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ON SUPERCONVERGENT SPLINE COLLOCATION METHODS FOR THE RADIOSITY EQUATION

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Dedicated to Professor Gheorghe Coman at his 70^{th} *anniversary*

Abstract. In this paper we study collocation methods based on piecewise polynomial interpolation for the radiosity equation. We give a brief outline of this equation and its properties. With a special choice of interior nodes, we show that interpolation of degree r of the solution leads to an error in the collocation method of $O\left(h^{r+1}\right)$, where h is the mesh size of the triangulation. We conclude the paper by giving superconvergence results, considering separately the case where r is odd and the case where r is even.

1. The radiosity equation

Radiosity is a method of describing illumination based on a detailed analysis of light reflections off diffuse surfaces. It is typically used to render images of the interior of buildings. In computer graphics, the computation of lighting can be done via radiosity.

1.1. Definition. Properties

Radiosity is defined as being the energy per unit solid angle that leaves a surface. The *radiosity equation* is a mathematical model for the brightness of a collection of one or more surfaces. The equation is

$$u(P) - \frac{\rho(P)}{\pi} \int\limits_S u(Q)G(P,Q)V(P,Q)dS_Q = E(P), \quad P \in S \tag{1}$$

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where $u(P)$ is the *radiosity*, or the brightness, at $P \in S$. $E(P)$ is the *emissivity* at $P \in S$, the energy per unit area emitted by the surface.

The function $\rho(P)$ gives the *reflectivity* at $P \in S$, i. e. the bidirectional reflection distribution function. We have that $0 \leq \rho(P) < 1$, with $\rho(P)$ being 0 where there is no reflection at all at P. The radiosity equation is derived from the rendering equation under the *radiosity assumption*: all surfaces in the environment are *Lambertian diffuse reflectors*. What this means is that the reflectivity $\rho(P)$ is independent of the incoming and outgoing directions and, hence, of the angle at which the reflection takes place. Thus, $\rho(P)$ can be taken out from under the integral of a more general formulation (the rendering equation, see Cohen and Wallace [5]), leading to (1).

The function G, a geometric term, is given by

$$\begin{aligned} G(P,Q) &= \frac{[(Q-P)\cdot \mathbf{n}_P]\,[(P-Q)\cdot \mathbf{n}_Q]}{|P-Q|^4} \\ &= \frac{\cos\theta_P \cdot \cos\theta_Q}{|P-Q|^2} \end{aligned} \tag{2}$$

where $\mathbf{n}_P$ is the inner unit normal to S at P, θ_P is the angle between $\mathbf{n}_P$ and $Q-P$, and $\mathbf{n}_Q$ and θ_Q are defined analogously.

The function $V(P,Q)$ is a *visibility* function. It is 1 if the points P and Q are "mutually visible" (meaning they can "see each other" along a straight line segment which does not intersect S at any other point), and 0 otherwise. Surfaces S for which $V \equiv 1$ on S are called *unoccluded*, and this is the case that we will consider here. More about the radiosity equation can be found in Cohen and Wallace [5].

We can write (1) in the form

$$u(P) - \int_S K(P,Q)u(Q)dS_Q = E(P), \quad P \in S \tag{3}$$

with

$$K(P,Q) = \frac{\rho(P)}{\pi} G(P,Q)V(P,Q), \quad P,Q \in S \tag{4}$$

or, in operator form

$$(I-K)u = E \tag{5}$$

Let S be a smooth surface, although not necessarily connected. Later on, more assumptions on the surface S will be made.

The function $G(P,Q)$ given in (2) has a singularity at $P=Q$ and is smooth otherwise. Since this function plays an important role in the study of the solvability of equation (1), we give in the next lemma some of its properties.

Lemma 1. *Let S be a smooth C^{i+1} surface to which the Divergence Theorem can be applied. Let $P \in S$. Then*

a) $|G(P,Q)| \le c_1, \quad P,Q \in S, \quad P \ne Q$;

b) $G(P,Q) \ge 0, \text{ for } Q \in S$;

c) $\displaystyle\int_S G(P,Q)\, dS_Q = \pi$;

d) if S is the unit sphere, then $G(P,Q) \equiv \dfrac{1}{4}$;

e) $\left|D_Q^i G(P,Q)\right| \le \dfrac{c_2}{|P-Q|^i}, \quad P \ne Q$, *$c_2$ independent of P and Q.*

For the proof, see [10].

Since the surface S is smooth and by Lemma 1, it is relatively easy to prove that the integral operator $\mathcal{K}$ of (5) is compact as an operator on either $C(S)$ or $L^2(S)$ into itself (see Mikhlin [13] pp. 160-162).

1.2. Solvability and Regularity of the Radiosity Equation

The solvability theory for the radiosity equation (1) is relatively straightforward, being based on the Geometric Series Theorem.

Let S be a smooth unoccluded surface (not necessarily connected). Thus the normal $\mathbf{n}_P$ is to be a continuous function of $P \in S$. In addition to the *radiosity assumption* (discussed in Section 1.1., we will also assume that the reflectivity function $\rho(P) \in C(S)$ and that it satisfies

$$\|\rho\|_\infty < 1 \tag{6}$$

From the physical point of view, what (6) means is that the surface does not reflect 100% of all the light that it receives, which is a reasonable assumption.

For the regularity of the solution of (1), we have

Lemma 2. *Let $m \geq 0$ be an integer, S a smooth unoccluded surface. Assume the reflectivity function $\rho \in C^{m+1}(S)$ and it satisfies (6). Then*

$$u \in C^m(S) \Rightarrow \mathcal{K}u \in C^{m+1}(S) \tag{7}$$

Theorem 3. *Let $m \geq 0$ be an integer. Let $\hat{S}$ be the boundary of a convex open set Ω, and assume $\hat{S}$ is a surface to which the Divergence Theorem can be applied. Assume S is a smooth (possibly disconnected) unoccluded surface $S \subset \hat{S}$. Also, assume $\rho, E \in C^m(S)$. Then*

(a) *The equation (1) is uniquely solvable for each E, with the solution $u(P)$ satisfying*

$$\|u\|_\infty \leq \frac{\|E\|_\infty}{1 - \|\mathcal{K}\|} \tag{8}$$

(b) *The solution $u \in C^m(S)$.*

For the proof, see [10].

2. Preliminaries for Collocation Methods

Let S be a smooth unoccluded surface in $\mathbb{R}^3$, which can be written as

$$S = S_1 \cup S_2 \cup ... \cup S_J \tag{9}$$

with each S_j the continuous image of a polygonal region in the plane

$$F_j : R_j \xrightarrow[onto]{1-1} S_j, \quad j = 1, ..., J \tag{10}$$

Generally, we will need to assume that the mappings F_j are several times continuously differentiable.

To create triangulations for S, we first triangulate each R_j and then map this triangulation onto S_j. Let $\{\widehat{\Delta}^j_{n,k} \mid k = 1, ..., n_j\}$ be a triangulation of R_j, and then define

$$\Delta^j_{n,k} = F_j(\widehat{\Delta}^j_{n,k})$$

This yields a triangulation of S, which we refer to collectively as $\mathcal{T}_n = \{\Delta_1, ..., \Delta_n\}$. Let

$$h \equiv h_n = \max_{1\le j\le J} \max_{1\le k\le n_j} diameter\left(\widehat{\Delta}^j_{n,k}\right) \tag{11}$$

be the mesh size of this triangulation. (The number of triangles n is to be understood implicitly; from now on, we dispense with it.)

We make the following assumptions concerning this triangulation:

T1. The set of all vertices of the surface S is a subset of the set of all vertices of the triangulation $\mathcal{T}_n$.

T2. The union of all edges of S is contained in the union of all edges of all triangles in $\mathcal{T}_n$.

T3. If two triangles in $\mathcal{T}_n$ have a nonempty intersection, then that intersection consists either of (i) a single common vertex, or (ii) all of a common edge.

We call triangulations satisfying T1 - T3 *conforming triangulations*.

Let Δ_k be some element from $\mathcal{T}_n$, and let it correspond to some $\widehat{\Delta}_k$, say $\widehat{\Delta}_k \subset R_j$ and $\Delta_k = F_j(\widehat{\Delta}_k)$. Let $\{\hat{v}_{k,1}, \hat{v}_{k,2}, \hat{v}_{k,3}\}$ denote the vertices of $\widehat{\Delta}_k$. Define $m_k : \sigma \xrightarrow[onto]{1-1} \Delta_k$ by

$$m_k(s,t) = F_j(u\hat{v}_{k,1} + t\hat{v}_{k,2} + s\hat{v}_{k,3}), \quad (s,t) \in \sigma, \quad u = 1 - s - t \tag{12}$$

(an affine mapping), where σ is the unit simplex $\sigma = \{(s,t) | 0 \le s, t, s + t \le 1\}$. Now we can define interpolation and numerical integration over a triangular surface element Δ by means of a similar formula over σ.

Let α be a given constant with $0 \le \alpha \le \dfrac{1}{3}$. Define the interpolation nodes by

$$q_{i,j} = \left(\frac{i + (r - 3i)\alpha}{r}, \frac{j + (r - 3j)\alpha}{r}\right), \quad i, j \ge 0, \quad i + j \le r \tag{13}$$

These $f_r = \dfrac{(r+1)(r+2)}{2}$ nodes form a uniform grid over σ. If $\alpha = 0$, some of these points are on the edges of σ. If $\alpha > 0$, then they are symmetrically placed points in the interior of σ. To avoid problems with the unit normal and with the nonsmoothness of the kernel, throughout this paper we want to consider only nodes that are interior to the triangular elements, so we will work with $0 < \alpha < \dfrac{1}{3}$.

Denote by $l_{i,j}(s,t)$ the corresponding Lagrange interpolation basis functions. Then for a given $g \in C(\sigma)$, the formula

$$p_r(s,t) = \sum_{0 \le i+j \le r} g(q_{i,j}) l_{i,j}(s,t) \tag{14}$$

is the unique polynomial of degree r that interpolates $g(s,t)$ at the nodes $\{q_{i,j} \mid i,j \ge 0, \quad i+j \le r\}$.
Denote the nodes and the basis functions collectively by $\{q_1, ..., q_{f_r}\}$ and $\{l_1, ..., l_{f_r}\}$. So, now we have the interpolation formula

$$g(s,t) \approx \sum_{j=1}^{f_r} g(q_j) l_j(s,t), \quad g \in C(S) \tag{15}$$

Integrating (15) over σ, we obtain the quadrature formula

$$\int_\sigma g(s,t) d\sigma \approx \sum_{j=1}^{f_r} \omega_j g(q_{i,j}) \tag{16}$$

where $\omega_j = \int_\sigma l_j(s,t) d\sigma$. Since the formula (15) is exact for all polynomials of degree $\le r$, formula (16) has degree of precision at least r.

Let

$$\mathcal{P}_n g(m_k(s,t)) = \sum_{j=1}^{f_r} g(m_k(q_j)) l_j(s,t), \quad P = m_k(s,t) \in \Delta_k \tag{17}$$

Define a collocation method using (17) (the collocation nodes coincide with the interpolation nodes). Substitute

$$\begin{aligned} u_n(P) &= \sum_{j=1}^{f_r} u_n(v_{k,j}) l_j(s,t), \quad P \in m_k(s,t) \in \Delta_k \\ v_{k,j} &= m_k(q_j), \quad k = 1, ..., n \end{aligned} \tag{18}$$

into (1). This leads to the linear system

$$\begin{aligned} u_n(v_i) &- \frac{\rho(P)}{\pi} \sum_{k=1}^{n} \sum_{j=1}^{nf_r} u_n(v_{k,j}) \int_\sigma G\left(v_i, m_k(s,t)\right) l_j(s,t) \\ &\cdot \quad \left|\left(D_s m_k \times D_t m_k\right)(s,t)\right| d\sigma = E(v_i), \quad i = 1, ..., nf_r \end{aligned} \tag{19}$$

This can be written abstractly as

$$(\mathcal{I} - P_n\mathcal{K})u_n = \mathcal{P}_n E \tag{20}$$

Also, introduce the iterated collocation solution

$$\hat{u}_n = E + \mathcal{K}u_n \tag{21}$$

We will give an error analysis based on standard projection operator theory (e. g. see Atkinson [2] Section 4.2). We have

Theorem 4. *Assume S is a smooth unoccluded surface in $\mathbb{R}^3$, and assume $S \subset \hat{S}$, with $\hat{S}$ the type of surface required in Lemma 1. Assume S satisfies (9) and (10) with each $F_j \in C^{r+2}$. Then for all sufficiently large n, say $n \geq n_0$, the operators $\mathcal{I} - P_n\mathcal{K}$ are invertible on $C(S)$ and have uniformly bounded inverses. Moreover, for the true solution u of (1) and the solution u_n of (20)*

$$\|u - u_n\|_\infty \leq \left\|(\mathcal{I} - P_n\mathcal{K})^{-1}\right\| \|(u - \mathcal{P}_n u)\|_\infty, \quad n \geq n_0 \tag{22}$$

Furthermore, if the emissivity $E \in C^{r+1}(S)$, then

$$\|u - u_n\|_\infty \leq O(h^{r+1}), \quad n \geq n_0 \tag{23}$$

3. Superconvergent Collocation Methods

So we know that under suitable assumptions, interpolation of degree r leads to an error of order $O(h^{r+1})$ in the collocation method associated with it. Sometimes at the collocation node points, the collocation method converges more rapidly than over all S, in which case

$$\lim_{n\to\infty} \frac{\max\limits_{1\leq i\leq nf_r} |u(v_i) - \hat{u}_n(v_i)|}{\|u - u_n\|_\infty} = 0 \tag{24}$$

Such methods are *superconvergent* at the collocation node points.

Let us examine more carefully the terms in (24). For simplicity, we work with the solution $\hat{u}_n$ of the iterated collocation equation (21). This should cause no problems, since we know that the convergence of $\hat{u}_n$ to u is at least as rapid as that of the solution of the collocation equation (20) to u. Moreover, $\hat{u}(v_i) = u_n(v_i)$ at all collocation nodes.

By looking at the linear system associated with

$$(\mathcal{I} - \mathcal{K}P_n)(u - \hat{u}_n) = \mathcal{K}(u - \mathcal{P}_n u) \tag{25}$$

we have

$$\max_{1 \le i \le nf_r} |u(v_i) - \hat{u}_n(v_i)| \le c \max_{1 \le i \le nf_r} |\mathcal{K}(\mathcal{I} - P_n)u(v_i)| \tag{26}$$

(see Atkinson [2] p. 449). So, to find superconvergent methods, now we focus on finding errors for $\mathcal{K}(I - P_n)u(v_i)$.

Let $\tau \subset \mathbb{R}^2$ be a planar triangle with vertices $\{v_1, v_2, v_3\}$ and define the mapping $m_\tau : \sigma \longrightarrow \tau$ as in (12). For $g \in C(\tau)$, define

$$\mathcal{L}_\tau g(x, y) = \sum_{j=1}^{f_r} g(m_\tau(q_j)) l_j(s, t) \tag{27}$$

which is a polynomial of degree r in the parametrization variables s and t, interpolating g at the nodes $\{m_\tau(q_1), ..., m_\tau(q_{f_r})\}$.

Define a numerical integration formula over τ by

$$\int_\tau g(x, y) d\tau \approx \int_\tau \mathcal{L}_\tau g(x, y) d\tau \tag{28}$$

which has degree of precision at least r. In what follows, for differentiable functions g, we will use the notation

$$|D^k g(x, y)| = \max_{0 \le i \le k} \left| \frac{\partial^k g(x, y)}{\partial x^i \partial y^{k-i}} \right| \tag{29}$$

In investigating superconvergent collocation methods based on interpolation r, we have to distinguish two cases: where r is odd and where r is even.

3.1. Interpolation of Odd Degree

Consider the quadrature formula (28), based on interpolation of degree r, an odd number. It has degree of precision at least r. Suppose we can find a value $0 < \alpha_0 < \frac{1}{3}$, such that for $\alpha = \alpha_0$, formula (28) has degree of precision $r + 1$. Then, if we extend it to a rectangle, it will have degree of precision $r + 2$. We have the following result.

Lemma 5. *Let τ_1 and τ_2 be planar right triangles that form a square R of length h on a side. Let $g \in C^{r+3}(R)$. Let $\Phi \in L^1(R)$ two times differentiable with derivatives of order 1 and 2 in $L^1(R)$. Assume $\alpha = \alpha_0$. Then*

$$\left| \int_R \Phi(x,y)(I - \mathcal{L}_\tau) g(x,y) d\tau \right| \le c h^{r+3} \left[\int_R (|\Phi| + |D\Phi| + |D^2\Phi|) d\tau \right] \max_{\substack{R \\ i=r+1,r+2,r+3}} \left\{ |D^i g| \right\} \tag{30}$$

with $\mathcal{L}_\tau g(x,y) \equiv \mathcal{L}_{\tau_i} g(x,y)$, where $(x,y) \in \tau_i$, $i = 1, 2$.

If integrating over a single triangle, the bound is given by

Lemma 6. *Let τ be a planar right triangle and assume the two sides which form the right angle have length h. Assume $\alpha = \alpha_0$. Let $g \in C^{r+2}(\tau), \Phi \in L^1(\tau)$ differentiable with first derivatives in $L^1(\tau)$. Then*

$$\left| \int_\tau \Phi(x,y)(\mathcal{I} - L_\tau) g(x,y) d\tau \right| \le c h^{r+2} \left[\int_\tau (|\Phi| + |D\Phi|) d\tau \right] \cdot \max_\tau \left\{ |D^{r+1} g|, |D^{r+2} g| \right\} \tag{31}$$

where c denotes a generic constant.

For the proofs, see [10].

Remark. These results can be extended to general triangles, but then the derivatives of g and Φ will involve the mapping m_τ from (12). Let $h(\tau)$ denote the diameter of τ and $h^*(\tau)$ the radius of the circle inscribed in τ and tangent to its sides. Define

$$r(\tau) = \frac{h(\tau)}{h^*(\tau)} \tag{32}$$

Assume that for our triangulations $\mathcal{T}_n = \{\Delta_{n,k}\}$, $n \ge 1$, we have

$$\sup_n \left[\max_{\Delta_{n,k} \in \mathcal{T}_n} r(\Delta_{n,k}) \right] < \infty \tag{33}$$

Condition (33) prevents the triangles $\Delta_{n,k}$ from having angles which approach 0 as $n \to \infty$.

Now, we want to apply these results to the individual subintegrals in

$$\begin{aligned} \mathcal{K}u(v_i) &= \frac{\rho(v_i)}{\pi} \sum_{k=1}^{n} \int_\sigma G\left(v_i, m_k(s,t)\right) u\left(m_k(s,t)\right) \\ &\cdot \quad \left| \left(D_s m_k \times D_t m_t\right)(s,t) \right| d\sigma, \quad i = 1, ..., 6n \end{aligned} \tag{34}$$

with

$$\begin{aligned} g(s,t) &= u\,(m_k(s,t))\,|(D_s m_k \times D_t m_t)\,(s,t)| \\ \Phi(s,t) &= G\,(v_i, m_k(s,t)) \end{aligned} \tag{35}$$

Theorem 7. *Assume the hypotheses of Theorem 4, with each $F_j \in C^{r+2}$. Assume $u \in C^{r+2}(S)$. Assume the triangulation $\mathcal{T}_n$ of S satisfies (33) and that it is symmetric. For those integrals in (34) for which $v_i \in \Delta_k$, assume that all such integrals are evaluated with an error of $O(h^{r+3})$. Assume $\alpha = \alpha_0$. Then*

$$\max_{1 \le i \le n f_r} |u(v_i) - \hat{u}_n(v_i)| \le c h^{r+3} \log h \tag{36}$$

Proof. We bound

$$\max_{1 \le i \le n f_r} |\mathcal{K}(I - P_n) u(v_i))|$$

By our assumption, the error in evaluating the integral of (34) over Δ^* will be $O(h^{r+3})$.

Partition $\mathcal{T}_n^*$ into parallelograms to the maximum extent possible. Denote by $\mathcal{T}_n^{(1)}$ the set of all triangles making up such parallelograms and let $\mathcal{T}_n^{(2)}$ contain the remaining triangles. Then

$$\mathcal{T}_n^* = \mathcal{T}_n^{(1)} \cup \mathcal{T}_n^{(2)}$$

It is easy to show that the number of triangles in $\mathcal{T}_n^{(1)}$ is $O(n) = O(h^{-2})$, and the number of triangles in $\mathcal{T}_n^{(2)}$ is $O(\sqrt{n}) = O(h^{-1})$.

It can be shown that all but a finite number of the triangles in $\mathcal{T}_n^{(2)}$, bounded independent of n, will be at a minimum distance from v_i. That means that the triangles in $\mathcal{T}_n^{(2)}$ are "far enough" from v_i, so that the function $G(v_i, Q)$ is uniformly bounded for Q being in a triangle in $\mathcal{T}_n^{(2)}$.

By Lemma 6, the contribution to the error coming from the triangles in $\mathcal{T}_n^{(2)}$ will be $O\left(h^{r+3} \|D^{r+2} u\|_\infty\right)$.

Using Lemma 5 we have that the contribution to the error coming from triangles in $\mathcal{T}_n^{(1)}$ is of order

$$ch^{r+3} \int\limits_{S-\Delta^*} \sum_{j=0}^{2} \frac{1}{|v_i - Q|^j} dS_Q \tag{37}$$

Using a local representation of the surface and then using polar coordinates, the expression in (37) is of order

$$ch^{r+3}\left(h^2 + h + \log h\right) = O(h^{r+3}\log h)$$

Combining the errors arising from the integrals over Δ^*, $\mathcal{T}_n^{(1)}$, and $\mathcal{T}_n^{(2)}$, we have (36).

3.2. Interpolation of Even Degree

Analogously, consider the quadrature formula (28), based on interpolation of degree r, an even number, which has degree of precision at least r. Considered over a rectangle formed by two symmetric triangles, it has degree of precision $r+1$, since r is an even number. Define a collocation method with it as before. We have:

Lemma 8. *Let τ_1 and τ_2 be planar right triangles that form a square R of length h on a side. Let $g \in C^{r+2}(R)$. Let $\Phi \in L^1(R)$ differentiable with first order derivatives in $L^1(R)$. Then*

$$\left| \int\limits_R \Phi(x,y)(I - \mathcal{L}_\tau)g(x,y)d\tau \right| \le ch^{r+2}\left[\int\limits_\tau (|\Phi| + |D\Phi|)d\tau\right] \cdot \max_{\substack{R \\ i=r+1,r+2}} \left\{|D^i g|\right\} \tag{38}$$

with $\mathcal{L}_\tau g(x,y) \equiv \mathcal{L}_{\tau_i} g(x,y)$, where $(x,y) \in \tau_i$, $i = 1,2$.

For integration over one triangle only, the term in h in (38) is only h^{r+1}. We use these results to prove the following superconvergence result.

Theorem 9. *Assume the hypotheses of Theorem 4, with each $F_j \in C^{r+2}$. Assume $u \in C^{r+2}(S)$. Assume the triangulation $\mathcal{T}_n$ of S satisfies (33) and that it is symmetric. For those integrals in (34) for which $v_i \in \Delta_k$, assume that all such integrals are evaluated with an error of $O(h^{r+2})$. Then*

$$\max_{1 \le i \le nf_r} |u(v_i) - \hat{u}_n(v_i)| \le ch^{r+2} \tag{39}$$

The proof of Theorem 9 is similar to that of Theorem 7.

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