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# TWO EXISTENCE RESULTS FOR VARIATIONAL INEQUALITIES

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*Dedicated to Professor Ştefan Cobzaş at his* $60^{th}$ *anniversary*

**Abstract**. In this paper we prove the existence of solutions for some variational inequalities, governed by two-variables set-valued mappings, in both stationary and evolution cases.

## 1. Introduction

Operators with two variables, having monotonicity properties with respect to one of the variables and continuity properties with respect to the other one, have been studied since more than 40 years (see [6], [13]). Such kind of operators appear in the theory of nonlinear elliptic operators in divergence form, which are monotone only in the highest order terms, and satisfy a compactness condition for the lower order terms (see [14]).

Existence results for variational inequalities governed by such operators were established in papers like [3], [5].

In this paper we continue some ideas from [5] for a more general class of variational problems, which includes, as a particular case, hemivariational inequalities.

The mathematical theory of hemivariational inequalities was introduced by P.D. Panagiatopoulos (see [11]) and studied by many authors (see for instance [9], [10]).

The main result of this paper is stated in Section 2 (Theorem 5). It gives sufficient conditions for the existence of solutions for a variational inequality governed

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by set-valued mappings of two variables, using the classical Ky Fan intersection theorem. Then, in Section 3, we use this result to study a class of evolution variational inequalities.

The same method that we applied can be used, when $\Phi = 0$, to prove a similar result, where (H1) is replaced by a condition of Karamardian pseudomontonicity (see [5]).

If $A$ is a mapping of one variable, Brézis pseudomonotone, similar results on evolution hemivariational inequalities were established in [7] and [8].

## 2. An existence result for a stationary variational inequality

In what follows, $V$ is a real Banach space, $V^*$ is its dual and $\langle \cdot, \cdot \rangle$ is the usual duality pairing.

Let $C \subset V$ be a nonempty, closed, convex set and let $A : C \times C \to 2^{V^*}$ be a set-valued mapping. Let $\Phi : C \times V \to \mathbb{R} \cup \{+\infty\}$ be a weakly upper semi-continuous function, sublinear in the second variable. We suppose that for each $u \in C$ and for each $v \in T_C(u) + u$ we have $(u, v-u) \in D(\Phi)$, where by $T_C(u)$ we denote the tangent cone of $C$ at $u$.

Consider the following variational problem:

$$(VI) \quad \text{Find } u \in C \text{ such that } \sup_{f \in A(u,u)} \langle v-u, f\rangle + \Phi(u, v-u) \geq 0, \quad \forall\, v \in C.$$

In what follows the set-valued mapping $A$ will have the following properties:

(H1) $\sup_{f \in A(v,v)} \langle u-v, f\rangle + \Phi(v, u-v) \geq 0$ implies that $\sup_{f \in A(u,v)} \langle u-v, f\rangle + \Phi(v, u-v) \geq 0$, for each $u, v \in C$,

(H2) For each $v \in C$, $A(\cdot, v) : C \to 2^{V^*}$ is upper semi-continuous from the line segments of $C$ to $V^*$,

(H3) For each $u \in C$, $A(u, \cdot) : C \to 2^{V^*}$ is weakly upper semi-continuous (from $V$ with the weak topology, to $V^*$ with the norm topology),

(H4) $A(u, v)$ is compact, for each $u, v \in C$.

**Remark 1.** *The hypothesis (H1) is true, for example, when it takes place*

*(H1')* $\sup_{f\in A(u,v)} \langle u-v, f\rangle \geq \sup_{f\in A(v,v)} \langle u-v, f\rangle$, *for each* $u, v \in C$,
*in particular, when* $A$ *is monotone with respect to the first variable, as it was considered in* [3].

**Remark 2.** *Several particular cases of (VI) are:*

*I) Suppose* $V$ *is a reflexive Banach space, densely and compactly embedded into a separable Hilbert space* $H$ *(then* $V \subset H \subset V^*$ *is an evolution triple). Let* $J : H \to \mathbb{R}$ *be a locally Lipschitz function, and by* $J^0(u;v)$ *denote the generalized Clarke derivative of* $J$*, at the point* $u$*, in the direction* $v$*:*

$$J^0(u;v) = \limsup_{w\to u,\ \varepsilon\to 0_+} \frac{J(w+\varepsilon v) - J(w)}{\varepsilon}.$$

*It is well known (see* [4]*) that* $J^0(\cdot;\cdot)$ *is sublinear in the second variable and globally upper semi-continuous. This means we can take* $\Phi = J|_V^0$*.*

*II) Consider the same evolution triple as above. Let* $\Omega$ *be a bounded and open subset of* $\mathbb{R}^N$*, let* $T : H \to L^2(\Omega, \mathbb{R}^k)$ *be a linear and continuous operator, and let* $j : \Omega\times\mathbb{R}^k \to \mathbb{R}$ *be a Caratheodory function, locally Lipschitz with respect to the second variable. Denote by* $j^0(x,y)(h)$ *the partial generalized Clarke derivative,*

$$j^0(x,y)(h) = \limsup_{y'\to y, t\to 0^+} \frac{j(x, y'+th) - j(x,y')}{t}.$$

*Suppose that there exist* $h_1 \in L^2(\Omega)$ *and* $h_2 \in L^\infty(\Omega)$ *such that*

$$\|z\| \leq h_1(x) + h_2(x)\|y\| \text{ , a.e. } x\in\Omega, \text{ for all } y\in\mathbb{R}^k, \ z\in\partial j(x,y), \text{ where}$$

$$\partial j(x,y) = \{z\in\mathbb{R}^k, \langle z, h\rangle \leq j^0(x,y)(h), \ \forall h\in\mathbb{R}^k\}.$$

*It is proved in* [12]*, that the mapping*
$(u,w) \in V\times V \mapsto \int_\Omega j^0(x, Tu(x))(Tw(x))dx$ *is weakly upper semi-continuous.*
*Then we can take* $\Phi(u,w) = \int_\Omega j^0(x, Tu(x))(Tw(x))dx$*.*

*III) An example of a single-valued mapping that satisfies (H1'), (H2)-(H4) is the following (see* [14]*):* $A : H_0^1(\Omega) \to (H_0^1(\Omega))^*$*, defined by*

$$\langle A(u,v), w\rangle = \int_\Omega G(x, v(x), \nabla u(x))\nabla w(x)dx + \int_\Omega g_0(x, v(x), \nabla v(x))w(x)dx,$$

*where* $\Omega$ *is a bounded domain in* $\mathbb{R}^N$ *and the functions* $g_j : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $G = (g_1, \dots, g_N)$ *have the properties:*

*(P1)* $g_j(x, \eta, \xi)$ *is measurable in* $x \in \Omega$ *and continuous in* $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N$*, for* $j = 0, \dots, N$*,*

*(P2)* $|g_j(x, \eta, \xi)| \leq c(k(x) + |\eta| + \|\xi\|)$*, with* $k \in L^2(\Omega)$*, a.e.* $x \in \Omega$*, for every* $\eta \in \mathbb{R}$*,* $\xi \in \mathbb{R}^N$*,*

*(P3)* $\sum_{j=1}^{N} (g_j(x, \eta, \xi) - g_j(x, \eta, \tilde{\xi}))(\xi_j - \tilde{\xi}_j) > 0$*, a.e.* $x \in \Omega$*, for every* $\eta \in \mathbb{R}$*,* $\xi \neq \tilde{\xi} \in \mathbb{R}^N$*.*

We formulate also a coercivity condition:

(H5) There exists $K \subset C$, weakly compact, and $u_0 \in C$ such that

$$\sup_{f \in A(u,u)} \langle u_0 - u, f \rangle + \Phi(u, u_0 - u) < 0,$$

for each $u \in C \setminus K$.

In the study of existence of a solution for the problem $(VI)$, the following lemmas will be needed.

**Lemma 3.** *(marginal function lemma)*[1] *Let* $X$ *and* $Y$ *be two topological spaces,* $G : X \to 2^Y$ *a set-valued mapping and* $g : X \times Y \to \mathbb{R}$*. Denote* $h : X \to \mathbb{R}$*,* $h(x) = \sup_{y \in G(x)} g(x, y)$ *the marginal function. If the following conditions*

*(a)* $g$ *is u.s.c. on* $X \times Y$*,*

*(b)* $G(x_0)$ *is compact for some* $x_0 \in X$*,*

*(c)* $G$ *is u.s.c. at* $x_0$*,*

*are satisfied, then* $h$ *is u.s.c. at* $x_0$*.*

**Lemma 4.** *[KyFan](see* [2]*) Let* $X$ *be a topological vector space,* $H$ *a subset of* $X$ *and* $F : H \to 2^X$ *a set-valued mapping with* $F(x)$ *closed for every* $x \in H$*, such that:*

*(a)* $F(x_0)$ *is compact for some* $x_0 \in H$*,*

*(b) for every* $x_1, x_2, \dots, x_n \in H$*,* $co\{x_1, x_2, \dots, x_n\} \subset \bigcup_{i=1}^{n} F(x_i)$*.*

*Then* $\bigcap_{x \in X} F(x) \neq \emptyset$*.*

**Theorem 5.** *In the hypotheses (H1)-(H5), the problem* $(VI)$ *has at least one solution.*

**Proof.** Following the idea from [5], we divide the proof in several steps:

a) For each $w \in C$ we denote:

$$A_1(w) = \{u \in C \mid \sup_{f \in A(u,u)} \langle w - u, f\rangle + \Phi(u, w - u) \geq 0\}$$

It is obvious that $\bigcap_{w \in C} A_1(w)$ is the set of solutions of the problem $(VI)$.

We verify the conditions of Ky Fan's Lemma for $F(w) = \text{w-cl}A_1(w)$ (the weak closure). Consider $u_0 \in C$, the element from the coercivity condition (H5). This condition implies that $A_1(u_0) \subset K$. But $K$ is weakly compact and so w-cl$A_1(u_0)$ is also weakly compact.

b) Let $w_1, \ldots, w_n \in C$.

We want to prove that $\text{co}\{w_1, \ldots, w_n\} \subset \bigcup_{i=1}^{n} A_1(w_i) \subset \bigcup_{i=1}^{n} \text{w-cl}A_1(w_i)$.

Suppose that this is not true, that is there exist $\lambda_1, \ldots, \lambda_n \geq 0$, with $\sum_{j=1}^{n} \lambda_j = 1$ such that $\bar{w} = \sum_{j=1}^{n} \lambda_j w_j \notin A_1(w_i)$, for every $i = \overline{1,n}$, which implies

$$\langle w_i - \bar{w}, f\rangle + \Phi(\bar{w}, w_i - \bar{w}) < 0,$$

for each $f \in A(\bar{w}, \bar{w})$ and $i = \overline{1,n}$.

For a fixed $f \in A(\bar{w}, \bar{w})$, we have, using the previous inequality and the sublinearity of $\Phi$,

$$\begin{aligned} 0 \leq& \langle \bar{w} - \bar{w}, f\rangle + \Phi(\bar{w}, \bar{w} - \bar{w}) \\ &= \langle \sum_{j=1}^{n} \lambda_j w_j - \sum_{j=1}^{n} \lambda_j \bar{w}, f\rangle + \Phi(\bar{w}, \sum_{j=1}^{n} \lambda_j w_j - \sum_{j=1}^{n} \lambda_j \bar{w}) \\ &\leq \sum_{j=1}^{n} (\lambda_j \langle w_j - \bar{w}, f\rangle + \lambda_j \Phi(\bar{w}, w_j - \bar{w}) < 0, \end{aligned}$$

which is a contradiction. This gives us that $\text{co}\{w_1, \ldots, w_n\} \subset \bigcup_{i=1}^{n} \text{w-cl}A_1(w_i)$. We obtain, by Lemma 4,

$$\bigcap_{w \in C} \text{w-cl}A_1(w) \neq \emptyset. \tag{1}$$

c) Denote, for $w \in C$

$$A_2(w) = \{u \in K \mid \sup_{f \in A(w,u)} \langle w - u, f\rangle + \Phi(u; w - u) \geq 0\}$$

We will prove that

$$\bigcap_{w\in C} A_2(w) \subset \bigcap_{w\in C} A_1(w). \tag{2}$$

Let $u \in C$, $u \in A_2(w)$, for every $w \in C$. Fix $v \in C$ and consider $v_t = tv+(1-t)u \in C$, for each $t \in [0,1]$. From $u \in A_2(v_t)$ we get:

$$\sup_{f\in A(v_t,u)} \langle v_t - u, f\rangle + \Phi(u, v_t - u) \geq 0, \qquad \forall t \in [0,1]$$

and further, using the fact that $\Phi(u,\cdot)$ is positively homogeneous and dividing by $t$

$$\sup_{f\in A(v_t,u)} \langle v - u, f\rangle + \Phi(u, v - u) \geq 0, \qquad \forall t \in (0,1]. \tag{3}$$

Define $G : [0,1] \to 2^{V^*}$, $G(t) = A(v_t,u)$. According to (H2), this is upper semi-continuous and according to (H4), $G(0) = A(u,u)$ is compact. The mapping $(f,v) \in V^* \times V \mapsto \langle v-u, f\rangle$ is continuous for $V$ with the weak topology and $V^*$ with the norm topology. Applying Lemma 3 we have that $h(t) = \sup_{f\in G(t)} \langle v-u, f\rangle$ is upper semi-continuous at 0, that is

$$\limsup_{t\to 0} \sup_{f\in A(v_t,u)} \langle v - u, f\rangle \leq \sup_{f\in A(u,u)} \langle v-u, f\rangle.$$

From this and from (3) it follows that $u$ is a solution for $(VI)$, that is

$$u \in \bigcap_{w\in C} A_1(w).$$

d) At the final step we prove that

$$\bigcap_{w\in C} \text{w-cl}A_1(w) \subset \bigcap_{w\in C} A_2(w). \tag{4}$$

From step (a) we have that $\bigcap_{w\in C} \text{w-cl}A_1(w) \subset K$.

Let $u \in C$, arbitrarily fixed and let $v \in \text{w-cl}A_1(u)$. We will prove that $v \in A_2(u)$. From $v \in \text{w-cl}A_1(u)$, there exists a net $\{v_j\}$ in $A_1(u)$ such that $v_j \rightharpoonup v$. The fact that $v_j \in A_1(u)$ means that

$$\sup_{f\in A(v_j,v_j)} \langle u - v_j, f\rangle + \Phi(v_j, u - v_j) \geq 0, \ \forall\, j \in I,$$

and using hypothesis (H1),

$$\sup_{f\in A(u,v_j)} \langle u - v_j, f\rangle + \Phi(v_j, u - v_j) \geq 0,\ \forall\, j \in I,$$

In order to use Lemma 3 we notice that the mapping $(f, z) \in V^* \times V \mapsto \langle u - z, f\rangle$ is continuous, $A(u, \cdot)$ is weakly upper semi-continuous and $A(u, v)$ is compact. It follows that $h(z) = \sup\limits_{f\in A(u,z)} \langle u - z, f\rangle$ is weakly upper semi-continuous at $v$, that implies

$$\limsup_{v_j \rightharpoonup v} \sup_{f\in A(u,v_j)} \langle u - v_j, f\rangle \leq \sup_{f\in A(u,v)} \langle u - v, f\rangle.$$

On the other side, $\limsup\limits_{v_j \rightharpoonup v} \Phi(v_j, u - v_j) \leq \Phi(v, u - v)$. Further on, we have

$$\begin{aligned} 0 &\leq \limsup_{v_j \rightharpoonup v} \{ \sup_{f\in A(u,v_j)} \langle u - v_j, f\rangle + \Phi(v_j, u - v_j)\} \\ &\leq \limsup_{v_j \rightharpoonup v} \sup_{f\in A(u,v_j)} \langle u - v_j, f\rangle + \limsup_{v_j \rightharpoonup v} \Phi(v_j, u - v_j) \\ &\leq \sup_{f\in A(u,v)} \langle u - v, f\rangle + \Phi(v, u - v), \end{aligned}$$

which means that $v \in A_2(u)$, for every $u \in C$. This proves (4). From (2), (1) and (4) we get $\bigcap\limits_{w\in C} A_1(w) \neq \emptyset$, which concludes the proof.

**Remark 6.** *(see [5]) If in addition to the previous hypotheses, $A(u, u)$ is a convex set, then $u$ is also a solution of the following problem:*

*Find $u \in C$ and $f \in A(u, u)$ such that $\langle v - u, f\rangle + \Phi(u, v - u) \geq 0,\ \forall\, v \in C$.*

**Remark 7.** *From step (d) of the proof, it is clear that hypothesis (H1) can be replaced with the supposition that the "diagonal" mapping $A(\cdot, \cdot)$ is weakly upper semi-continuous (from $V$ with the weak topology to $V^*$ with the norm topology).*

**Remark 8.** *A similar result can be obtained, for variational inequalities, by considering $C \subset V^*$, $A : C \times C \to V$, where $V^*$ is equipped with the weak* topology and $V$ with the norm topology (see [5]).*

## 3. An evolution variational inequality

Consider the following evolution variational inequality:

$$(EVI) \quad u \in C \quad \langle v-u, Lu\rangle + \sup_{f\in A(u,u)} \langle v-u, f\rangle + \Phi(u, v-u) \geq 0, \quad \forall\, v \in C,$$

where $C \subset V$ is nonempty, convex, closed,

$A : V \times V \to 2^{V^*}$ satisfies the hypotheses (H1'), (H2)-(H4) and $L : D(L) \subset V \to V^*$ is a closed densely linear maximal monotone operator.

It is known that, in these conditions, $W = D(L)$, endowed with the graph norm $\|u\|_W = \|u\|_V + \|Lu\|_{V^*}$, is a reflexive Banach space. Denote $\tilde{C} = C \cap D(L)$; it is a convex, closed, nonempty set.

Hypothesis (H5) will be replaced by

(H5') There exists $K \subset \tilde{C}$, weak compact, and $u_0 \in \tilde{C}$ such that

$$\langle u_0 - u, Lu\rangle + \sup_{f\in A(u,u)} \langle u_0 - u, f\rangle + \Phi(u, u_0 - u) < 0,$$

for each $u \in \tilde{C} \setminus K$.

**Theorem 9.** *In the hypotheses (H1'), (H2)-(H4) and (H5'), the problem $(EVI)$ has at least one solution.*

**Proof.** We have that $W$ is densely embedded in $V$. Denoting $i : W \to V$ the natural embedding of $W$ in $V$ and $i^* : V^* \to W^*$ its adjoint, we define the operator $B : \tilde{C} \times \tilde{C} \to 2^{W^*}$ by

$$B(u, v) = \tilde{L}(u) + \tilde{A}(u, v),\ \forall\, u, v \in \tilde{C},$$

where $\tilde{L} : W \to W^*$, $\tilde{L} = i^* \circ L \circ i$, that is

$$\langle v, \tilde{L}(u)\rangle_{W\times W^*} = \langle v, i^*(L(iu))\rangle_{W\times W^*} = \langle iv, L(iu)\rangle_{V\times V^*},\ \forall\, u, v \in W.$$

The same, $\tilde{f} \in \tilde{A}(u, v)$ means that $\tilde{f} = i^* f$, with $f \in A(iu, iv) \subset V^*$, that is $\langle w, \tilde{f}\rangle_{W\times W^*} = \langle w, i^* f\rangle_{W\times W^*} = \langle iw, f\rangle_{V\times V^*}$.

With these notations, problem $(EVI)$ can be written:

$$u \in \tilde{C} = C \cap D(L) \text{ such that } \sup_{g\in B(u,u)} \langle v-u, g\rangle + \Phi_{|W}(u, v-u) \geq 0,\ \forall\, v \in \tilde{C}$$

We will prove that the operator $B$ defined above satisfies the hypotheses (H1')-(H4), in the space $W$ with the weak topology and $W^*$ with the norm topology.

(H1') Let $u, v \in \tilde{C}$, fixed. If $g \in B(u,v)$ then $g = \tilde{L}(u) + f$, with $f \in \tilde{A}(u,v)$ that is $f = i^*h$, $h \in A(iu, iv)$. We have, taking account of the monotonicity of $L$ and of (H1):

$$\sup_{g\in B(u,v)} \langle u-v, g\rangle_{W\times W^*} = \langle iu - iv, L(iu)\rangle_{V\times V^*} + \sup_{h\in A(iu,iv)} \langle iu-iv, h\rangle_{V\times V^*}$$

$$\geq \langle iu - iv, L(iv)\rangle_{V\times V^*} + \sup_{h\in A(iv,iv)} \langle iu-iv, h\rangle_{V\times V^*} = \sup_{g\in B(v,v)} \langle u-v, g\rangle_{W\times W^*}.$$

(H2) For each $v \in W$ fixed, $B(\cdot, v) : \tilde{C} \to 2^{W^*}$ is upper semi-continuous from the line segments in $\tilde{C}$ to $W^*$.

Indeed, let $u_1, u_2 \in \tilde{C}$ arbitrarily fixed, $u_t = tu_1 + (1-t)u_2$, for $t \in [0,1]$ and define $\tilde{G} : [0,1] \to W^*$ by $\tilde{G}(t) = B(u_t, v) = \tilde{L}(u_t) + \tilde{A}(u_t, v)$.

The upper semi-continuity of $\tilde{G}$ at $t = 0$ follows from the upper semi-continuity of $A(\cdot, iv)$ and from the continuity of $\tilde{L}$ from $W$ to $W^*$.

(H3) The weak upper semi-continuity of $B(u,\cdot) : \tilde{C} \to 2^{W^*}$, for $u \in \tilde{C}$ fixed, is a consequence of the fact that $\tilde{L}(u)$ does not depend on $v$ and of the weak upper semi-continuity of $\tilde{A}(u,\cdot)$ at an arbitrary point of $W$.

(H4) We want to prove that, for each $u, v \in \tilde{C}$, $B(u,v)$ is compact. Consider a sequence $\{f_n\}$ in $B(u,v) \subset W^*$, $f_n = \tilde{L}(u) + g_n$, with $g_n = i^*h_n$, $h_n \in A(iu, iv)$. Since $A(iu, iv)$ is compact, there exists a subsequence (denoted in the same way), $h_n \to h \in A(iu, iv)$, in $V^*$. Then $f_n \to \tilde{L}(u) + i^*h \in B(u,v)$.

The fact that $\Phi$ is weakly upper semi-continuous in the topology of $V$ implies directly that it is also upper semi-continuous in the topology of $W$, because $u_j \rightharpoonup u$ in $W$ means $u_j \rightharpoonup u$ in $V$ and $L(u_j) \rightharpoonup L(u)$ in $V^*$.

All the hypotheses (H1'), (H2)-(H4) being satisfied and having also (H5'), we can apply Theorem 5 and conclude the proof.

**Remark 10.** *The following particular case is frequently used: Let $U$ be a real reflexive Banach space, densely and compactly imbedded into a separable Hilbert space $H$, $U \subset H \subset U^*$, i.e. an evolution triple. (For example $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$, where $\Omega$*

*is a bounded domain in* $\mathbb{R}^N$, $H_0^1(\Omega)$ *is the well known Sobolev space and* $H^{-1}(\Omega)$ *is its dual). Let* $V = L^2(0,\tau;U)$ *and* $V^* = L^2(0,\tau;U^*)$ *the dual of* $V$. *In this case, L can be the differentiation operator* $\frac{d^2\cdot}{dt^2}$.

## References

[1] Aubin, J.P., Ekeland, I., *Applied Nonlinear Analysis*, John Wiley and Sons, 1984.

[2] Brézis, H., Nirenberg, L., Stampacchia, G., *A remark on Ky Fan's Minimax Principle*, Bollettino U.M.I. (4)6, (1972), pag. 293-300.

[3] Chen, Y.Q., *On semi-monotone operator theory and applications*, J. Math. Anal. Appl. 231 (1999), 177-192.

[4] Clarke, F.H., *Optimization and Nonsmooth Analysis*, John Wiley & Sons, New-York, (1983).

[5] Kassay, G., Kolumbán, J., *Variational inequalities given by semi-pseudomonotone mappings*, Nonlinear Analysis Forum 5 (2000), pag. 35-50.

[6] Leray, J., Lions, J.L., *Quelques résultats de Višik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder*, Bull. Soc. Math. France 93 (1965), 97-107.

[7] Liu, Z.H., *Existence Results for Evolution Noncoercive Hemivariational Inequalities*, J. Optimization Theory and Applications, vol. 120, no.2, pag. 417-427 (2004).

[8] Liu, Z.H., *A class of evolution hemivariational inequalities*, Nonlinear Analysis 36 (1999), 91-100.

[9] Motreanu, D., Panagiotopoulos, P.D., *Minimax Theorems and Qualitative Properties of the Solutions of Hemivariational Inequalities*, Kluwer Academic Publishers, 1999.

[10] Naniewicz, Z., Panagiotopoulos, P.D., *Mathematical Theory of Hemivariational Inequalities and Applications*, Marcel Dekker, New-York, 1995.

[11] Panagiotopoulos, P.D., *Hemivariational Inequalities. Applications in Mechanics and Engineering*, Springer-Verlag, Berlin, 1993.

[12] Panagiotopoulos, P.D., Fundo, M., Rădulescu, V., *Existence theorems of Hartman-Stampacchia type for hemivariational inequalities and applications*, J. of Global Optimization, 15 (1999), 41-54.

[13] Pascali, D., Sburlan, S., *Nonlinear mappings of monotone type*, Editura Academiei, Bucureşti, Romania and Sijthoff and Noordhoff International Publishers, The Netherlands, 1978.

[14] Showalter, R.E., *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, American Mathematical Society, Mathematical Surveys and Monographs, 49, 1997.

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