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# ON CERTAIN NUMERICAL CUBATURE FORMULAS FOR A TRIANGULAR DOMAIN

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*Dedicated to Professor D. D. Stancu on his* $80^{th}$ *birthday*

**Abstract**. The purpose of this article is to discuss certain cubature formulas for the approximation of the value of a definite double integral extended over a triangular domain. We start by using the Biermann's interpolation formula [5], [19]. Then we consider also the results obtained by D.V. Ionescu in the paper [12], devoted to the construction of some cubature formulas for evaluating definite double integrals over an arbitrary triangular domain. In the recent papers [6] and [7] there were investigated some homogeneous cubature formulas for a standard triangle. In the case of the triangle having the vertices $(0,0)$, $(1,0)$, $(0,1)$ there were constructed by H. Hillion [11] several cubature formulas by starting from products of Gauss-Jacobi formulas and using an affine transformation, which can be seen in the book of A.H. Stroud [22].

## 1. **Use of Biermann's interpolation formula**

**1.1.** Let us consider a triangular grid of nodes $M_{i,k}(x_i, y_k)$, determined by the intersection of the distinct straight lines $x = x_i$, $y = y_k$ $(0 \le i + k \le m)$. We assume that $a = x_0 < x_1 < \cdots < x_{m-1} < x_m = b$ and $c = y_0 < y_1 < \cdots < y_{m-1} < y_m = d$. We denote by $T = T_{a,b,c}$ the triangle having the vertices $A(a, c)$, $B(b, c)$, $C(a, d)$.

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It is known that the Biermann's interpolation formula [5], [19], which uses the triangular array of base points $M_{i,k}$ can be written in the following form:

$$f(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{m-i}u_{i-1}(x)v_{j-1}(y)\left[\begin{array}{c}x_0,x_1,\dots,x_i\\ y_0,y_1,\dots,y_j\end{array};\ f\right]+(r_mf)(x,y), \qquad (1.1)$$

where

$$u_{i-1}(x)=(x-x_0)(x-x_1)\dots(x-x_{i-1}),\ u_{-1}(x)=1,\ u_m(x)=u(x)$$

and

$$v_{j-1}(y)=(y-y_0)(y-y_1)\dots(y-y_{j-1}),\ v_{-1}(y)=1,\ v_m(y)=v(y),$$

with the notations

$$u(x)=(x-x_0)(x-x_1)\dots(x-x_m),$$

$$v(y)=(y-y_0)(y-y_1)\dots(y-y_m).$$

The brackets used above represent the symbol for bidimensional divided differences. By using a proof similar to the standard methods for obtaining the expression of the remainder of Taylor's formula for two variables, Biermann has shown in [5] that this remainder $(r_mf)(x,y)$ may be expressed under the remarkable form:

$$(r_mf)(x,y)=\frac{1}{(m+1)!}\sum_{k=0}^{m+1}\binom{m+1}{k}u_{k-1}(x)v_{m-k}(y)f^{(k,m+1-k)}(\xi,\eta)$$

where $(\xi,\eta)\in T$ and we have used the notation for the partial derivatives:

$$g^{(p,q)}(x,y)=\frac{(\partial^{p+q}g)(x,y)}{\partial^p x^p\partial^q y^q}.$$

**1.2.** By using the interpolation formula (1.1) we can construct cubature formulas for the approximation of the value of the double integral

$$I(f)=\int\int_T f(x,y)dxdy.$$

We obtain a cubature formula of the following form:

$$J(f)=\sum_{i=0}^{m}\sum_{j=0}^{m-i}A_{i,j}\left[\begin{array}{c}x_0,x_1,\dots,x_i\\ y_0,y_1,\dots,y_j\end{array};\ f\right]+R_m(f), \qquad (1.2)$$

where the coefficients have the expressions:

$$A_{i,j} = \int\int_{\Gamma} u_{i-1}(x)v_{j-1}(y)dxdy. \tag{1.3}$$

The remainder of the cubature formula (1.2) can be expressed as follows

$$R_m(f) = \sum_{k=0}^{m+1} \frac{1}{k!(m+1-k)!} \cdot A_{k,m+1-k} f^{(k,m+1-k)}(\xi, \eta).$$

Developing the computation in (1.2), (1.3) we can obtain the following form for our cubature formula:

$$J(f) = \sum_{i=1}^{m+1} \sum_{j=1}^{m+1-i} C_{i,j} f(x_i, y_j) + R_m(f). \tag{1.4}$$

This formula has, in general, the degree of exactness $D = m$, but by special selections of the nodes one can increase this degree of exactness.

A necessary and sufficient condition that the degree of exactness to be $m+p$ is that

$$K_{n,s} = \int\int_T x^r y^s u_{i-1}(x) v_{m-i}(y) dxdy, \quad i,j = \overline{0,m}$$

vanishes if $r+s = 0, 1, \ldots, p-1$ and to be different from zero if $r+s = p$. This result can be established if we take into account the expression of the remainder $R_m(f)$.

**1.3.** For illustration we shall give some examples. First we introduce a notation:

$$I_{p,q} = \int\int_T x^p y^q dxdy,$$

where $p$ and $q$ are nonnegative integers.

If we take $m = 0$ then we obtain the interpolation formula

$$f(x,y) = f(x_0,y_0) + (x-x_0)\begin{bmatrix} x_0, x & ; & f \\ y_0 & & \end{bmatrix} + (y-y_0)\begin{bmatrix} x_0 & ; f \\ y_0, y & \end{bmatrix}.$$

Imposing the conditions

$$\int\int_T (x-x_0)dxdy = 0,$$

$$\int\int_T (y-y_0)dxdy = 0$$

we deduce that $x_0 = I_{1,0}/I_{0,0}$, $y_0 = I_{0,1}/I_{0,0}$ and thus we get a cubature formula of the form

$$\int\int_T f(x,y)dxdy = Af(G) + R_0(f),$$

where $G = (x_G, y_G)$ is the barycentre of the triangle $T$.

The remainder has an expression of the following form

$$R_0(f) = Af^{(2,0)}(\xi,\eta) + 2Bf^{(1,1)} + Cf^{(0,2)}(\xi,\eta)$$

with

$$A = \frac{1}{2}(I_{0,0}I_{2,0} - I_{1,0}^2)/I_{0,0}$$
$$B = \frac{1}{2}(I_{0,0}I_{1,1} - I_{1,0}I_{0,1})/I_{0,0}$$
$$C = \frac{1}{2}(I_{0,0}I_{0,2} - I_{0,1}^2)/I_{0,0}$$

where $(\xi,\eta)$ are points from the interior of the triangle $T$.

**1.4.** If we take $m = 1$ and we determine $x_1$ and $y_1$ by imposing the conditions

$$\int\int_T (x-a)(x-x_1)dxdy = 0$$
$$\int\int_T (y-c)(y-y_1)dxdy = 0$$

we can deduce that we must have: $x_1 = \dfrac{a+b}{2}$, $y_1 = \dfrac{c+d}{2}$.

So we obtain a cubature formula of the form

$$\int\int_T f(x,y)dxdy$$
$$= \frac{(b-a)(d-c)}{6}\left[f\left(\frac{a+b}{2},c\right) + f\left(a,\frac{c+d}{2}\right) + f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right] + R_1(f),$$

where the remainder has the expression:

$$R_1(f) = \frac{(b-a)^4(d-c)}{720}f^{(3,0)}(\xi,\eta) - \frac{(b-a)^3(d-c)^2}{480}f^{(2,1)}(\xi,\eta)$$
$$-\frac{(b-a)^2(d-c)^3}{480}f^{(1,2)}(\xi,\eta) + \frac{(b-a)(d-c)^4}{720}f^{(0,3)}(\xi,\eta).$$

By starting from this cubature formula we can construct a numerical integration formula, with five nodes, for a rectangle domain $D = [a,b]\times[c,d]$, namely

$$\int\int_D f(x,y)dxdy = \frac{(b-a)(d-c)}{6}\left[f\left(a,\frac{c+d}{2}\right) + f\left(\frac{a+b}{2},c\right)\right.$$

$$+2f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},d\right)+f\left(b,\frac{c+d}{2}\right)\Bigg]+R(f).$$

For the remainder of this formula we are able to obtain the following estimation

$$|R(f)|\le\frac{h^2k^3}{80}\left[\frac{h}{3}M_1+\frac{k}{2}M_2\right],$$

where $h=b-a$, $k=d-c$ and $M_1=\sup\limits_D|f^{(2,2)}(x,y)|$, $M_2=\sup\limits_D|f^{(1,3)}(x,y)|$.

As was indicated by S.E. Mikeladze [13] this formula was obtained earlier by N.K. Artmeladze in the paper [1].

**1.5.** Now we establish a cubature formula for the triangle $T=T_{a,b,c}$ having the total degree of exactness equal with three. Imposing the conditions that the remainder vanishes for the monomials $e_{i,j}(x,y)=x^iy^j$ $(0\le i+j\le 3)$, one finds the equations

$$\int\int_T(x-a)(x-x_2)(x-b)dxdy=0$$

$$\int\int_T(x-a)(x-x_2)(y-c)dxdy=0$$

$$\int\int_T(x-a)(y-c)(y-y_2)dxdy=0$$

$$\int\int_T(y-c)(y-y_2)(y-d)dxdy=0$$

This system of equations has the solution $x_2=\dfrac{3x+2b}{5}$, $y_2=\dfrac{3c+2d}{5}$.

Consequently we obtain a cubature formula with six nodes, having the degree of exactness three, namely:

$$\int\int_T f(x,y)dxdy=\frac{(b-a)(d-c)}{288}\Bigg[3f(a,c)+8f(a,d)+8f(b,c)+25f\left(a,\frac{3c+2d}{5}\right)$$

$$+25f\left(\frac{3a+2b}{5},c\right)+75f\left(\frac{3a+2b}{5},\frac{3c+2d}{5}\right)\Bigg]+R_3(f),$$

where the remainder has the expression:

$$R_3(f)=\frac{1}{7200}[-h^5kf^{(4,0)}(\xi,\eta)+2h^4f^{(3,1)}(\xi,\eta)-2h^3k^3f^{(2,2)}(\xi,\eta)$$

$$+2h^2k^4f^{(1,3)}(\xi,\eta)-hk^5f^{(0,4)}(\xi,\eta)].$$

If we choose $m = 3$ and we want to obtain a cubature formula of degree of exactness four, it is necessary and sufficient to be satisfied the conditions:

$$\int\int_T (x-a)(x-x_2)(x-x_3)(x-b)dxdy = 0$$

$$\int\int_T (x-a)(x-x_2)(x-x_3)(y-c)dxdy = 0$$

$$\int\int_T (x-a)(x-x_2)(y-c)(y-y_2)dxdy = 0$$

$$\int\int_T (x-a)(y-c)(y-y_2)(y-y_3)dxdy = 0$$

$$\int\int_T (y-c)(y-y_2)(y-y_3)(y-d)dxdy = 0$$

This system of equations has the following solution:

$$x_2 = \frac{7a+2b}{9}, \quad x_3 = \frac{3a+5b}{8}, \quad y_2 = \frac{3c+d}{4}, \quad y_3 = \frac{c+2d}{3}$$

By using it we arrive at the following cubature formula with ten nodes and degree of exactness four:

$$\int\int_T f(x,y)dxdy \cong \frac{(b-a)(d-c)}{4384800}\left[34713f(a,c) + 83835f\left(\frac{7a+2b}{9}, c\right)\right.$$

$$+77952f\left(a, \frac{3c+d}{4}\right) + 172032f\left(\frac{3a+5b}{8}, c\right) + 653184f\left(\frac{7a+2b}{9}, \frac{3c+d}{4}\right)$$

$$+147987f\left(a, \frac{c+2d}{3}\right) + 38280f(b,c) + 516096f\left(\frac{3a+5b}{8}, \frac{c+2d}{4}\right)$$

$$\left.+443961f\left(\frac{7a+2b}{9}, \frac{c+2d}{3}\right) + 24360f(a,d)\right]$$

Now we present a cubature formula more simple:

$$\int\int_T f(x,y)dxdy \cong \left\{\frac{hk}{45} - \{8[f(a-2h,b) + f(a,b-2k) + f(a,b)]\right.$$

$$+32[f(a-h,b-2k) + f(a-2h,b-k) + f(a+h,b-2k) + f(a-2h,b+k)$$

$$\left.+f(a+h,b-k) + f(a-h,b+k)] + 64[f(a-h,b) - f(a,b-k) + f(a-h,b-k)]\}\right\},$$

**1.6.** By using a similar procedure we can construct formulas of global degree of exactness equal with five. For this purpose we have to resolve the equations:

$$\int\int_T (x-a)(x-x_2)(x-x_3)(x-x_4)(x-b)dxdy = 0$$

$$\int\int_T (x-a)(x-x_2)(x-x_3)(x-x_4)(y-c)dxdy = 0$$
$$\int\int_T (x-a)(x-x_2)(x-x_3)(y-c)(y-y_2)dxdy = 0$$
$$\int\int_T (x-a)(x-x_2)(y-c)(y-y_2)(y-y_3)dxdy = 0$$
$$\int\int_T (x-a)(y-c)(y-y_2)(y-y_3)(y-y_4)dxdy = 0$$
$$\int\int_T (y-c)(y-y_2)(y-y_3)(y-y_4)(y-d)dxdy = 0$$

with six unknown quantities. We mention that only four of these equations are distinct.

Now we can take advantage of having two degree of liberty and to construct a cubature formula using 13 nodes and having the degree of exactness equal with five, but we prefer to establish a cubature formula with 15 nodes represented by rational numbers.

By using the following solution of the preceding system of equations:

$$x_2 = \frac{3a+b}{2}, \quad x_3 = \frac{5a+2b}{7}, \quad x_4 = \frac{3a+5b}{8}$$
$$I_2 = \frac{3c+d}{4}, \quad I_3 = \frac{5c+2d}{7}, \quad I_4 = \frac{3c+5d}{8},$$

we obtain the following cubature formula with 15 nodes:

$$\int\int_T f(x,y)dxdy \cong \frac{(b-a)(d-c)}{2872800}\Bigg[-11571f(a,c) + 493696f\left(\frac{3a+b}{4}, c\right)$$
$$+493696f\left(a, \frac{3c+d}{4}\right) - 424977f\left(\frac{5a+2b}{7}, c\right) - 424977f\left(a, \frac{5c+2b}{7}\right)$$
$$-4085760f\left(\frac{3a+b}{4}, \frac{3c+d}{4}\right) + 211456f\left(\frac{3a+5b}{8}, c\right) + 211456f\left(a, \frac{3c+5d}{8}\right)$$
$$+211456f\left(\frac{3a+5b}{8}, c\right) + 211456f\left(a, \frac{3c+5d}{8}\right) + 4609920f\left(\frac{5a+2b}{7}, \frac{3c+d}{4}\right)$$
$$+4609920f\left(\frac{3a+b}{4}, \frac{5c+2d}{7}\right) + 24472f(b,c) + 24472f(a,d)$$
$$+247296f\left(\frac{3a+5b}{8}, \frac{3c+d}{4}\right) + 247296f\left(\frac{3a+b}{4}, \frac{3c+5d}{8}\right)$$
$$-4789995f\left(\frac{5a+2b}{7}, \frac{5c+2d}{7}\right)\Bigg].$$

## 2. Use of a method of D.V. Ionescu for numerical evaluation double integrals over an arbitrary triangular domain

**2.1.** In the paper [12] D.V. Ionescu has given a method for construction certain cubature formulas for an arbitrary triangular domain $D$, with vertices $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$.

One denotes by $L$, $M$ and $N$ the barycentres of the masses $(\alpha, 1, 1)$, $(1, \alpha, 1)$, $(1, 1, \alpha)$ situated in the vertices $A, B, C$ of triangle $D$. The new triangle $LMN$ is homotethic with the triangle $ABC$. Giving to $\alpha$ the real values $\alpha_1, \alpha_2, \dots, \alpha_n$ one obtains the nodes $L_i, M_i, N_i$ $(i = \overline{1, n})$. In the paper [12] were considered cubature formulas with the fixed nodes $L_i, M_i, N_i$ of the following form:

$$\int\int_D f(x, y)dxdy = \sum_{i=1}^{n} A_i[f(L_i) + f(M_i) + f(N_i)]. \tag{2.1}$$

The coefficients $A_i$ will be determined so that this cubature formula to have the degree of exactness equal with $n$.

It was proved that this problem is possible if and only if $n \leq 5$.

One observes that in the special case $\alpha = 1$ and $n = 1$, one obtains the cubature formula

$$\int\int_D f(x, y)dxdt = Sf(G),$$

where $G$ is the barycentre of the triangle $T$, while $S$ is the area of this triangle.

For $n = 2$ and $\alpha_2 = 1$ we get the cubature formula

$$\int\int_D f(x, y)dxdy = \frac{S}{12}\left\{\frac{(\alpha_1 + 2)^2}{(\alpha_1 - 1)^2}[f(L_1) + f(M_1) + f(N_1)] + 9\frac{\alpha_1^2 - 4a_1}{(\alpha_1 - 1)^2}f(G)\right\}.$$

If $\alpha_1 = 0$ we obtain the cubature formula:

$$\int\int_D f(x, y)dxdy = \frac{S}{3}[f(A') + f(B') + f(C')],$$

where $A', B', C'$ are the middles of the sides of the triangle $T$.

In the case $n = 3$, $\alpha_2 = 3$ and $\alpha_3 = 1$ we find the cubature formula:

$$\int\int_T f(x, y)dxdy = \frac{S}{48}\{25[f(L) + f(M) + f(N)] - 27f(G)\},$$

where $G$ is the barycenter of the masses (3,1,1) placed in the vertices of the triangle $D$.

It should be mentioned that the above cubature formulas can be used for the numerical calculation of an integral extended to a polygonal domain, since it can be decomposed in triangles and then we can apply these particular cubature formulas.

The above cubature formulas can be extended to three or more dimensions.

We mention that Hortensia Roşcău [15] has proved that the problem is possible only for $n \leq 3$ in the case when the masses are distinct.

## 3. **Some recent methods for numerical calculation of a double integral over a triangular domain**

**3.1.** One can make numerical evaluation of a double integral extended over a triangular domain $T$, having the vertices (0,0), (0,1) and (1,0) using as basic domain the square $D = [0,1] \times [0,1]$.

Let $T$ and $D$ be related to each other by means of the transformations $x = g(u,v)$, $y = h(u,v)$.

It will be assumed that $g$ and $h$ have continuous partial derivatives and that the Jacobian $J(u,v)$ does not vanish in $D$. We have:

$$I(f) = \int \int_T f(x,y)dxdy = \int \int_D f(g(u,v),h(u,v))J(u,v)dudv$$

For $g = xu$, $h = x(1-u)$ we have $J(u,v) = x$ and the integral $I(f)$ becomes:

$$I(f) = \int_0^1 \int_0^1 xf(xu, x(1-u))dxdy \tag{3.1}$$

**3.2.** Several classes of numerical integration formulas can be obtained by products of Gauss-Jacobi quadrature formulas based on the transformation (3.1). In the paper [11] by Hillion Pierre [11] has been described some applications of this transformation, including one to the solution of a Dirichlet problem using the finite element method.

**3.3.** Since for any triangle from the plane $xOy$ there exists an affine transformation which leads to the standard triangle:

$$T_h = \{(x,y) \in \mathbb{R}^2;\ x \geq 0,\ y \geq 0,\ x + y \leq h,\ h \in \mathbb{R}_+\},$$

Gh. Coman has considered and investigated, in the paper [6] (see also the book [7]), the so called homogeneous cubature formulas of interpolation type for the triangle $T_h$, characterized by the fact that each term of the remainder has the same order of approximation.

A simple example of such a cubature formula is represented by

$$\int\int_{T_h} f(x,y)dxdy = \frac{h^2}{120}\left[3f(0,0)+3f(h,0)+3f(0,h)\right.$$

$$\left.+8f\left(\frac{h}{2},0\right)+8f\left(\frac{h}{2},\frac{h}{2}\right)+8f\left(0,\frac{h}{2}\right)+27f\left(\frac{h}{3},\frac{h}{3}\right)\right]+R_3(f),$$

having the degree of exactness equal with three.

The remainder can be evaluated by using the partial derivatives of the function $f$ of order (4,0), (3,1), (1,3), (0,4) and (2,2).

Some new homogeneous cubature formulas for a triangular domains were investigated in the recent paper [14] by I. Pop-Purdea.

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