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METRIC SPACE WITH FIXED POINT PROPERTY WITH RESPECT TO CONTRACTIONS

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Dedicated to Professor Ştefan Cobzaş at his 60^{th} *anniversary*

Abstract. In this paper we present some equivalent statements with the fixed point property of a metric space with respect to contractions. These statements are in terms of completeness, Picard operators, fractal operators, minimal displacement, well posedness of fixed point problem and the limit shadowing property.

1. **Introduction**

Let X be a nonempty set and $(X, S(X), M)$ a fixed point structure on X ([18] and [19]). Let $S_1(X) \subset P(X)$ such that $S(X) \subset S_1(X)$. By definition $(X, S(X), M)$ is maximal in $S_1(X)$ if we have

$$S(X) = \{A \in S_1(X) \mid f \in M(A) \ \Rightarrow \ F_f \neq \emptyset\}.$$

Is an open problem to establish if a given fixed point structure is maximal or not. For example in some concrete structured sets this problem take the following forms:

- Characterize the ordered sets with fixed point property with respect to increasing operators ([3], [5], [10], [11], [18], [19], [22]-[24]).
- Characterize the metric space with fixed point property with respect to continuous operators ([1], [2], [6], [10]).

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- Characterize the metric space with fixed point property with respect to contractions ([4], [21], [6]-[10]).
- Characterize the Banach space X with the following property ([10], [19], [20]):

$$Y \in P_{b,cl,cv}(X),\ f : Y \to Y \text{ nonexpansive } \Rightarrow F_f \neq \emptyset.$$

- Characterize the Banach space with the following property ([10], [19], [20]):

$$Y \in P_{wcp,cv}(X),\ f : Y \to Y \text{ nonexpansive } \Rightarrow F_f \neq \emptyset.$$

The aim of this paper is to present some equivalent statements with the fixed point property of a metric space with respect to contractions.

2. **Notations and notions**

Let (X, d) be a metric space and $(P_{cp}(X), H_d, \subset)$ the corresponding ordered metric space of fractals. In what follow we shall use the following notations. We denote

$$CT(X, X) := \{f : X \to X \mid f \text{ is a contraction}\}.$$

If $f \in CT(X, X)$ then we denote by $\widehat{f} : P_{cp}(X) \to P_{cp}(X),\ A \mapsto f(A) := \bigcup_{a \in A} f(a)$, the corresponding fractal operator.

$$(UF)_{\widehat{f}} := \{A \in P_{cp}(X) \mid \widehat{f}(A) \subset A\},$$

$$(LF)_{\widehat{f}} := \{A \in P_{cp}(X) \mid \widehat{f}(A) \supset A\}.$$

For an operator $f : X \to X$ we denote by $d(f) := \inf\{d(x, f(x)) \mid x \in X\}$ the minimal displacement of f (K. Goebel (1973) ([10], 586)).

To present our results we need the following notions:

Definition 2.1. (F.S. De Blasi and J. Myjak (1989) ([17])). Let (X, d) be a metric space and $f : X \to X$ an operator. The fixed point problem for f is well posed iff

(a) $F_f = \{x^*\}$;

(b) if $x_n \in X$, $n \in \mathbb{N}$ and $d(x_n, f(x_n)) \to 0$ as $n \to \infty$, then $d(x_n, x^*) \to 0$ as $n \to \infty$.

Definition 2.2. ([13]) An operator $f : X \to X$ has the limit shadowing property iff $x_n \in X$, $n \in \mathbb{N}$, $d(x_{n+1}, f(x_n)) \to 0$ as $n \to \infty$ imply that there exists $x \in X$ such that $d(x_n, f^n(x)) \to 0$ as $n \to \infty$.

Definition 2.3. A metric space is complete with respect to $CT(X, X)$ iff $f \in CT(X, X)$ implies that $(f^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$.

Remark 2.1. If $f \in CT(X, X)$ then $(f^n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence for all $x \in X$.

3. Equivalent statements

The main result of this paper is the following

Theorem 3.1. *Let (X, d) be a metric space. The following statements are equivalent:*

(i) (X, d) has the fixed point property with respect to $CT(X, X)$.

(ii) $f \in CT(X, X)$ implies that f is Picard operator.

(iii) (X, d) is complete with respect to $CT(X, X)$.

(iv) $f \in CT(X, X)$ implies that there exists $x_f^* \in X$ such that $d(f) = d(x_f^*, f(x_f^*))$.

(v) $f \in CT(X, X)$ implies that the fixed point problem for f is well posed.

(vi) $f \in CT(X, X)$ implies that $F_{\widehat{f}} \neq \emptyset$.

(vii) $f \in CT(X, X)$ implies that $(UF)_{\widehat{f}} \neq \emptyset$.

(viii) $f \in CT(X, X)$ implies that $(LF)_{\widehat{f}} \neq \emptyset$.

(ix) $f \in CT(X, X)$ implies that there exists $x \in X$ such that $(f^n(x))_{n\in\mathbb{N}}$ converges.

Proof. (i) $\Rightarrow$ (ii). Let f be an α-contraction with $F_f = \{x^*\}$. Then $d(f^n(x), x^*) = d(f^n(x), f^n(x^*)) \leq \alpha^n d(x, x^*) \to 0$ as $n \to \infty$, for all $x \in X$. So, f is Picard operator.

(ii) $\Rightarrow$ (iii). Follows from the definition of Picard operators.

(iii) $\Rightarrow$ (iv). Let $f \in CT(X,X)$. It is clear that (iii) implies (ii). So, x_f^* is the fixed point of f.

(iv) $\Rightarrow$ (v). Let f be an α-contraction and $d(f) = d(x_f^*, f(x_f^*))$. If $x_f^* \neq f(x_f^*)$, then we have

$$d(f(x_f^*), f^2(x_f^*)) \leq \alpha d(x_f^*, f(x_f^*)) < d(x_f^*, f(x_f^*)).$$

This implies that $F_f = \{x_f^*\}$. Let $x_n \in X$, $n \in \mathbb{N}$, be such that $d(x_n, f(x_n)) \to 0$ as $n \to \infty$. We have

$$\begin{aligned} d(x_n, x_f^*) &\leq d(x_n, f(x_n)) + d(f(x_n), x_f^*) \\ &\leq d(x_n, f(x_n)) + \alpha d(x_n, x_f^*). \end{aligned}$$

Hence,

$$d(x_n, x^*) \leq \frac{1}{1-\alpha} d(x_n, f(x_n)) \to 0 \text{ as } n \to \infty.$$

(v) $\Rightarrow$ (vi). By a theorem of Nadler f contraction implies that $\widehat{f}$ is contraction. Let $f \in CT(X,X)$ with $F_f = \{x^*\}$. Then by definition of $\widehat{f}$, $\{x^*\} \in F_{\widehat{f}}$. But, $\widehat{f}$ contraction implies $F_{\widehat{f}} = \{\{x^*\}\}$.

(vi) $\Rightarrow$ (vii). Let $f \in CT(X,X)$. The condition (vi) implies $F_{\widehat{f}} = \{A^*\}$. But $\delta(\widehat{f}(A^*)) = \delta(A^*) \leq \alpha\delta(A^*)$. This implies $A^* = \{a^*\}$. We remark that $A^* \in (UF)_{\widehat{f}} \neq \emptyset$.

(vii) $\Rightarrow$ (viii). Let $f \in CT(X,X)$ and $A^* \in (UF)_{\widehat{f}}$. These imply $A^* = \{a^*\}$. So, $A^* \in (LF)_{\widehat{f}}$.

(viii) $\Rightarrow$ (ix). $f \in CT(X,X)$ and $A^* \in (LF)_{\widehat{f}}$ imply $A^* = \{a^*\}$. So, $F_f = \{x^*\}$ and $f^n(x) \to x^*$ as $n \to \infty$, for all $x \in X$.

(ix) $\Rightarrow$ (i). Let $f \in CT(X,X)$, and $x \in X$ such that $f^n(x) \to y^*$. From the continuity of f we have that $y^* \in F_f$. So, $F_f = \{y^*\}$.

Remark 3.1. It is well known that there exist incomplete metric spaces with fixed point property with respect to contraction (see [4], [21]). On the other hand there exists some equivalent statements with completeness ([1], [5], [7]-[9], [12],...).

Remark 3.2. Condition (ii) implies that each $f \in CT(X,X)$ has the limit shadowing property.

Indeed, let f be an α-contraction with $F_f = \{x^*\}$ and $x_n \in X$, $n \in \mathbb{N}$, such that

$$d(x_{n+1}, f(x_n)) \to 0 \text{ as } n \to \infty.$$

We have

$$\begin{aligned} d(x_n, x^*) &\leq d(x_n, f(x_{n-1})) + d(f(x_{n-1}), x^*) \\ &\leq d(x_n, f(x_{n-1})) + \alpha d(x_{n-1}, x^*) \leq \dots \\ &\leq d(x_n, f(x_{n-1})) + \alpha d(x_{n-1}, f(x_{n-2})) + \dots \\ &+ \alpha^{n-1} d(x_1, f(x_0)) + \alpha^n d(x_0, x^*). \end{aligned}$$

From the Cauchy's lemma we have that

$$d(x_n, x^*) \to 0 \text{ as } n \to \infty.$$

So, $d(x_n, f^n(x_0)) \leq d(x_n, x^*) + d(x^*, f^n(x_0)) \to 0$ as $n \to \infty$.

Remark 3.3. Let X be a nonempty and $f : X \to X$ an operator. We suppose that there exists $A \in P(X)$ such that

$$A \subset f(A).$$

For the fixed point theory for such operators see J. Andres [2] and the references therein.

The above considerations give rise to

Open problem 3.1. Extend the results of this paper to generalized metric spaces.

Open problem 3.2. Extend the results of this paper to generalized contractions.

Open problem 3.3. Extend the results of this paper to the case of multivalued generalized contractions.

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