,k} f \tag{1.1}$$

where $T_{n,k} : C[0,1] \to \mathbb{R}$ are positive linear functionals with the property that $T_{n,k}e_0 = 1$ for $k = 0, 1, \ldots, n$ and $e_i(t) = t^i$, $i \in \mathbb{N}$.

So, for $T_{n,k}f = f\left(\frac{k}{n}\right)$ we obtain Bernstein's polynomial of degree n and for

$$T_{n,k} f = f\left(\frac{k+\alpha}{n+\beta}\right)$$

where $0 \le \alpha \le \beta$ the operator A_n becomes Bernstein-Stancu operator $P_n^{(\alpha,\beta)}$.

In [4] C. Mortici and I. Oancea defined a new class of operators of Bernstein-Stancu type operators. They considered the non-nonegative real numbers $\alpha_{n,k}$, $\beta_{n,k}$ so that

$$\alpha_{n,k} \le \beta_{n,k}.$$

They define an approximation operator denoted by

$$P_n^{(A,B)} : C[0,1] \to C[0,1]$$

with the formula

$$(P_n^{(A,B)} f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k+\alpha_{n,k}}{m+\beta_{n,k}}\right).$$

In [4] the following theorem was proved:

Theorem 1.1. *Given the infinite dimensional lower triangular matrices*

$$A = \begin{pmatrix} \alpha_{00} & 0 & \ldots & & \\ \alpha_{10} & \alpha_{11} & 0 & \ldots & \\ \alpha_{20} & \alpha_{21} & \alpha_{22} & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

and

$$B = \begin{pmatrix} \beta_{00} & 0 & \dots & & \\ \beta_{10} & \beta_{11} & 0 & \dots & \\ \beta_{20} & \beta_{21} & \beta_{22} & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

with the following properties:

a) $0 \le \alpha_{n,k} \le \beta_{n,k}$ *for every non-negative integers* n *and* $k \le n$

b) $\alpha_{n,k} \in [a,b]$, $\beta_{n,k} \in [c,d]$ *for every non-negative integers* n *and* k, $k \le n$ *and for some non-negative real numbers* $0 \le a < b$ *and* $0 \le c < d$. *Then for every continuous function* $f \in C[0,1]$, *we have*

$$\lim_{m\to\infty} P_n^{(\alpha,\beta)} f = f, \text{ uniformly on } [0,1].$$

In the following, by the definition, an operator of the form (1.1), where

$$T_{n,k} f = f(x_{k,n}), \quad k \le n, \quad k,n \in \mathbb{N}$$

is an operator of the Bernstein-Stancu type.

2. Main results

First we characterize the Bernstein-Stancu operators which transform the polynomial of degree one into the polynomials of degree one.

Theorem 2.1. *Let* $A_n : C[0,1] \to C[0,1]$ *an operator of the Bernstein-Stancu type.*

Then

$$x_{k,n} = \alpha_n \frac{k}{n} + \beta_n, \quad k \le n$$

where α_n, β_n *are positive numbers such that*

$$\alpha_n + \beta_n \le 1.$$

Proof. By the definition of the operator A_n of the Bernstein-Stancu type we have

$$A_n(e_0)(x) = \sum_{k=0}^{n} p_{n,k}(x) = 1.$$

Let us suppose that

$$A_n(e_1)(x) = \alpha_n x + \beta_n.$$

From the equality

$$\sum_{k=0}^{n} p_{n,k}(x)\frac{k}{n} = x$$

we get

$$\sum_{k=0}^{n} p_{n,k}(x) x_{k,n} = \sum_{k=0}^{n} p_{n,k}(x)\left(\alpha_n \frac{k}{n} + \beta_n\right). \tag{2.1}$$

Because the set $\{p_{n,k}\}_{k\in\{0,1,\dots,n\}}$ forms a basis in Π_n we get

$$x_{k,n} = \alpha_n \frac{k}{n} + \beta_n.$$

By the condition $x_{k,n} \in [0,1]$, $0 \le k \le n$, $k, n \in \mathbb{N}$ we obtain

$$\alpha_n, \beta_n \ge 0 \text{ and } \alpha_n + \beta_n \le 1.$$

Remark. There exist operators of the Bernstein-Stancu type which don't transform polynomials of degree one into the polynomials of the same degree.

An interesting operator of Bernstein-Stancu type, which maps e_2 into e_2 is the following:

$$B_n^*(f)(x) = \sum_{k=0}^{n} p_{n,k} f\left(\sqrt{\frac{k(k-1)}{n(n-1)}}\right), \quad n \in \mathbb{N},\ n > 1. \tag{2.2}$$

For the operator B_n^* verifies the following relations:

$$B_n^*(e_0) = e_0$$

$$B_n^*(e_2) = e_2$$

$$\frac{nx-1}{n-1} - \frac{1}{n} p_{n,1}(x) \le B_n(e_1)(x) \le x.$$

The following result describes the fact that $(A_n)_{n\in\mathbb{N}}$ given by (1.1) is a positive linear approximation process.

Theorem 2.2. *Let $(A_n)_{n\in\mathbb{N}}$ be defined as in (1.1) and $f \in C[0,1]$. Then*

$$\lim_{n\to\infty} \|f - A_n f\|_\infty = 0 \tag{2.3}$$

if and only if

$$\lim_{n\to\infty} \|\Delta_n\|_\infty = 0 \tag{2.4}$$

where

$$\Delta_n(x) := \sum_{k=0}^{n} p_{n,k}(x) T_{n,k} \left(\cdot - \frac{k}{n}\right)^2 . \tag{2.5}$$

Proof. $(\Rightarrow)$: For the validity of (2.4) it is sufficient to verify the assumption of Popoviciu-Bohman-Korovkin theorem. We first notice that

$$|\Delta_n(x)| = \left| \sum_{k=0}^{n} p_{n,k}(x) T_{n,k}(e_2) - 2 \sum_{k=0}^{n} p_{n,k}(x) \frac{k}{n} T_{n,k}(e_1) + x^2 + \frac{x(1-x)}{n} \right| \tag{2.6}$$

and if for all $f \in C[0,1]$

$$\lim_{n\to\infty} \|f - A_n f\|_\infty = 0,$$

we get

$$\lim_{n\to\infty} \sum_{k=0}^{n} p_{n,k}(x) T_{n,k}(e_2) = \lim_{n\to\infty} A_n(e_2)(x) = x^2$$

and

$$\lim_{n\to\infty} \left\{ \sum_{k=0}^{n} p_{n,k}(x) \frac{k}{n} T_{n,k}(e_1) - x^2 \right\} = \lim_{n\to\infty} \sum_{k=0}^{n} p_{n,k}(x) \frac{k}{n} \{T_{n,k}(e_1) - x\}.$$

Now, we can estimate

$$\left| \sum_{k=0}^{n} p_{n,k}(x) \frac{k}{n} \{T_{n,k}(e_1) - x\} \right| \le \sum_{k=0}^{n} p_{n,k}(x) T_{n,k}(|e_1 - x|) \le \sqrt{A_n(\cdot - x)^2(x)}.$$

From this and (2.6) it follows that

$$|\Delta_n(x)| \le |A_n(e_2)(x) - x^2| + 2\sqrt{A_n(1-x)^2(x)} + \frac{x(1-x)}{n}$$

and therefore one obtains

$$\lim_{n\to\infty} \|\Delta_n\|_\infty = 0.$$

$(\Leftarrow)$: Suppose now that (2.4) holds with the following two estimates

$$|A_n(e_1)(x) - x| \le \sqrt{\Delta_n(x)}$$

and

$$\begin{aligned}|A_n(e_2)(x)-x^2| &= \left|\sum_{k=0}^{n} p_{n,k}(x)T_{n,k}\left(\cdot-\frac{k}{n}\right)\left(\cdot+\frac{k}{n}\right)+\frac{x(1-x)}{n}\right| \\ &\le 2\sum_{k=0}^{n} p_{n,k}(x)T_{n,k}\left(\left|\cdot-\frac{k}{n}\right|\right)+\frac{x(1-x)}{n} \\ &\le 2\sqrt{\Delta_n(x)}+\frac{x(1-x)}{n}\end{aligned}$$

and finishes the proof of this theorem.

Remarks. 1. Theorem 2.2 can be find in [2].

2. Theorem 1.1 ([4]) follows from the following estimate:

$$\begin{aligned}\Delta_n(x) &= \sum_{k=0}^{n} p_{n,k}(x)\left(\frac{k+\alpha_{n,k}}{n+\beta_{n,k}}-\frac{k}{n}\right)^2 \\ &= \sum_{k=0}^{n} p_{n,k}(x)\frac{(n\alpha_{n,k}-k\beta_{n,k})^2}{n^2(n+\beta_{n,k})^2} \\ &\le \sum_{k=0}^{n} p_{n,k}(x)\frac{(b+d)^2}{(n+a)^2}=\frac{(b+d)^2}{(n+a)^2}\end{aligned}$$

Theorem 2.3. *Let A_n be an operator of the form (1.1) such that*

$$A_n e_1 = \alpha_n e_1 + \beta_n.$$

We denote by L_n the operator of Bernstein-Stancu type given by

$$(L_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\alpha_n \frac{k}{n} + \beta_n\right).$$

Then, for all $x \in [0,1]$ and for all convex functions f we have

$$f(\alpha_n x + \beta_n) \le (L_n f)(x) \le A_n(f)(x).$$

Moreover, if f is a strict convex function and $L_n(f)(x_0) = A_n(f(x_0))$ for some $x_0 \in (0,1)$, if and only if $L_n = A_n$.

Proof. Because $(p_{n,k})_{k=0,n}$ is a basis in Π_n by the condition

$$A_n e_1 = \alpha_n e_1 + \beta_n$$

we obtain that

$$T_{n,k}e_1 = \alpha_n \frac{k}{n} + \beta_n.$$

Let f be a convex function. From Jensen's inequality we have

$$T_{n,k}(f) \geq f(T_{n,k}(e_1)) = f\left(\alpha_n \frac{k}{n} + \beta_n\right) \tag{2.7}$$

By (2.7) we get

$$\sum_{k=0}^{n} p_{n,k}(x) T_{n,k}(f) \geq \sum_{k=0}^{n} p_{n,k}(x) f\left(\alpha_n \frac{k}{n} + \beta_n\right) \geq f(\alpha_n x + \beta_n)$$

or

$$A_n(f)(x) \geq (L_n f)(x) \geq f(\alpha_n x + \beta_n).$$

Let us suppose that

$$L_n(f)(x_0) = A_n(f)(x_0). \tag{2.8}$$

The equality (2.8) can be written as:

$$\sum_{k=0}^{n} p_{n,k}(x_0) \left(T_{n,k}(f) - f\left(\alpha_n \frac{k}{n} + \beta_n\right)\right) = 0.$$

Because

$$p_{n,k}(x_0) \geq 0, \quad k = 0, 1, \ldots, n$$

follows that

$$T_{n,k}(f) - f\left(\alpha_n \frac{k}{n} + \beta_n\right) = 0, \quad k = 0, 1, \ldots, n. \tag{2.9}$$

It is known the following result [3]:

Let A be a linear positive functional, $A : C[0,1] \to \mathbb{R}$. Then, there exists the distinct points $\xi_1, \xi_2 \in [0,1]$ such that

$$A(f) - f(a_1) = [a_2^2 - a_1^2]\left[\xi_1, \frac{\xi - 1 + \xi_2}{2}, \xi_2; f\right] \tag{2.10}$$

where $a_i = A(e_i)$, $i \in \mathbb{N}$.

By (2.9) and (2.10) we obtain

$$(T_{n,k}(e_2)) - T_{n,k}^2(e_1) = 0, \quad k = 0, 1, \ldots, n. \tag{2.11}$$

From (2.11) we get

$$T_{n,k}(f) = f(T_{n,k}(e_1)) = f\left(\alpha_n \frac{k}{n} + \beta_n\right), \quad k = 0, 1, \ldots, n$$

for every continuous function f.

This finished the proof.

Remark. This extremal relation for the Bernstein-Stancu operators was considered in [1] in particular case when $f = e_2$.

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