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A NOTE ON MULTIVALUED MEIR-KEELER TYPE OPERATORS

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Dedicated to Professor Gheorghe Coman at his 70^{th} *anniversary*

Abstract. The purpose of this work is to present some fixed point results for multivalued generalized Meir-Keeler type operators.

1. **Introduction**

Throughout this paper, the standard notations and terminologies in nonlinear analysis (see [12], [13], [8]) are used. For the convenience of the reader we recall some of them.

Let (X,d) be a metric space. By $\tilde{B}(x_0;r)$ we denote the closed ball centered in $x_0 \in X$ with radius $r > 0$.

Also, we will use the following symbols:

$P(X) := \{Y \subset X|\ Y \text{ is nonempty}\}$, $P_{cl}(X) := \{Y \in P(X)|\ Y \text{ is closed}\}$,

$P_b(X) := \{Y \in P(X)|\ Y \text{ is bounded }\}$, $P_{b,cl}(X) := P_{cl}(X) \bigcap P_b(X)$.

Let A and B be nonempty subsets of the metric space (X,d). The gap between these sets is

$$D(A,B) = \inf\{d(a,b)|\ a \in A,\ b \in B\}.$$

In particular, $D(x_0,B) = D(\{x_0\},B)$ (where $x_0 \in X$) is called the distance from the point x_0 to the set B.

Also, if $A,B \in P_b(X)$, then one denote

$$\delta(A,B) := \sup\{d(a,b)|\ a \in A,\ b \in B\}.$$

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The Pompeiu-Hausdorff generalized distance between the nonempty closed subsets A and B of the metric space (X, d) is defined by the following formula:

$$H(A,B) := \max\{\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(a,b)\}.$$

The symbol $T : X \multimap Y$ means $T : X \to P(Y)$, i. e. T is a set-valued operator from X to Y. We will denote by $Graf(T) := \{(x,y) \in X \times Y | y \in T(x)\}$ the graph of T.

For $T : X \to P(X)$ the symbol $Fix(T) := \{x \in X | \ x \in T(x)\}$ denotes the fixed point set of the set-valued operator T. Also, for $x \in X$, we denote $F^n(x) := F(F^{n-1}(x))$, $n \in \mathbb{N}^*$, where $F^0(x) := x$.

Definition 1.1. If $f : X \to X$ is an single-valued operator, let us consider the following conditions:

i) α-contraction condition:

(1) $\alpha \in [0,1[$ and for $x, y \in X \Rightarrow d(f(x), f(y)) \leq \alpha d(x,y)$;

ii) contractive condition:

(2) $x, y \in X, \ x \neq y \Rightarrow d(f(x), f(y)) < d(x,y)$;

iii) Meir-Keeler type condition:

(3) for each $\eta > 0$ there exists $\delta > 0$ such that $x, y \in X$, $\eta \leq d(x,y) < \eta+\delta \Rightarrow d(f(x), f(y)) < \eta$;

iv) ϵ-locally Meir-Keeler type condition (where $\epsilon > 0$)

(4) for each $0 < \eta < \epsilon$ there is $\delta > 0$ such that $x, y \in X$, $\eta \leq d(x,y) < \eta+\delta \Rightarrow d(f(x), f(y)) < \eta$.

Let us observe that, condition (iii) implies (ii), (iii) implies (iv) and each of these conditions implies the continuity of f.

Definition 1.2. If $F : X \to P_{cl}(X)$ is a multi-valued operator then F is said to be:

i) α-contraction if:

(5) $\alpha \in [0,1[$ and for $x, y \in X \Rightarrow H(F(x), F(y)) \leq \alpha d(x,y)$;

ii) contractive if:

(6) $x, y \in X, \ x \neq y \Rightarrow H(F(x), F(y)) < d(x,y)$;

iii) Meir-Keeler type operator if:

(7) for each $\eta > 0$ there exists $\delta > 0$ such that $x, y \in X$, $\eta \leq d(x, y) < \eta + \delta \Rightarrow H(F(x), F(y)) < \eta$;

iv) ϵ-locally Meir-Keeler type operator (where $\epsilon > 0$) if:

(8) for each $0 < \eta < \epsilon$ there is $\delta > 0$ such that $x, y \in X$, $\eta \leq d(x, y) < \eta + \delta \Rightarrow H(F(x), F(y)) < \eta$.

It is easily to see that condition (iii) implies (ii), (iii) implies (iv) and each of these conditions implies the upper semi-continuity of F.

The following theorems are fundamental in the theory of Meir-Keeler type operators.

The first result is known as Meir-Keeler fixed point principle for self single-valued operators.

Theorem 1.3. (Meir-Keeler [5]) *Let (X, d) be a complete metric space and f an operator from X into itself. If f satisfies the Meir-Keeler type condition (3) then f has a unique fixed point, i.e. $F_f = \{x^*\}$. Moreover, for any $x \in X$, we have $\lim_{n\to\infty} f^n(x) = x^*$.*

For the multivalued case, a similar result was proved by S. Reich, as follows.

Theorem 1.4. (Reich [9]) *Let (X, d) be a complete metric space and $F : X \to P_{cp}(X)$ be a multivalued operator. If F satisfies the Meir-Keeler type condition (7), then F has at least one fixed point.*

For the case of a multivalued contractive operator Smithson proved:

Theorem 1.5. (Smithson [14]) *Let (X, d) be a compact metric space and $F : X \to P_{cl}(X)$ be a multivalued contractive operator. Then F has at least one fixed point.*

The purpose of this work is to consider a generalized Meir-Keeler type multivalued operator and to discuss some connections with the classical one. Some fixed point results are also given. Two open problems are pointed out. Our results are in connections with some theorems given in S. Reich [9], R. P. Agarwal, D. O'Regan, N.

Shahzad [1], S. Leader [4], S. Park, W. K. Kim [7], T. Cardinali, P. Rubbioni [2], I. A. Rus [10], [11], etc.

2. Main Results

Let (X, d) be a metric space and $F : X \to P_{cl}(X)$ be a multivalued operator. For $x, y \in X$, let us denote

$$M(x, y) := \max\{d(x, y), D(x, F(x)), D(y, F(y)), \frac{1}{2}[D(x, F(y)) + D(y, F(x))]\}.$$

Consider the following two Meir-Keeler type conditions on F:

(9) for each $\eta > 0$ there exists $\delta > 0$ such that $x, y \in X$, $\eta \leq M(x, y) < \eta + \delta \Rightarrow H(F(x), F(y)) < \eta$;

(10) for each $\eta > 0$ there exists $\delta > 0$ such that $x, y \in X$, $M(x, y) < \eta + \delta \Rightarrow H(F(x), F(y)) < \eta$.

Our first remark is that (9) $\Leftrightarrow$ (10).

This follows from the following two lemmas.

Lemma 2.1. *If (X, d) is a metric space and $F : X \to P_{cl}(X)$ satisfies* (9), *then $H(F(x), F(y)) \leq M(x, y)$, for each $x, y \in X$.*

Proof. We discuss two cases:

1) $M(x, y) = 0$; Then $x = y$ and we are done.

2) $M(x, y) > 0$; Let $\eta > 0$ and $\delta > 0$ such that (9) holds. Suppose, by contradiction, that $H(F(x), F(y)) > M(x, y)$. Then $H(F(x), F(y)) > M(x, y) \geq \eta$, a contradiction with (9). $\square$

Lemma 2.2. *Let (X, d) be a metric space and $F : X \to P_{cl}(X)$. Then* (9) $\Leftrightarrow$ (10).

Proof. (10) $\Rightarrow$ (9) is obviously. For the reverse implication, let us consider $\eta > 0$ and $x, y \in X$ such that (9) holds. We have the following two situations:

1) $M(x, y) < \eta$; Then from Lemma 2.1 we get that $H(F(x), F(y)) < \eta$.

1) $M(x, y) \geq \eta$; Then from (9) we have $H(F(x), F(y)) < \eta$. $\square$.

Also, we have:

Lemma 2.3. *If (X,d) is a metric space and $F : X \to P_{cl}(X)$ satisfies* (10), *then $H(F(x),F(y)) < M(x,y)$, for each $x,y \in X$, with $x \neq y$.*

Proof. If there exist $x \neq y \in X$ such that $H(F(x),F(y)) \geq M(x,y)$, then we contradict (9). □

Lemma 2.4. *Let (X,d) be a metric space and $F : X \to P_{cl}(X)$ be a multivalued operator such that* (9) *(or equivalently* (10)*) holds. Then:*

(11) *for each $\eta > 0$ there exists $\delta > 0$ such that $(x,y) \in GrafF$, $\eta \leq d(x,y) < \eta + \delta \Rightarrow H(F(x),F(y)) < \eta$.*

Proof. For $\eta > 0$ let $\delta > 0$ be such that (9) holds. Let $y \in F(x)$ be arbitrary. Then $M(x,y) = \max\{d(x,y), D(x,F(x)), D(y,F(y)), \frac{1}{2}D(x,F(y))\}$.

Since $D(x,F(x)) \leq d(x,y)$ and $D(x,F(y)) \leq d(x,y) + D(y,F(y))$ it follows that $M(x,y) = \max\{d(x,y), D(y,F(y))\}$.

If $M(x,y) = D(y,F(y))$, then from (9) we have the following contradiction: $\eta \leq D(y,F(y)) \leq H(F(x),F(y)) < \eta$. So $M(x,y) = d(x,y)$. □

In a similar way as above, we have:

Lemma 2.5. *Let (X,d) be a metric space and $F : X \to P_{cl}(X)$ be a multivalued operator. Consider the following condition:*

(12) *for each $\eta > 0$ there exists $\delta > 0$ such that $(x,y) \in GrafF$, $d(x,y) < \eta + \delta \Rightarrow H(F(x),F(y)) < \eta$.*

Then (11) $\Leftrightarrow$ (12).

The following result is an easy consequence of the above lemmas.

Lemma 2.6. *If (X,d) is a metric space and $F : X \to P_{cl}(X)$ satisfies*

(9') for each $\eta > 0$ there exists $\delta > 0$ such that $(x,y) \in GrafF$, $\eta \leq M(x,y) < \eta + \delta \Rightarrow H(F(x),F(y)) < \eta$

or

(10') for each $\eta > 0$ there exists $\delta > 0$ such that $(x,y) \in GrafF$, $M(x,y) < \eta + \delta \Rightarrow H(F(x),F(y)) < \eta$,

then $H(F(x),F(y)) < d(x,y)$, for each $(x,y) \in GrafF$, with $x \neq y$.

Open Problem A. *Establish fixed point results for (generalized) Meir-Keeler multivalued operators on graphic, i. e. satisfying the condition* (9′) *or* (10′).

For example, from the above results and Theorem 1.5. we immediately obtain:

Theorem 2.7. *Let* (X, d) *be a compact metric space and* $F : X \to P_{cp}(X)$ *be a multivalued operator, such that it satisfies the contractive condition* (6) *for each* $(x, y) \in (X \times X) \setminus GrafF$. *Suppose that* F *satisfies the following generalized Meir-Keeler type condition:*

(13) *for each* $\eta > 0$ *there exists* $\delta > 0$ *such that* $(x, y) \in GrafF$, $M(x, y) < \eta + \delta \Rightarrow H(F(x), F(y)) < \eta$,

then F *has at least one fixed point.*

Proof. The assumption (13) implies (12) which is equivalent to (11). From the above remark, F satisfies the contractive condition (6) for each $(x, y) \in GrafF$. Hence, F is contractive on X. The rest of the proof follows from Theorem 1.5. □

Open Problem B. *Establish results of the above type for the case of a locally (generalized) Meir-Keeler multivalued operator, see also the conditions* (4) *and* (8).

The following theorem is a slight modification of a result established by R. P. Agarwal, D. O'Regan, N. Shahzad in [1].

Theorem 2.8. *Let* (X, d) *be a complete metric space,* $x_0 \in X$, $r > 0$ *and* $f : \widetilde{B}(x_0; r) \to X$ *an operator. Suppose that:*

i) for each $\eta > 0$ *there exists* $\delta > 0$ *such that* $x, y \in \widetilde{B}(x_0; r)$, $\eta \leq d(x, y) < \eta + \delta \Rightarrow d(f(x), f(y)) < \eta$.

ii) $d(x_0, f^n(x_0)) < r$, *for each* $n \in \mathbb{N}^*$.

Then $Fixf = \{x^*\}$.

An extension to the multivalued case is the following:

Theorem 2.9. *Let* (X, d) *be a complete metric space,* $x_0 \in X$, $r > 0$ *and* $F : \widetilde{B}(x_0; r) \to P_{cp}(X)$ *be a multivalued operator. Suppose that:*

i) for each $\eta > 0$ *there exists* $\delta > 0$ *such that* $x, y \in \widetilde{B}(x_0; r)$, $\eta \leq d(x, y) < \eta + \delta \Rightarrow H(F(x), F(y)) < \eta$.

ii) $\delta(x_0, F^n(x_0)) < r$, *for each* $n \in \mathbb{N}^*$.

Then $Fix F \neq \emptyset$.

Sketch of the proof. Let us consider the operator $G : \widetilde{B}(\{x_0\}; r) \subset (P_{cp}(X), H) \to (P_{cp}(X), H)$, given by $G(Y) := \bigcup_{x \in Y} F(x)$. Then G satisfies all the hypothesis of Theorem 2.8. Hence, there exists $Y^* \in P_{cp}(X)$ such that $Y^* = G(Y^*)$. Define $h : Y^* \to \mathbb{R}_+$, by $h(a) := D(a, F(a))$. Since F is contractive on $\widetilde{B}(x_0; r)$ it follows that F is upper semicontinuous on $\widetilde{B}(x_0; r)$. Thus h is lower semicontinuous on Y^*. Since Y^* is compact, there exists $b \in Y^*$ and $c \in F(b)$ such that $\inf_{a \in Y^*} h(a) = d(b, c)$. If we suppose that $d(b, c) > 0$ then we get a contradiction: $h(c) = D(c, F(c)) \leq H(F(b), F(c)) < d(b, c)$. Hence $b = c$ and so the conclusion follows. $\square$

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