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FIRST ORDER ITERATIVE FUNCTIONAL-DIFFERENTIAL EQUATION WITH PARAMETER

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Dedicated to Professor D. D. Stancu on his 80^{th} *birthday*

Abstract. We consider the following first order iterative functional-differential equation with parameter

$$x'(t) = f(t, x(t), x(x(t))) + \lambda, \quad t \in [a,b];$$
$$x(t) = \varphi(t), \quad a_1 \le t \le a,$$
$$x(t) = \psi(t), \quad b \le t \le b_1.$$

Using the Schauder's fixed point theorem we first establish an existence theorem, then by means of the contraction principle state an existence and uniqueness theorem, and after that a data dependence result. Finally, we give some examples which illustrate our results.

1. Introduction

Although many works on functional-differential equation exist (see for example J. K. Hale and S. Verduyn Lunel [9], V. Kalmanovskii and A. Myshkis [10] and T. A. Burton [3] and the references therein), there are a few on iterative functional-differential equations ([2], [4], [5], [8], [12], [13], [16], [17], [19]).

In this paper we consider the following problem:

$$x'(t) = f(t, x(t), x(x(t))) + \lambda, \quad t \in [a,b]; \tag{1.1}$$

$$x|_{[a_1,a]} = \varphi, \qquad x|_{[b,b_1]} = \psi. \tag{1.2}$$

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where

(C_1) $a, b, a_1, b_1 \in \mathbb{R}$, $a_1 \le a < b \le b_1$;

(C_2) $f \in C([a,b] \times [a_1, b_1]^2, \mathbb{R})$;

(C_3) $\varphi \in C([a_1, a], [a_1, b_1])$ and $\psi \in C([b, b_1], [a_1, b_1])$;

The problem is to determine the pair (x, λ),

$$x \in C([a_1, b_1], [a_1, b_1]) \cap C^1([a, b], [a_1, b_1]), \quad \lambda \in \mathbb{R},$$

which satisfies (1.1)+(1.2).

In this paper, using the Schauder's fixed point theorem we first establish an existence theorem, then by means of the contraction principle state an existence and uniqueness theorem, and after that a data dependence result. Finally, we take an example to illustrate our results.

2. **Existence**

We begin our considerations with some remarks.

Let (x, λ) be a solution of the problem (1.1)+(1.2). Then this problem is equivalent with the following fixed point equation

$$x(t) = \begin{cases} \varphi(t), & t \in [a_1, a], \\ \varphi(a) + \displaystyle\int_a^t f(s, x(s), x(x(s)))\,\mathrm{d}s + \lambda(t-a), & t \in [a, b], \\ \psi(t), & t \in [b, b_1]. \end{cases} \tag{2.3}$$

From the condition of continuity of x in $t = b$, we have that

$$\lambda = \frac{\psi(b) - \varphi(a)}{b - a} - \frac{1}{b-a} \int_a^b f(s, x(s), x(x(s)))\,\mathrm{d}s. \tag{2.4}$$

Now we consider the operator

$$A : C([a_1, b_1], [a_1, b_1]) \to C([a_1, b_1], \mathbb{R}),$$

where

$$A(x)(t) := \begin{cases} \varphi(t), & t \in [a_1, a], \\ \varphi(a) + \dfrac{t-a}{b-a}(\psi(b) - \varphi(a)) - \dfrac{t-a}{b-a}\displaystyle\int_a^b f(s, x(s), x(x(s)))\,\mathrm{d}s + \\ \qquad + \displaystyle\int_a^t f(s, x(s), x(x(s)))\,\mathrm{d}s, & t \in [a, b], \\ \psi(t), & t \in [b, b_1]. \end{cases} \tag{2.5}$$

It is clear that (x, λ) is a solution of the problem (1.1)+(1.2) iff x is a fixed point of the operator A and λ is given by (2.4).

So, the problem is to study the fixed point equation

$$x = A(x).$$

We have

Theorem 2.1. *We suppose that*

(i) the conditions $(\mathrm{C}_1) - (\mathrm{C}_3)$ *are satisfied;*

(ii) $m_f \in \mathbb{R}$ *and* $M_f \in \mathbb{R}$ *are such that* $m_f \leq f(t, u_1, u_2) \leq M_f,\ \forall\ t \in [a, b],\ u_i \in [a_1, b_1],\ i = 1, 2,$ *and we have:*

$$a_1 \leq \min\left(\varphi(a), \psi(b)\right) - \max\left(0, M_f(b-a)\right) + \min\left(0, m_f(b-a)\right),$$

and

$$\max\left(\varphi(a), \psi(b)\right) - \min\left(0, m_f(b-a)\right) + \max\left(0, M_f(b-a)\right) \leq b_1.$$

Then the problem $(1.1) + (1.2)$ *has in* $C([a_1, b_1], [a_1, b_1])$ *at least a solution.*

Proof. In what follow we consider on $C([a_1, b_1], \mathbb{R})$ the Chebyshev norm, $||\cdot||_C$.

Condition (ii) assures that the set $C([a_1, b_1], [a_1, b_1])$ is an invariant subset for the operator A, that is, we have

$$A(C([a_1, b_1], [a_1, b_1])) \subset C([a_1, b_1], [a_1, b_1]).$$

Indeed, for $t \in [a_1, a] \cup [b, b_1]$, we have $A(x)(t) \in [a_1, b_1]$. Furthermore, we we obtain

$$a_1 \leq A(x)(t) \leq b_1, \ \forall t \in [a, b],$$

if and only if

$$a_1 \leq \min_{t \in [a,b]} A(x)(t) \tag{2.6}$$

and

$$\max_{t \in [a,b]} A(x)(t) \leq b_1 \tag{2.7}$$

hold. Since

$$\min_{t \in [a,b]} A(x)(t) = \min\left(\varphi(a), \psi(b)\right) - \max\left(0, M_f(b-a)\right) + \min\left(0, m_f(b-a)\right),$$

respectively

$$\max_{t \in [a,b]} A(x)(t) = \max\left(\varphi(a), \psi(b)\right) - \min\left(0, m_f(b-a)\right) + \max\left(0, M_f(b-a)\right),$$

the requirements (2.6) and (2.7) are equivalent with the conditions appearing in (ii).

So, in the above conditions we have a selfmapping operator

$$A : C([a_1, b_1], [a_1, b_1]) \to C([a_1, b_1], [a_1, b_1]).$$

It is clear that A is completely continuous and the set $C([a_1, b_1], [a_1, b_1]) \subseteq C([a_1, b_1], \mathbb{R})$ is a bounded convex closed subset of the Banach space $(C([a_1, b_1], \mathbb{R}), \| \cdot \|_C)$. By Schauder's fixed point theorem the operator A has at least a fixed point. □

3. Existence and uniqueness

Let $L > 0$, and introduce the following notation:

$$C_L([a_1, b_1], [a_1, b_1]) := \{x \in C([a_1, b_1], [a_1, b_1]) | \ |x(t_1) - x(t_2)| \leq L|t_1 - t_2|,$$

$$\forall \ t_1, t_2 \in [a_1, b_1]\}.$$

Remark that $C_L([a_1, b_1], [a_1, b_1]) \subset (C([a_1, b_1], \mathbb{R}), \| \cdot \|_C)$ is a complete metric space.

We have

Theorem 3.1. *We suppose that*

(i) *the conditions* $(C_1) - (C_3)$ *are satisfied;*

(ii) *there exists* $L_f > 0$ *such that:*

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \leq L_f \left(|u_1 - v_1| + |u_2 - v_2|\right),$$

for all $t \in [a,b]$, $u_i, v_i \in [a_1,b_1]$, $i = 1,2$;

(iii) $\varphi \in C_L([a_1,a],[a_1,b_1])$, $\psi \in C_L([b,b_1],[a_1,b_1])$;

(iv) $m_f, M_f \in \mathbb{R}$ *are such that* $m_f \leq f(t,u_1,u_2) \leq M_f$, $\forall\, t \in [a,b]$, $u_i \in [a_1,b_1]$, $i = 1,2$, *and we have:*

$$a_1 \leq \min\left(\varphi(a),\psi(b)\right) - \max\left(0, M_f(b-a)\right) + \min\left(0, m_f(b-a)\right),$$

and

$$\max\left(\varphi(a),\psi(b)\right) - \min\left(0, m_f(b-a)\right) + \max\left(0, M_f(b-a)\right) \leq b_1;$$

(v) $2\max\{|m_f|,|M_f|\} + \left|\dfrac{\psi(b)-\varphi(a)}{b-a}\right| \leq L$;

(vi) $2L_f(L+2)(b-a) < 1$.

Then the problem (1.1)+(1.2) *has in* $C_L([a_1,b_1],[a_1,b_1])$ *a unique solution. Moreover, if we denote by* (x^*,λ^*) *the unique solution of the Cauchy problem, then it can be determined by*

$$x^* = \lim_{n\to\infty} A^n(x), \text{ for all } x \in X,$$

and

$$\lambda^* = \frac{\psi(b)-\varphi(a)}{b-a} - \frac{1}{b-a}\int_a^b f(s,x^*(s),x^*(x^*(s)))\,ds.$$

Proof. Consider the operator $A : C_L([a_1,b_1],[a_1,b_1]) \to C([a_1,b_1],\mathbb{R})$ given by (2.5).

Conditions (iii) and (iv) imply that $C_L([a_1,b_1],[a_1,b_1])$ is an invariant subset for A. Indeed, from the Theorem 2.1 we have

$$a_1 \leq A(x)(t) \leq b_1,\ x(t) \in [a_1,b_1]$$

for all $t \in [a_1,b_1]$.

Now, consider $t_1,t_2 \in [a_1,a]$. Then,

$$|A(x)(t_1) - A(x)(t_2)| = |\varphi(t_1) - \varphi(t_2)| \leq L|t_1 - t_2|,$$

as $\varphi \in C_L([a_1, a], [a_1, b_1])$, due to (iii).

Similarly, for $t_1, t_2 \in [b, b_1]$

$$|A(x)(t_1) - A(x)(t_2)| = |\psi(t_1) - \psi(t_2)| \le L|t_1 - t_2|,$$

that follows from (iii), too.

On the other hand, if $t_1, t_2 \in [a, b]$, we have,

$$\begin{aligned}
&|A(x)(t_1) - A(x)(t_2)| = \\
&= \left|\varphi(a) + \frac{t_1 - a}{b - a}(\psi(b) - \varphi(a)) - \frac{t_1 - a}{b - a}\int_a^b f(s, x(s), x(x(s)))\,\mathrm{d}s\right. \\
&+ \int_a^{t_1} f(s, x(s), x(x(s)))\,\mathrm{d}s - \varphi(a) - \frac{t_2 - a}{b - a}(\psi(b) - \varphi(a)) \\
&\left.+\frac{t_2 - a}{b - a}\int_a^b f(s, x(s), x(x(s)))\,\mathrm{d}s - \int_a^{t_2} f(s, x(s), x(x(s)))\,\mathrm{d}s\right| \\
&= \left|\frac{t_1 - t_2}{b - a}[\psi(b) - \varphi(a)] - \frac{t_1 - t_2}{b - a}\int_a^b f(s, x(s), x(x(s)))\,\mathrm{d}s - \int_{t_1}^{t_2} f(s, x(s), x(x(s)))\,\mathrm{d}s\right| \\
&\le |t_1 - t_2| \left[\left|\frac{\psi(b) - \varphi(a)}{b - a}\right| + 2\max\{|m_f|, |M_f|\}\right] \le L|t_1 - t_2|.
\end{aligned}$$

Therefore, due to (v), the operator A is L-Lipschitz and, consequently, it is an invariant operator on the space $C_L([a_1, b_1], [a_1, b_1])$.

From the condition (v) it follows that A is an L_A-contraction with

$$L_A := 2L_f(L + 2)(b - a).$$

Indeed, for all $t \in [a_1, a] \cup [b, b_1]$, we have $|A(x_1)(t) - A(x_2)(t)| = 0$.

Moreover, for $t \in [a,b]$ we get

$$
\begin{aligned}
&|A(x_1)(t) - A(x_2)(t)| \leq \\
\leq & \left| \frac{t-a}{b-a} \int_a^b [f(s, x_1(s), x_1(x_1(s))) - f(s, x_2(s), x_2(x_2(s)))] \, \mathrm{ds} \right| + \\
+ & \left| \int_a^t [f(s, x_1(s), x_1(x_1(s))) - f(s, x_2(s), x_2(x_2(s)))] \, \mathrm{ds} \right| \leq \\
\leq & \max_{t \in [a,b]} \left| \frac{t-a}{b-a} \right| \cdot L_f \int_a^b (|x_1(s) - x_2(s)| + |x_1(x_1(s)) - x_2(x_2(s))|) \, \mathrm{ds} + \\
+ & L_f \int_a^t (|x_1(s) - x_2(s)| + |x_1(x_1(s)) - x_2(x_2(s))|) \, \mathrm{ds} \leq \\
\leq & L_f \left[(b-a)||x_1 - x_2||_C + \int_a^b |x_1(x_1(s)) - x_1(x_2(s)) + x_1(x_2(s)) - x_2(x_2(s))| \mathrm{ds} \right] + \\
+ & L_f \left[(t-a)||x_1 - x_2||_C + \int_a^t |x_1(x_1(s)) - x_1(x_2(s)) + x_1(x_2(s)) - x_2(x_2(s))| \mathrm{ds} \right] \leq \\
\leq & 2L_f(b-a) \left(||x_1 - x_2||_C + L||x_1 - x_2||_C + ||x_1 - x_2||_C \right) = \\
= & 2L_f(L+2)(b-a)||x_1 - x_2||_C.
\end{aligned}
$$

By the contraction principle the operator A has a unique fixed point, that is the problem (1.1) + (1.2) has in $C_L([a_1,b_1],[a_1,b_1])$ a unique solution (x^*, λ^*).

Obviously, x^* can be determined by

$$x^* = \lim_{n \to \infty} A^n(x), \text{ for all } x \in X,$$

and, from (2.4) we get

$$\lambda^* = \frac{\psi(b) - \varphi(a)}{b-a} - \frac{1}{b-a} \int_a^b f(s, x^*(s), x^*(x^*(s))) \, \mathrm{d}s.$$

□

4. **Data dependence**

Consider the following two problems

$$\begin{cases} x'(t) = f_1(t, x(t), x(x(t))) + \lambda_1, & t \in [a,b] \\ x(t) = \varphi_1(t), & t \in [a_1, a] \\ x(t) = \psi_1(t), & t \in [b, b_1] \end{cases} \tag{4.8}$$

and

$$\begin{cases} x'(t) = f_2(t, x(t), x(x(t))) + \lambda_2, & t \in [a,b] \\ x(t) = \varphi_2(t), & t \in [a_1, a] \\ x(t) = \psi_2(t), & t \in [b, b_1] \end{cases} \tag{4.9}$$

Let f_i, φ_i and ψ_i, $i = 1, 2$ be as in the Theorem 3.1.

Consider the operators $A_1, A_2 : C_L([a_1, b_1], [a_1, b_1]) \to C_L([a_1, b_1], [a_1, b_1])$ given by

$$A_i(x)(t) := \begin{cases} \varphi_i(t), & t \in [a_1, a], \\ \varphi_i(a) + \dfrac{t-a}{b-a}(\psi_i(b) - \varphi_i(a)) - \dfrac{t-a}{b-a}\displaystyle\int_a^b f_i(s, x(s), x(x(s)))\,\mathrm{d}s + \\ \qquad + \displaystyle\int_a^t f_i(s, x(s), x(x(s)))\,\mathrm{d}s, & t \in [a,b], \\ \psi_i(t), & t \in [b, b_1], \end{cases} \tag{4.10}$$

$i = 1, 2$.

Thus, these operators are contractions. Denote by x_1^*, x_2^* their unique fixed points.

We have

Theorem 4.1. *Suppose we are in the conditions of the Theorem 3.1, and, moreover*

(i) *there exists η_1 such that*

$$|\varphi_1(t) - \varphi_2(t)| \le \eta_1, \quad \forall t \in [a_1, a],$$

and

$$|\psi_1(t) - \psi_2(t)| \le \eta_1, \quad \forall t \in [b, b_1];$$

(ii) *there exists* $\eta_2 > 0$ *such that*

$$|f_1(t,u_1,u_2) - f_2(t,u_1,u_2)| \leq \eta_2, \ \forall\ t \in [a,b], \ \forall\ u_i \in [a_1,b_1], \ i=1,2.$$

Then

$$||x_1^* - x_2^*||_C \leq \frac{3\eta_1 + 2(b-a)\eta_2}{1 - 2L_f(L+2)(b-a)},$$

and

$$|\lambda_1^* - \lambda_2^*| \leq \frac{2\eta_1}{b-a} + \eta_2,$$

where $L_f = \max(L_{f_1}, L_{f_2})$, *and* (x_i^*, λ_i^*), $i=1,2$ *are the solutions of the corresponding problems* (4.8), (4.9).

Proof. It is easy to see that for $t \in [a_1,a] \cup [b,b_1]$ we have

$$\|A_1(x) - A_2(x)\|_C \leq \eta_1.$$

On the other hand, for $t \in [a,b]$, we obtain

$$|A_1(x)(t) - A_2(x)(t)| = \Big|\varphi_1(a) - \varphi_2(a) + \frac{t-a}{b-a}\left[\psi_1(b) - \psi_2(b) - (\varphi_1(a) - \varphi_2(a))\right] -$$

$$-\frac{t-a}{b-a}\int_a^b [f_1(s,x(s),x(x(s))) - f_2(s,x(s),x(x(s)))]\,\mathrm{d}s +$$

$$+\int_a^t [f_1(s,x(s),x(x(s))) - f_2(s,x(s),x(x(s)))]\,\mathrm{d}s\Big| \leq$$

$$\leq |\varphi_1(a) - \varphi_2(a)| + \frac{t-a}{b-a}\left[|\psi_1(b) - \psi_2(b)| + |\varphi_1(a) - \varphi_2(a)|\right] +$$

$$+\frac{t-a}{b-a}\int_a^b |f_1(s,x(s),x(x(s))) - f_2(s,x(s),x(x(s)))|\,\mathrm{d}s +$$

$$+\int_a^t |f_1(s,x(s),x(x(s))) - f_2(s,x(s),x(x(s)))|\,\mathrm{d}s \leq$$

$$\leq \eta_1 + \max_{t\in[a,b]}\frac{t-a}{b-a}\cdot[2\eta_1 + \eta_2(b-a)] + \eta_2 \cdot \max_{t\in[a,b]}(t-a) =$$

$$= 3\eta_1 + 2(b-a)\eta_2$$

So, we have

$$\|A_1(x) - A_2(x)\|_C \leq 3\eta_1 + 2(b-a)\eta_2, \ \forall\ x \in C_L([a_1,b_1],[a_1,b_1]).$$

Consequently, from the data dependence theorem we obtain

$$\|x_1^* - x_2^*\|_C \leq \frac{3\eta_1 + 2(b-a)\eta_2}{1 - 2L_f(L+2)(b-a)}.$$

Moreover, we get

$$|\lambda_1^* - \lambda_2^*| =$$

$$= \left| \frac{\psi_1(b) - \varphi_1(a)}{b-a} - \frac{1}{b-a}\int_a^b f_1(s, x(s), x(x(s)))\,\mathrm{d}s - \frac{\psi_2(b) - \varphi_2(a)}{b-a} + \right.$$

$$\left. + \frac{1}{b-a}\int_a^b f_2(s, x(s), x(x(s)))\,\mathrm{d}s \right| \leq$$

$$\leq \frac{1}{b-a}\Bigg[|\psi_1(b) - \psi_2(b)| + |\varphi_1(a) - \varphi_2(a)| +$$

$$+ \int_a^b |f_1(s, x(s), x(x(s))) - f_2(s, x(s), x(x(s)))|\,\mathrm{d}s \Bigg] \leq$$

$$\leq \frac{1}{b-a}[\eta_1 + \eta_1 + \eta_2(b-a)] = \frac{2\eta_1}{b-a} + \eta_2,$$

and the proof is complete. □

5. **Examples**

Consider the following problem:

$$x'(t) = \mu x(x(t)) + \lambda; \quad t \in [0,1], \;\; \mu \in \mathbb{R}_+^*, \;\; \lambda \in \mathbb{R} \tag{5.11}$$

$$x|_{[-h,0]} = 0; \quad x|_{[1,1+h]} = 1, \quad h \in \mathbb{R}_+^* \tag{5.12}$$

with $x \in C([-h, 1+h], [-h, 1+h]) \cap C^1([0,1], [-h, 1+h])$.

We have

Proposition 5.1. *We suppose that*

$$\mu \leq \frac{h}{1+2h}.$$

Then the problem (5.11) *has in* $C([-h, 1+h], [-h, 1+h])$ *at least a solution.*

Proof. First of all notice that accordingly to the Theorem 2.1 we have $a = 0$, $b = 1$, $\psi(b) = 1, \varphi(a) = 0$ and $f(t, u_1, u_2) = \mu u_2$. Moreover, $a_1 = -h$ and $b_1 = 1 + h$ can be taken. Therefore, from the relation

$$m_f \le f(t, u_1, u_2) \le M_f, \ \forall t \in [0, 1], \forall u_1, u_2 \in [-h, 1+h],$$

we can choose $m_f = -h\mu$ and $M_f = (1+h)\mu$.

For these data it can be easily verified that the conditions (ii) from the Theorem 2.1 are equivalent with the relation

$$\mu \le \frac{h}{1+2h},$$

consequently we have the proof. □

Let $L > 0$ and consider the complete metric space $C_L([-h, h+1], [-h, h+1])$ with the Chebyshev norm $\|\cdot\|_C$.

Another result reads as follows.

Proposition 5.2. *Consider the problem* (5.11). *We suppose that*

(i) $\mu \le \frac{h}{1+2h}$;

(ii) $2(1+h)\mu + 1 \le L$

(iii) $2\mu(L+2) < 1$

Then the problem (5.11) *has in* $C_L([-h, h+1], [-h, h+1])$ *a unique solution.*

Proof. Observe that the Lipschitz constant for the function $f(t, u_1, u_2) = \mu u_2$ is $L_f = \mu$.

By a common check in the conditions of Theorem 3.1 we can make sure that

$$2\max\{|m_f|, |M_f|\} + \left|\frac{\psi(b) - \varphi(a)}{b-a}\right| \le L \Longleftrightarrow 2(1+h)\mu + 1 \le L,$$

and

$$2L_f(L+2)(b-a) < 1 \Longleftrightarrow 2\mu(L+2) < 1.$$

Therefore, by Theorem 3.1 we have the proof. □

Now take the following problems

$$x'(t) = \mu_1 x(x(t)) + \lambda; \quad t \in [0, 1], \ \mu_1 \in \mathbb{R}_+^*, \ \lambda \in \mathbb{R} \tag{5.13}$$

$$x|_{[-h,0]} = \varphi_1; \quad x|_{[1,1+h]} = \psi_1, \quad h \in \mathbb{R}_+^* \tag{5.14}$$

$$x'(t) = \mu_2 x(x(t)) + \lambda; \quad t \in [0,1], \ \mu_2 \in \mathbb{R}_+^*, \ \lambda \in \mathbb{R} \tag{5.15}$$

$$x|_{[-h,0]} = \varphi_2; \quad x|_{[1,1+h]} = \psi_2, \quad h \in \mathbb{R}_+^*. \tag{5.16}$$

Suppose that we have satisfied the following assumptions

(H_1) $\varphi_i \in C_L([-h,0],[-h,1+h])$, $\psi_i \in C_L([1,1+h],[-h,1+h])$, such that $\varphi_i(0) = 0$, $\psi_i(1) = 1$, $i = 1,2$;

(H_2) we are in the conditions of Proposition 5.2 for both of the problems (5.13) and (5.15).

Let (x_1^*, λ_1^*) be the unique solution of the problem (5.13) and (x_2^*, λ_2^*) the unique solution of the problem (5.15). We are looking for an estimation for $\|x_1^* - x_2^*\|_C$.

Then, build upon Theorem 4.1, by a common substitution one can make sure that we have

Proposition 5.3. *Consider the problems* (5.13), (5.15) *and suppose the requirements* $H_1 - H_2$ *hold. Additionally,*

(i) *there exists* η_1 *such that*

$$|\varphi_1(t) - \varphi_2(t)| \le \eta_1, \quad \forall t \in [-h,0],$$

$$|\psi_1(t) - \psi_2(t)| \le \eta_1, \quad \forall t \in [1,1+h];$$

(ii) *there exists* $\eta_2 > 0$ *such that*

$$|\mu_1 - \mu_2| \cdot |u_2| \le \eta_2, \ \forall \ t \in [0,1], \ \forall \ u_2 \in [-h, 1+h].$$

Then

$$\|x_1^* - x_2^*\|_C \le \frac{3\eta_1 + 2\eta_2}{1 - 2(L+2) \cdot \max\{\mu_1, \mu_2\}},$$

and

$$|\lambda_1^* - \lambda_2^*| \le 2\eta_1 + \eta_2.$$

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