STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LII**, Number 4, December 2007

BERNSTEIN-STANCU OPERATORS

VOICHIŢA ADRIANA CLECIU

Dedicated to Professor D. D. Stancu on his 80^{th} birthday

Abstract. The purpose of this paper is to investigate the modifications operators $C_n : Y \rightarrow \Pi_n$

$$(C_n f)(x) = \sum_{k=0}^{n} \frac{k!}{n^k} \binom{n}{k} \mathbf{m}_{k,n} \left[0, \frac{1}{n}, ..., \frac{k}{n}; f\right] x^k, f \in Y,$$

where the real numbers $(m_{k,n})_{k=0}^{\infty}$ are selected in order to preserve some important properties of Bernstein operators. For $\mathbf{m}_{j,n} = \frac{(a_n)_j}{j!}, a_n \in (0,1]$ we obtain Bernstein-Stancu operators

$$(\overline{C}_n f)(x) = \sum_{k=0}^{n} \frac{(a_n)_k}{n^k} \binom{n}{k} \left[0, \frac{1}{n}, ..., \frac{k}{n}; f\right] x^k, \ f \in Y$$

and we study some of their properties.

1. Introduction

Let Π_n be the linear space of all real polynomials of degree $\leq$ n and denote by Y the linear space of all functions $[0,1] \rightarrow \mathbb{R}$.

Consider the sequence of Bernstein operators $B_n : Y \rightarrow \Pi_n$ where

$$(B_n f)(x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right), \ b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \quad, f \in Y.$$

Because for $j \in \{0, 1, ..., n\}$

$$\frac{1}{j!} \frac{d^j (B_n f)(x)}{dx^j} = \binom{n}{j} \frac{j!}{n^j} \sum_{k=0}^{n-j} b_{n-j,k}(x) \left[\frac{k}{n}, \frac{k+1}{n}, ..., \frac{k+j}{n}; f\right],$$

Received by the editors: 06.03.2006.

2000 *Mathematics Subject Classification.* 41A10, 41A36.

Key words and phrases. approximation by positive linear operators, Bernstein operators, Bernstein basis, Bernstein-Stancu operators.

the following well-known formula holds

$$(B_n f)(x) = \sum_{k=0}^{n} \frac{k!}{n^k} \binom{n}{k} \left[0, \frac{1}{n}, ..., \frac{k}{n}; f\right] x^k. \tag{1}$$

Starting with (1), we investigate the following modifications $C_n : Y \rightarrow \Pi_n$

$$(C_n f)(x) = \sum_{k=0}^{n} \frac{k!}{n^k} \binom{n}{k} \mathbf{m}_{k,n} \left[0, \frac{1}{n}, ..., \frac{k}{n}; f\right] x^k, f \in Y, \tag{2}$$

where the real numbers $(m_{k,n})_{k=0}^{\infty}$ are selected in order to preserve some important properties of Bernstein operators. Observe that from (2)

$$\begin{cases} C_n e_0 = \mathbf{m}_{0,n} \\ C_n e_1 = \mathbf{m}_{1,n} e_1 \\ C_n e_2 = \mathbf{m}_{2,n} e_2 + \dfrac{e_1}{n} (\mathbf{m}_{1,n} - \mathbf{m}_{2,n} e_1) \\ (C_n \Omega_{2,x})(x) = (\mathbf{m}_{2,n} - 2\mathbf{m}_{1,n} + \mathbf{m}_{0,n}) x^2 + \dfrac{x}{n} (\mathbf{m}_{1,n} - \mathbf{m}_{2,n} x), \end{cases} \tag{3}$$

where $e_j(t) = t^j$ and $\Omega_{2,x} = (t-x)^2$. In the following, we shall consider that $\mathbf{m}_{0,n} = 1$.

The following problem arises to emphasize numbers $\mathbf{m}_{k,n}$, $k \in \mathbb{N}$, for which the linear transformations $(C_n)_{n=1}^{\infty}$ are *positive operators* and moreover

$$\lim_{n\to\infty} \mathbf{m}_{1,n} = 1 \qquad \textit{and} \qquad \lim_{n\to\infty} \mathbf{m}_{2,n} = 1.$$

Denote by Π_s the set of all real polynomial functions of exact degree s.

Lemma 1. *If* $p \in \Pi_s$ *and* $\mathbf{m}_{s,n} \neq 0$, *then* $C_n p \in \Pi_s, n \geq s$.

Proof. Use the fact that

$$\left[0, \frac{1}{n}, ..., \frac{k}{n}; f\right] = \begin{cases} 0, k > j \\ 1, k = j \end{cases}.$$

Therefore, if $p(x) = a_0 x^s + ... + a_{s-1} x + a_s, a_0 \neq 0$, then from (2) one finds

$$(C_n p)(x) = b_0 x^s + ... + b_s,$$

with

$$b_s := \frac{s!}{n^s} \binom{n}{s} \mathbf{m}_{s,n}, b_0 \neq 0.$$ □

Lemma 2. *If C_n is a positive operator with $\mathbf{m}_{0,n} = 1$, then $\mathbf{m}_{1,n} \in [0,1]$ and*

$$0 \leq \mathbf{m}_{1,n} - \mathbf{m}_{2,n} \leq \frac{n}{n-1} \left(1 - \mathbf{m}_{1,n}\right).$$

Proof. For the proof it is enough to observe that

$$0 \leq e_1 \left(t\right) \leq 1, t \in [0,1]$$

implies

$$0 \leq \left(C_n e_1\right)\left(x\right) = \mathbf{m}_{1,n} x \leq \mathbf{m}_{0,n} = 1, \forall x \in [0,1],$$

that is $\mathbf{m}_{1,n} \in [0,1]$. Further, from $t\left(1-t\right) \geq 0, \forall t \in [0,1]$, we have

$$\left(C_n e_1\right)\left(x\right) - \left(C_n e_2\right)\left(x\right) \geq 0$$

for any x from $[0,1]$, that is $\mathbf{m}_{2,n} x \leq \mathbf{m}_{1,n}$. To complete the proof it is sufficient to use the fact that $\left(C_n \Omega_{2,x}\right)\left(x\right)$ must be non-negative on $[0,1]$. □

Lemma 3. *Suppose that C_n is a positive operator with $\mathbf{m}_{0,n} = 1$.*

1) If $\mathbf{m}_{2,n} = 1$, then $\mathbf{m}_{1,n} = 1$;

2) If $\mathbf{m}_{1,n} = 1$, then $\mathbf{m}_{2,n} = 1$.

Proof. Use Lemma 2. □

Lemma 4. *Suppose that $f : [0,1] \to \mathbb{R}$ is convex on $[0,1]$. If C_n is a positive operator with $\mathbf{m}_{0,n} = 1$, then*

$$f\left(m_{1,n} x\right) \leq \left(C_n f\right)\left(x\right), \forall x \in [0,1].$$

Proof. It is known that for a convex function $f : [0,1] \to \mathbb{R}$ and a linear positive operator $T : Y \to Y$, we have

$$f\left(\left(Te_1\right)\left(x\right)\right) \leq \left(Tf\right)\left(x\right), \ \forall x \in [0,1] \qquad \text{(see [7] and [8]).}$$ □

Lemma 5. *Suppose that $\left(C_n\right)_{n=1}^{\infty}$ are positive operators with $\mathbf{m}_{0,n} = 1$. If*

$$\lim_{n\to\infty} \mathbf{m}_{1,n} = 1,$$

then

$$\lim_{n\to\infty} \mathbf{m}_{2,n} = 1.$$

Proof. From $0 \le \mathbf{m}_{1,n} - \mathbf{m}_{2,n} \le \frac{n}{n-1}\left(1 - \mathbf{m}_{1,n}\right)$, see Lemma 2. □

Further we consider the uniform norm $\|g\| := \max\limits_{x\in[0,1]} |g(t)|$.

Lemma 6. *Suppose that* $\mathbf{m}_{0,n} = 1$, $\forall n \in \mathbb{N}^*$. *If* $(C_n)_{n=1}^{\infty}$ *is a sequence of positive operators, then* $\lim\limits_{n\to\infty} \mathbf{m}_{1,n} = 1$ *implies*

$$\lim_{n\to\infty} \|f - C_n f\| = 0, \quad \forall f \in C[0,1].$$

2. The Bernstein form of the operator C_n

Theorem 7. *Suppose that* C_n *is defined as in (2). Then for* $f : [0,1] \to \mathbb{R}$

$$(C_n f)(x) = \sum_{k=0}^{n} b_{n,k}(x)\, C_{k,n}[f], \tag{4}$$

with

$$C_{k,n}[f] = \sum_{j=0}^{k} \binom{k}{j} f\left(\frac{j}{n}\right) \sum_{\nu=0}^{k-j} (-1)^{\nu} \binom{k-j}{\nu} \mathbf{m}_{\nu+j,n}.$$

Proof. Observe that

$$\left[0, \frac{1}{n}, ..., \frac{k}{n}; f\right] = \frac{k!}{n^k} \sum_{\nu=0}^{k} (-1)^{k-\nu} \binom{k}{\nu} f\left(\frac{\nu}{n}\right).$$

From (2)

$$(C_n f)(x) = \sum_{k=0}^{n} A_k x^k,$$

with

$$A_k := \mathbf{m}_{k,n} \binom{n}{k} \sum_{\nu=0}^{k} (-1)^{k-\nu} \binom{k}{\nu} f\left(\frac{\nu}{n}\right).$$

Further, using the rule

$$\sum_{k=0}^{n} C_k \sum_{j=k}^{n} D_{k,j} = \sum_{k=0}^{n} \sum_{j=0}^{k} C_j D_{j,k},$$

we get (see [9])

$$(C_n f)(x) = \sum_{k=0}^{n} A_k x^k \left((1-x) + x\right)^{n-k} = \sum_{k=0}^{n} A_k \sum_{j=k}^{n} \binom{n-k}{j-k} x^j (1-x)^{n-j} =$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} C_{k,n}[f],$$

where

$$C_{k,n}[f] := \sum_{j=0}^{k} A_j \frac{(n-j)!k!}{(k-j)!n!}.$$

Therefore

$$C_{k,n}[f] := \sum_{j=0}^{k} \binom{k}{j} \mathbf{m}_{j,n} \sum_{\nu=0}^{j} (-1)^{\nu-j} \binom{j}{\nu} f\left(\frac{\nu}{n}\right) =$$

$$= \sum_{j=0}^{k} \binom{k}{j} f\left(\frac{j}{n}\right) \sum_{\nu=0}^{k-j} (-1)^{\nu} \binom{k-j}{\nu} \mathbf{m}_{\nu+j,n}$$

and we conclude with (4). □

3. Bernstein - Stancu operators: the case $\mathbf{m}_{j,n} = \frac{(a_n)_j}{j!}, a_n \in (0,1]$

Further, for $k \in \mathbb{N}$, $z \in \mathbb{C}$, let $(z)_0 = 1$ and $(z)_k = z(z+1)\dots(z+k-1)$.

Then the operator C_n from (2), denoted further by $\overline{C}_n$, becomes

$$\left(\overline{C}_n f\right)(x) = \sum_{k=0}^{n} \frac{(a_n)_k}{n^k} \binom{n}{k} \left[0, \frac{1}{n}, ..., \frac{k}{n}; f\right] x^k, \; f \in Y. \tag{5}$$

Lemma 8. *Assume that $\overline{C}_n$ is a positive operator, i.e. $a_n \in (0,1]$. Then*

$$\begin{cases} \left(\overline{C_n} e_0\right)(x) = 1 \\ \left(\overline{C_n} e_1\right)(x) = a_n x = x - (1-a_n)x \\ \left(\overline{C_n} e_2\right)(x) = x^2 + \dfrac{x(1-x)}{n} a_n + \dfrac{1-a_n}{2}\left(\frac{a_n}{n} - (2+a_n)\right)x^2 \\ \left(\overline{C_n}\Omega_{2,x}\right)(x) = \dfrac{x(1-x)}{n} a_n + x^2(1-a_n)\left(\dfrac{2-a_n}{2} + \dfrac{a_n}{2n}\right). \end{cases} \tag{6}$$

Moreover

$$\left|\left(\overline{C_n}\Omega_{2,x}\right)(x)\right| \le \frac{a_n}{4n} + (1-a_n), \quad \forall x \in [0,1]. \tag{7}$$

Proof. The above assertions follow using (3):

$$\begin{aligned}
&\left(\overline{C_n}e_0\right)(x) = \frac{(a_n)_0}{0!} = \\
&\left(\overline{C_n}e_1\right)(x) = \frac{(a_n)_1}{1!}e_1 = a_n e_1 = a_n x = x - (1-a_n)\,x \\
&\left(\overline{C_n}e_2\right)(x) = \frac{(a_n)_2}{2!}e_2 + \frac{e_1}{n}\left(\frac{(a_n)_1}{1!} - \frac{(a_n)_2}{2!}e_1\right) = \\
&= \frac{a_n\,(a_n+1)}{2}e_2 + \frac{a_n}{n}e_1\left(1 - \frac{a_n+1}{2}e_1\right) = \\
&= \frac{a_n\,(a_n+1)}{2}x^2 + \frac{a_n}{n}x\left(1 - \frac{a_n+1}{2}x\right) = \\
&= \frac{x(1-x)}{n}a_n + x^2\left(\frac{a_n^2+a_n}{2} + \frac{a_n-a_n^2}{2n}\right) = \\
&= \frac{x(1-x)}{n}a_n + x^2 + x^2\left(\frac{(a_n+2)\,(a_n-1)}{2} + \frac{a_n\,(1-a_n)}{2n}\right) = \\
&= x^2 + \frac{x(1-x)}{n}a_n + \frac{1-a_n}{2}\left(\frac{a_n}{n} - (2+a_n)\right)x^2
\end{aligned}$$

and

$$\begin{aligned}
&\left(\overline{C_n}\Omega_{2,x}\right)(x) = \left(\frac{(a_n)_2}{2!} - 2\frac{(a_n)_1}{1!} + 1\right)x^2 + \frac{x}{n}\left(\frac{(a_n)_1}{1!} - \frac{(a_n)_2}{2!}x\right) = \\
&= \left(\frac{a_n\,(a_n+1)}{2} - 2a_n + 1\right)x^2 + \frac{x}{n}\left(a_n - \frac{a_n\,(a_n+1)}{2}x\right) = \\
&= \left(\frac{a_n\,(a_n-3)}{2} + 1\right)x^2 + \frac{a_n}{n}x\left(1 - \frac{a_n+1}{2}x\right) \\
&= \frac{a_n}{n}x + x^2\left(-\frac{a_n}{n} + \frac{a_n^2-3a_n+2}{2} + \frac{a_n-a_n^2}{2n}\right) = \\
&= \frac{x(1-x)}{n}a_n + x^2\,(1-a_n)\left(\frac{2-a_n}{2} + \frac{a_n}{2n}\right).
\end{aligned}$$

□

Lemma 9. *Assume that $\overline{C}_n$ is a positive operator, i.e. $a_n \in (0,1]$. Then*

$$\begin{aligned}
\left(\overline{C_n}e_3\right)(x) &= \frac{(a_n)_3}{n^3}\begin{pmatrix} n \\ 3 \end{pmatrix}x^3 + \frac{3\,(a_n)_2}{n^3}\begin{pmatrix} n \\ 2 \end{pmatrix}x^2 + \frac{a_n}{n^2}x = \\
&= x^3 + \frac{3(n-1)}{2n^2}\,(a_n)_2\,x^2 + \frac{2-3n}{6n^2}\,(a_n)_3\,x^3 + \frac{a_n}{n^2}x + \frac{(a_n)_3-6}{6}x^3
\end{aligned}$$

$$\left(\overline{C_n}e_4\right)(x) = \frac{(a_n)_4}{n^4}\begin{pmatrix} n \\ 4 \end{pmatrix} x^4 + \frac{6\,(a_n)_3}{n^4}\begin{pmatrix} n \\ 3 \end{pmatrix} x^3 + \frac{7\,(a_n)_2}{n4}\begin{pmatrix} n \\ 2 \end{pmatrix} x^2 + \frac{a_n}{n^3}x =$$

$$= x^4 + \frac{(n-1)(n-2)}{n^3}(a_n)_3\,x^3 - \frac{6n^2-11n+6}{24n^3}(a_n)_4\,x^4 + \frac{a_n}{n^3}x +$$

$$+ \frac{7(n-1)}{2n^3}(a_n)_2\,x^2 - \frac{(1-a_n)\left(a_n^3+7a_n^2+18a_n+24\right)}{24}x^4$$

$$\left(\overline{C_n}\Omega_{4,x}\right)(x) = \left[\frac{(a_n)_4}{n^4}\begin{pmatrix} n \\ 4 \end{pmatrix} - \frac{4\,(a_n)_3}{n^3}\begin{pmatrix} n \\ 3 \end{pmatrix} + \frac{6\,(a_n)_2}{n^2}\begin{pmatrix} n \\ 2 \end{pmatrix} - 4a_n + 1\right] x^4 +$$

$$+ \left[\frac{6\,(a_n)_3}{n^4}\begin{pmatrix} n \\ 3 \end{pmatrix} - \frac{12\,(a_n)_2}{n^3}\begin{pmatrix} n \\ 2 \end{pmatrix} + \frac{6a_n}{n}\right] x^3 +$$

$$+ \left[\frac{7\,(a_n)_2}{n^4}\begin{pmatrix} n \\ 2 \end{pmatrix} - \frac{4a_n}{n^2}\right] x^2 + \frac{a_n}{n^3}x$$

$$= -\left(e_4(x) - \left(\overline{C_n}e_4\right)(x)\right) + 4x\left(e_3(x) - \left(\overline{C_n}e_3\right)(x)\right) -$$

$$- 6x^2\left(e_2(x) - \left(\overline{C_n}e_2\right)(x)\right) + 4x^3\left(e_1(x) - \left(\overline{C_n}e_1\right)(x)\right)$$

Proof. Using (5) we have:

$$\left(\overline{C_n}e_3\right)(x) = \frac{a_n}{n}\begin{pmatrix} n \\ 1 \end{pmatrix}\left[0, \frac{1}{n}; e_3\right] x + \frac{(a_n)_2}{n^2}\begin{pmatrix} n \\ 2 \end{pmatrix}\left[0, \frac{1}{n}, \frac{2}{n}; e_3\right] x^2 +$$

$$+ \frac{(a_n)_3}{n^3}\begin{pmatrix} n \\ 3 \end{pmatrix}\left[0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}; e_3\right] x^3 = \frac{(a_n)_3}{n^3}\begin{pmatrix} n \\ 3 \end{pmatrix} x^3 +$$

$$+ \frac{3\,(a_n)_2}{n^3}\begin{pmatrix} n \\ 2 \end{pmatrix} x^2 + \frac{a_n}{n^2}x = \frac{a_n}{n^2}x + \frac{3(n-1)}{2n^2}(a_n)_2 x^2 + \frac{(n-1)(n-2)}{6n^2}(a_n)_3\,x^3 =$$

$$= x^3 + \frac{3(n-1)}{2n^2}(a_n)_2\,x^2 + \frac{2-3n}{6n^2}(a_n)_3\,x^3 + \frac{a_n}{n^2}x + \frac{(a_n)_3 - 6}{6}x^3$$

$$\left(\overline{C_n}e_4\right)(x) = \frac{a_n}{n}\begin{pmatrix} n \\ 1 \end{pmatrix}\left[0, \frac{1}{n}; e_4\right] x + \frac{(a_n)_2}{n^2}\begin{pmatrix} n \\ 2 \end{pmatrix}\left[0, \frac{1}{n}, \frac{2}{n}; e_4\right] x^2 +$$

$$+ \frac{(a_n)_3}{n^3}\begin{pmatrix} n \\ 3 \end{pmatrix}\left[0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}; e_4\right] x^3 + \frac{(a_n)_4}{n^4}\begin{pmatrix} n \\ 4 \end{pmatrix}\left[0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \frac{4}{n}; e_4\right] x^4$$

$$= \frac{(a_n)_4}{n^4}\begin{pmatrix} n \\ 4 \end{pmatrix} x^4 + \frac{6\,(a_n)_3}{n^4}\begin{pmatrix} n \\ 3 \end{pmatrix} x^3 + \frac{7\,(a_n)_2}{n4}\begin{pmatrix} n \\ 2 \end{pmatrix} x^2 + \frac{a_n}{n^3}x$$

$$= \frac{a_n}{n^3}x + \frac{7(n-1)}{2n^3}(a_n)_2 x^2 + \frac{(n-1)(n-2)}{n^3}(a_n)_3\, x^3 + \frac{(n-1)(n-2)(n-3)}{24n^3}(a_n)_4\, x^4$$

$$= x^4 + \frac{(n-1)\,(n-2)}{n^3}(a_n)_3\, x^3 - \frac{6n^2-11n+6}{24n^3}(a_n)_4\, x^4 + \frac{a_n}{n^3}x + \frac{7(n-1)}{2n^3}(a_n)_2\, x^2$$

$$-\frac{(1-a_n)\left(a_n^3+7a_n^2+18a_n+24\right)}{24}x^4.$$

We use the fact that

$$\left(\overline{C_n}\Omega_{4,x}\right)(x) = \left(\overline{C_n}e_4\right)(x) - 4x\left(\overline{C_n}e_3\right)(x) + 6x^2\left(\overline{C_n}e_2\right)(x) - 4x^3\left(\overline{C_n}e_1\right)(x) + x^4$$

to obtain the above assertions. □

Theorem 10. *The linear operator $\overline{C}_n$ from (5) may be written in the Bernstein basis in the form*

$$\left(\overline{C}_n f\right)(x) = \sum_{k=0}^{n} b_{n,k}\,(x)\,\overline{C}_{k,n}\,[f]\,, \tag{8}$$

with

$$\overline{C}_{k,n}\,[f] = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j} f\left(\frac{j}{n}\right)(a_n)_j\,(1-a_n)_{k-j}$$

Proof. Let us find a convenient form of the coefficients $\overline{C}_{k,n}\,[f]$ from (4). In our case we have

$$\overline{C}_{k,n}\,[f] = \sum_{j=0}^{k}\binom{k}{j} f\left(\frac{j}{n}\right)\sum_{\nu=0}^{k-j}(-1)^{\nu}\binom{k-j}{\nu}\frac{(a_n)_{\nu+j}}{(\nu+j)!} =$$

$$= \sum_{j=0}^{k}\binom{k}{j} f\left(\frac{j}{n}\right)\frac{(a_n)_j}{j!}\sum_{\nu=0}^{k-j}\frac{(-k+j)_{\nu}\,(j+a_n)_{\nu}}{(j+1)_{\nu}\,\nu!} =$$

$$= \sum_{j=0}^{k}\binom{k}{j} f\left(\frac{j}{n}\right)\frac{(a_n)_j}{(j)!}\cdot {}_2F_1\left(\,-k+j, j+a_n; j+1; 1\right).$$

Because ${}_2F_1\left(\,-m, b; c; 1\right) = \dfrac{(c-b)_m}{(c)_m}$ for $m \in \mathbb{N}^*$, we have

$$\overline{C}_{k,n}\,[f] = \sum_{j=0}^{k}\binom{k}{j} f\left(\frac{j}{n}\right)\frac{(a_n)_j\,(1-a_n)_{k-j}}{j!\,(j+1)_{k-j}},$$

in other words

$$\overline{C}_{k,n}\left[f\right]=\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}f\left(\frac{j}{n}\right)\left(a_{n}\right)_{j}\left(1-a_{n}\right)_{k-j}.$$

When $a_n\in(0,1)$, it is clear that $f\geq 0$ on $[0,1]$ implies $\overline{C}_{k,n}\left[f\right]\geq 0$, that is $\overline{C}_n$ is a linear positive operator. □

For $g:[0,1]\to\mathbb{R}$ the Stancu operators $S_k:g\to S_kg$, $k\in\mathbb{N}$, are defined as $\left(S_0^{<b>}g\right)(x)=g(0)$ and for $k\in\{1,2,...\}$ (see [17], [18] and [4]):

$$\left(S_0^{<b>}g\right)(x)=\frac{1}{(b)_k}\sum_{j=0}^{k}\binom{k}{j}(bx)_j\,(b-bx)_{k-j}\,g\left(\frac{j}{k}\right),\quad x\in[0,1],$$

where $b\in[0,1]$ is a parameter. Observe that $\overline{C}_0 f=\overline{C}_{0,0}\left[f\right]:=f(0)$ and

$$\left(S_k^{<1>}g\right)(a_n)=\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(a_n)_j\,(1-a_n)_{k-j}\,g\left(\frac{j}{k}\cdot\frac{k}{n}\right),\ k\geq 1.$$

Therefore,

$$\overline{C}_{k,n}\left[f\right]=\left(S_k^{<1>}g_{n,k}^{<f>}\right)(a_n)$$

with

$$g_{n,k}^{<f>}(t)=f\left(t\frac{k}{n}\right),\ k\geq 1.$$

Definition 11. *The linear transformations $\overline{C}_{k,n}:Y\to\mathbb{R}$, $k\in\{0,1,2,...,n\}$, $n\in\mathbb{N}^*$, are the Stancu functionals. When $a_n\in(0,1)$, the linear positive transformations $\overline{C}_n:Y\to\Pi_n$, $n\in\mathbb{N}^*$, are called* ***Bernstein-Stancu operators****.*

Using the Chu-Vandermonde identity

$$\sum_{k=0}^{n}\binom{n}{k}(a)_k\,(b)_{n-k}=(a+b)_n$$

it is possible to find the images of Stancu functionals $\overline{C}_{k,n}$ at some monomials.

Next we use the following proposition

Lemma 12. *(A.Lupas [9], pag. 205). Let n be fixed,* $1 \leq s \leq n$*, and* $\|b_{n,k}\|$ *be the Bernstein basis. Suppose that A is a linear mapping defined on the algebra of polynomials such that* $\Pi_{s-1} \subseteq Ker\,(A)$*. If*

$$p(x) = \sum_{k=0}^{n} a_k b_{n,k}(x),$$

then

$$A(p) = \sum_{j=0}^{n-s} A\left(\psi_{j,s}\right) \Delta^s a_j,$$

where

$$\Delta^s a_j = \sum_{\nu=0}^{s} (-1)^{s-\nu} \binom{s}{\nu} a_{j+\nu}$$

and

$$\psi_{j,s}\left(x\right) = \binom{n}{s+j} x^{s+j}_{2} F_1\left(-n+s+j, j+1; s+j+1; x\right) =$$
$$= s\binom{n}{s} \int_0^x \left(x-y\right)^{s-1} b_{n-s,j}(y) dy.$$

Moreover,

$$\frac{1}{s!} \cdot \frac{d^s}{dx^s} \psi_{j,s}\left(x\right) = \binom{n}{s} b_{n-s,j}(x).$$

Using this proposition one can prove:

Theorem 13. *Let* $a_n \in (0,1)$ *and*

$$I_{n,j,\nu} = \int_0^1 t^{j-1+a_n} \left(1-t\right)^{-a_n} b_{n-j,\nu}(xt) dt.$$

Then

$$\frac{d^j}{dx^j}\left(\overline{C}_n f\right)(x) = \binom{n}{j} \frac{\left(j!\right)^2}{n^j} \cdot \frac{\sin\left(\pi a_n\right)}{\pi} \sum_{\nu=0}^{n-j} \left[\frac{j}{n}, \frac{j+1}{n}, ..., \frac{j+\nu}{n}; f\right] I_{n,j,\nu}.$$

Because the integrals $I_{n,j,\nu},\ j \in \{0,1,2,...,n\}$, are positive it follows:

Corollary 14. *Let j,n* $\in \mathbb{N}^*$*,* $0 \leq j \leq n-2$*. The operator* $\overline{C}_n$ *preserves the convexity of order j.*

The asymptotic behavior of the sequence $\left(\overline{C}_n\right)_{n=1}^{\infty}$ on a certain subspaces of $C[-1,1]$ is given in the following proposition:

Theorem 15. *Suppose* $x_0 \in [0,1]$ *and* $f''(x_0)$ *exists. If* $a_n \in (0,1)$, $\lim_{n\to\infty} a_n = 1$ *and* $L := \lim_{n\to\infty} n(1-a_n)$ *exists, then*

$$\lim_{n\to\infty} n\left[f(x_0) - \left(\overline{C}_n f\right)(x_0)\right] = -\frac{x(1-x)}{2} f''(x_0) + \left[x_0 f'(x_0) - \frac{x_0^2}{4} f''(x_0)\right] L.$$

Proof. We apply a version of a general proposition given by R. G. Mamedov (see [7]). More precisely, let $\varphi : \mathbb{N} \to \mathbb{R}$, $\lim_{n\to\infty} \varphi(n) = \infty$, such that

$$\lim_{n\to\infty} \varphi(n)\left[e_k(x_0) - \left(\overline{C}_n e_k\right)(x_0)\right] = r_k(x_0),$$

for $k \in \{1,2\}$.

In our case

$$\begin{aligned}
&n\left[e_1(x_0) - \left(\overline{C}_n e_1\right)(x_0)\right] = n(1-a_n)x_0\\
&n\left[e_2(x_0) - \left(\overline{C}_n e_2\right)(x_0)\right] = -x_0(1-x_0)a_n -\\
&-\frac{a_n(1-a_n)x_0^2}{2} + \frac{n(1-a_n)(2-a_n)x_0^2}{2}\\
&n\left[e_3(x_0) - \left(\overline{C}_n e_3\right)(x_0)\right] = \frac{3(1-n)}{2n}(a_n)_2 x_0^2 +\\
&+\frac{3n-2}{6n}(a_n)_3 x_0^3 - \frac{a_n}{n}x_0 - n\frac{(a_n)_3 - 6}{6}x_0^3 =\\
&= \frac{3(1-n)}{2n}(a_n)_2 x_0^2 + \frac{3n-2}{6n}(a_n)_3 x_0^3 -\\
&-\frac{a_n}{n}x_0 + \frac{n(1-a_n)(a_n^2+4a_n+6)}{6}x_0^3\\
&n\left[e_4(x_0) - \left(\overline{C}_n e_4\right)(x_0)\right] = -\frac{(n-1)(n-2)}{n^2}(a_n)_3 x_0^3 +\\
&+\frac{6n^2-11n+6}{24n^2}(a_n)_4 x_0^4 + \frac{a_n}{n^2}x_0 + \frac{7(n-1)}{2n^2}(a_n)_2 x_0^2 -\\
&-\frac{n(1-a_n)\left(a_n^3+7a_n^2+18a_n+24\right)}{24}x_0^4.
\end{aligned}$$

Therefore

$$\begin{aligned}
&r_1(x_0) = Lx_0,\\
&r_2(x_0) = -x_0(1-x_0) + \frac{3L}{2}x_0^2,\\
&r_3(x_0) = -3x_0^2(1-x_0) + \frac{11L}{6}x_0^3,\\
&r_4(x_0) = -6x_0^3(1-x_0) + \frac{25L}{12}x_0^4.
\end{aligned}$$

If $\Omega_{4,x} = (t-x)^4$, then

$$n\left(\overline{C_n}\Omega_{4,x}\right)(x) = -n\left(e_4(x) - \left(\overline{C_n}e_4\right)(x)\right) + 4nx\left(e_3(x) - \left(\overline{C_n}e_3\right)(x)\right) - \\ -6nx^2\left(e_2(x) - \left(\overline{C_n}e_2\right)(x)\right) + 4nx^3\left(e_1(x) - \left(\overline{C_n}e_1\right)(x)\right) \Rightarrow$$

$$\lim_{n\to\infty} n\left(\overline{C}_n\Omega_{4,x}\right)(x) = -r_4(x) + 4xr_3(x) - 6x^2r_2(x) + 4x^3r_1(x) = \frac{Lx^4}{4}$$

and

$$\lim_{n\to\infty}\varphi(n)\left(\overline{C}_n\Omega_{4,x_0}\right)(x_0) = 0,$$

then

$$\lim_{n\to\infty}\varphi(n)\left[f(x_0) - \left(\overline{C}_nf\right)(x_0)\right] = \left[f'(x_0) - x_0f''(x_0)\right]r_1(x_0) + \frac{r_2(x_0)}{2}f''(x_0) \quad (*)$$

and from $(*)$ we complete the proof. □

References

[1] Brass, H., *Eine Verallgemeinerung der Bernsteinschen Operatoren*, Abhandl. Math. Sem. Hamburg **36**(1971), 11-222.

[2] Cheney, E.W., Sharma, A., *On a generalization of Bernstein polynomials*, Riv. Mat. Univ. Parma (2) **5**(1964), 77-82.

[3] Cleciu, V.A., *About a new class of linear operators wich preserve the properties of Bernstein operators*, The proceedings of the international conference "The impact of european integrationon the national economy", Ed Risoprint, Cluj-Napoca, 2005, 45-54.

[4] Della Vechia, B., *On the approximation of functions by means of the operators of D.D. Stancu*, Studia Univ. Babes-Bolyai, Mathematica,**37**(1992), 3-36.

[5] Gavrea, I., Gonska, H.H., Kacso, D.P., *Positive linear operators with equidistant nodes*, Comput. Math. Appl., **8**(1996), 23-32.

[6] Ismail, M.E.H., *Polynomials of binomial type and approximation theory*, J. Approx. Theory, **32**(1978), 177-186.

[7] Lupaş, A., *Contributions to the theory of approximation by linear operators*, (Romanian), Doctoral Thesis, Univ. Babes-Bolyai, Cluj-Napoca, 1976.

[8] Lupaş, A., *A generalization of Hadamard inequalities for convex functions*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz., no **544-576**(1976), 115-121.

[9] Lupaş, A., *The approximation by means of some positive linear operators*, in approximation Theory (IDOMAT 95, Proc.International Dormund Meeting on Approximation Theory 1995) Editors: M.W. Muller et al.), Berlin, Akademie Verlag 1995, 201-229.

[10] Lupaş, A., *On the Remainder Term in some Approximation Formulas*, General Mathematics **3**, no 1-2, (1995), 5-11.

[11] Lupaş, A., *Approximation Operators of Binomial Type*, in New Developments in Approximation Theory, ISNM, vol 132, Birkhauser Verlag, Basel, 1999, 175-198.

[12] Lupaş, L., Lupaş, A., *Polynomials of binomial type and approximation operators*, Studia Univ. Babes-Bolyai, Mathematica, **XXXII**, 4, (1987), 61-63.

[13] Moldovan, Gr., *Generalizari ale polinoamelor lui S.N. Bernstein*, Teza de doctorat, Cluj-Napoca, 1971.

[14] Muhlbach, G., *Operatoren von Bernsteinschen Typ*, J. Approx. Theory, **3**(1970), 274-292.

[15] Popoviciu, T., *Remarques sur les polynomes binomiaux*, Mathematica **6**(1932), 8-10.

[16] Stancu, D.D., *Evaluation of the remainder term in approximation formulas by Bernstein polynomials*, Math. Comp. **83**(1963), 270-278.

[17] Stancu, D.D., *Approximation of functions by a new class of linear positive operators*, Rev. Roum. Math. Pures et Appl. **13**(1968), 1173-1194.

[18] Stancu, D.D., *Approximation of functions by means of some new classes of positive linear operators*, "Numerische Methoden der Approximationstheorie", Proc. Conf. Oberwolfach 1971 ISNM vol 16, Birkhauser Verlag, Basel, 1972, 187-203.

Universitatea "Babeş-Bolyai",
Facultatea de Ştiinte Economice şi Gestiunea Afacerilor,
str. T. Mihali nr. 58-60, 400591 Cluj-Napoca, Romania
E-mail address: vcleciu@econ.ubbcluj.ro