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ON CERTAIN PROPERTIES OF THE FRÉCHET DIFFERENTIAL OF HIGHER ORDER

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Dedicated to Professor Ştefan Cobzaş at his 60^{th} *anniversary*

Abstract. In this paper we propose to give detailed proofs for different generalizations of the Leibnitz formula for the calculation of the derivative of the order n, with $n \in \mathbb{N}$, of the functions' product. We will consider the Fréchet derivative of certain composed functions with the help of certain multilinear mappings.

1. **Introduction**

The idea of this paper has its origin the well-known Leibniz's formula concerning the calculation of the derivative of the product of two real functions with real variables.

So, given the number $n \in \mathbb{N}$, the interval $\mathbb{I} \subseteq \mathbb{R}$ and the functions $f, g : \mathbb{I} \to \mathbb{R}$ that have the derivative of the order n, then the product function $fg : \mathbb{I} \to \mathbb{R}$ admits the derivative of the order n as well, and:

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}, \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!\,(n-k)!},$$

for any function $h : \mathbb{I} \to \mathbb{R}$, $h^{(i)} : \mathbb{I} \to \mathbb{R}$ represents the derivative of the order i of the considered mapping.

A first generalization of this formula appears by considering the case of m functions with $m \in \mathbb{N}$, $f_1, \ldots, f_m : \mathbb{I} \to \mathbb{R}$. In this way, if these functions have

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derivatives of the order n, the same fact is true for the product function $f_1 \dots f_m : \mathbb{I} \to \mathbb{R}$ and:

$$(f_1 \dots f_m)^{(n)} = \sum_{\alpha_1+\dots+\alpha_m=n} \frac{n!}{\alpha_1! \dots \alpha_m!} f_1^{(\alpha_1)} \dots f_m^{(\alpha_m)}.$$

We can raise the issue of extending these formulas to the case of using functions defined between linear normed spaces.

Of course in this case it is necessary to find a "substitute" for the notion of product, but it will be necessary to specify the definition used for the extension of the notion of derivative.

To begin with, we have:

Remark 1.1. *For the linear normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ let us denote by $(X,Y)^*$ the set of the linear and continuous mappings $T : X \to Y$. The set $(X,Y)^*$ can be organized as a linear normed space with the usual operations that are the mappings' addition and multiplication with a real number, and the norm that for $T \in (X,Y)^*$ is defined through:*

$$\|T\| = \sup_{h \in X, \|h\|_X = 1} \|T(h)\|_Y .$$

It is easy to show that if $(Y, \|\cdot\|_Y)$ is a Banach space, then the space $\left((X,Y)^*, \|\cdot\|\right)$ is a Banach space as well.

Let us recall the following definition.

Definition 1.2. *Let be given the linear normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, the set $D \subseteq X$, the function $f : D \to Y$ and the point $x \in int(D)$.*
The considered function is differentiable in the point x in the Fréchet meaning that there exists a linear and continuous mapping $T_x \in (X,Y)^*$ and a mapping $R_x : X \to Y$ with:

$$\lim_{h \to \theta_X} \|R_x(h)\|_Y = 0$$

so that for every $h \in X$ the equality:

$$f(x+h) - f(x) = T_x(h) + \|h\|_X R_x(h)$$

is true.

Now we have:

Remark 1.3. *For the linear normed spaces* $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ *and a fixed element* $x \in X$ *let be the set:*

$$\mathcal{D}_x(X,Y) = \{ f / \exists_D \; D \subseteq X, x \in int(D), f : D \to Y, f \text{ differentiable at } x \}.$$

We can easily prove that if $f \in \mathcal{D}_x(X,Y)$ *the mapping* $T_x \in (X,Y)^*$ *exists with a unique determination. We will denote:*

$$f'(x) := T_x$$

and this mapping will be called a Fréchet differential of the mapping f *in the point* x.

Starting from the **definition 1.2** and using the successive differentiation and mathematical induction, we can introduce differentials of an order n, where $n \in \mathbb{N}$.

In order to clarify these questions, for $m \in \mathbb{N}$ we denote by $\left(X^{(m)}, Y\right)^*$ the set of the $m-$linear and continuous mappings which are defined from X^m to Y, where $X^m = \underbrace{X \times \cdots \times X}_{m \text{ times}}$.

We have:

Remark 1.4. *For any* $m \in \mathbb{N}$, *the set* $\left(X^{(m)}, Y\right)^*$ *can also be organized as a linear normed space using the mapping's addition and multiplication with a number. The norm in* $\left(X^{(m)}, Y\right)^*$ *for* $T \in \left(X^{(m)}, Y\right)^*$ *is defined through:*

$$\|T\| = \sup_{h_1,\ldots,h_n \in X, \|h_1\|_X = \cdots = \|h_n\|_X = 1} \|T(h_1, \ldots, h_n)\|_Y,$$

in addition, if $(Y, \|\cdot\|_Y)$ *is a Banach space,* $\left(X^{(m)}, Y\right)^*$ *is a Banach space as well.*

Therefore we have:

Definition 1.5. *In addition to the facts from the* ***definition 1.2*** *let us consider a number* $n \in \mathbb{N}$, $n \geq 2$. *If:*

- **a):** *there exists a neighbourhood* V *of the points* x, *so that for every* $y \in V \cap D$ *it exists the differential of the order* $n-1$ *of the function* f *at the point* y *and* $f^{(n-1)}(y) \in \left(X^{(n-1)}, Y\right)^*$,

b): *the function* $f^{(n-1)} : V \cap D \to \left(X^{(n-1)}, Y\right)^*$ *is also differentiable at the point* x,

then $\left(f^{(n-1)}\right)'(x) \in \left(X^{(n)}, Y\right)^*$, *mapping which we will denote by* $f^{(n)}(x)$ *is called the differential of the order* n *of the function* f *at the point* x.

It is necessary to remind one more case. Let us consider the linear normed spaces:

$$\left(X_1, \|\cdot\|_{X_1}\right), \ \ldots \ , \left(X_m, \|\cdot\|_{X_m}\right), \ \left(Y, \|\cdot\|_Y\right)$$

and a mapping $T : X_1 \times \ \ldots \ \times X_m \to Y$. We can say that this mapping is an $m-$linear and continuous mapping, if this mapping is linear and continuous after every argument.

We denote by $(X_1, \ldots, X_m; Y)^*$ the set of all mappings that verify the aforementioned properties.

For $h = (h_1, \ldots, h_m) \in X_1 \times \ \ldots \ \times X_m$ we can define:

$$\|h\| = \max\left\{\|h_1\|_{X_1}, \ldots, \|h_m\|_{X_m}\right\}$$

and so $\left((X_1, \ldots X_m; Y)^*, \|\cdot\|\right)$ is a linear normed space. In the case if $\left(Y, \|\cdot\|_Y\right)$ is a Banach space, then $\left((X_1, \ldots X_m; Y)^*, \|\cdot\|\right)$ is a Banach space as well.

2. **A generalization of Leibnitz's formula of derivation**

Let us consider the linear normed spaces:

$$\left(X, \|\cdot\|_X\right), \ \left(Y_1, \|\cdot\|_{Y_1}\right), \ \ldots \ , \ \left(Y_m, \|\cdot\|_{Y_m}\right), \ \left(Z, \|\cdot\|_Z\right),$$

the set $D \subseteq X$, the nonlinear mappings $f_i : D \to Y_i; \ i = \overline{1, m}$ and the $m-$ linear mapping $L \in (Y_1, \ldots, Y_m; Z)^*$.

With the help of these elements we build the function:

$$F : D \to Z, \ \ F(x) = L\left(f_1(x), \ldots, f_m(x)\right). \tag{1}$$

Our goal is to conclude, in the hypothesis of the differentiability of the functions $f_i : D \to Y_i; \ i = \overline{1, m}$, on the differentiability of the function (1) establishing connections between the differentials.

To start with, we have the following:

Lemma 2.1. *If the non-linear mappings $f_i : D \to Y_i;\ i = \overline{1,m}$, are differentiable at the point $x \in int\,(D)$, then the function* (1) *is also differentiable at the same point x and for any $h \in X$ we have the relation:*

$$F'(x)\,h = \\ = \sum_{k=1}^{m} L\left(f_1(x), \ldots, f_{k-1}(x), f_k'(x)\,h, f_{k+1}(x), \ldots, f_m(x)\right). \tag{2}$$

Proof. From the differentiability of the functions $f_i : D \to Y_i;\ i = \overline{1,m}$ at the point $x \in int\,(D)$ we deduce the existence, for any $i \in \{1, 2, \ldots, m\}$, of the linear mappings $f_i'(x) \in (X, Y_i)^*$ and of the non-linear mappings $R_x^{(i)} : X \to Y_i$, so that for any $h \in X$ we have:

$$f_i(x+h) = f_i(x) + f_i'(x)\,h + \|h\|_X\, R_x^{(i)}(h)\,,\ \lim_{h\to\theta_X} \left\|R_x^{(i)}(h)\right\|_{Y_i} = 0.$$

So it is clear that:

$$F(x+h) = L\left(f_1(x+h), \ldots, f_m(x+h)\right)$$

is in fact the value of the mapping $L \in (Y_1, \ldots, Y_m; Z)^*$ on the arguments:

$$f_1(x) + f_1'(x)\,h + \|h\|_X\, R_x^{(1)}(h)\,,\ \ldots\ ,\ f_m(x) + f_m'(x)\,h + \|h\|_X\, R_x^{(m)}(h)\,.$$

In this way:

$$F(x+h) = L\left(f_1(x), \ldots, f_m(x)\right) + \\ + \sum_{k=1}^{m} L\left(f_1(x), \ldots, f_{k-1}(x), f_k'(x)\,h, f_{k+1}(x), \ldots, f_m(x)\right) +$$

$$+ \|h\|_X \sum_{k=1}^{m} L\left(f_1(x), \ldots, f_{k-1}(x), R_x^{(k)}(h), f_{k+1}(x), \ldots, f_m(x)\right) + \\ + \sum_{k=2}^{m} \sum_{1\le i_1<\cdots<i_k\le m} E_{i_1,\ldots,i_k}^{(k)}(f; x, h)\,,$$

where $E_{i_1,\ldots,i_k}^{(k)}(f;x,h) \in Z$ represents the value of the mapping $L \in (Y_1,\ldots,Y_m;Z)^*$ on the arguments $f_1(x),\ldots,f_m(x)$, with the exception of the positions $i_1,\ldots,i_k \in \{1,2,\ldots,m\}$ for which we have the arguments:

$$f'_{i_j}(x)h + \|h\|_X R_x^{(i_j)}(h), \quad j=\overline{1,k};\ k=\overline{2,m}.$$

It is clear that if we define $F'(x) \in (X,Z)^*$, through the equality (2), and the mapping $R_x : X \to Z$ through:

$$R_x(h) = \begin{cases} \theta_Z & \text{for} \quad h = \theta_X, \\ P(x,h) + \dfrac{1}{\|h\|_X} Q(x,h) & \text{for} \quad h \neq \theta_X, \end{cases}$$

where we have denoted:

$$P(x,h) = \sum_{k=1}^{m} L\left(f_1(x),\ldots,f_{k-1}(x), R_x^{(k)}(h), f_{k+1}(x),\ldots,f_m(x)\right) \in Z$$

and:

$$Q(x,h) = \sum_{k=2}^{m} \sum_{1\le i_1<\cdots<i_k\le m} E_{i_1,\ldots,i_k}^{(k)}(f;x,h) \in Z,$$

we will have:

$$F(x+h) - F(x) = F'(x)h + \|h\|_X R_x(h). \tag{3}$$

It is clear that:

$$\|P(x,h)\|_Z \le \|L\| \sum_{k=1}^{m} \left(\left\|R_x^{(k)}(h)\right\|_{Y_k} \cdot \prod_{j=\overline{1,m};j\neq k} \|f_j(x)\|_{Y_j} \right)$$

and from $\lim\limits_{h\to\theta_X} \left\|R_x^{(k)}(h)\right\|_{Y_k} = 0$ we deduce:

$$\lim_{h\to\theta_X} \|P(x,h)\|_Z = 0. \tag{4}$$

Concerning the expression of $Q(x,h)$ we deduce:

$$\|Q(x,h)\|_Z \le \sum_{k=2}^{m} \sum_{1\le i_1<\cdots<i_k\le m} \left\|E_{i_1,\ldots,i_k}^{(k)}(f;x,h)\right\|_Z$$

and for any $k \in \{2,3,\ldots,m\}$ and $i_1,\ldots,i_k \in \{1,2,\ldots,m\}$ with $1 \leq i_1 < \cdots < i_k \leq m$ we have:

$$\left\| E^{(k)}_{i_1,\ldots,i_k}(f;x,h) \right\|_Z \leq$$

$$\leq \|L\| \cdot \prod_{j\in\{1,\ldots,m\}\setminus\{i_1,\ldots,i_k\}} \|f_j(x)\|_{Y_j} \times \prod_{j=1}^{k} \left\| f'_{i_j}(x)\,h + \|h\|_X\, R^{(i_j)}_x(h) \right\|_{Y_{i_j}} \leq$$

$$\leq \|L\| \cdot \|h\|^k_X \cdot \mathbf{C}^{(k)}_{i_1,\ldots,i_k}(x,h),$$

where:

$$\mathbf{C}^{(k)}_{i_1,\ldots,i_k}(x,h) = \prod_{j\in\{1,\ldots,m\}\setminus\{i_1,\ldots,i_k\}} \|f_j(x)\|_{Y_j} \times \prod_{j=1}^{k} \left(\left\| f'_{i_j}(x) \right\| + \left\| R^{(i_j)}_x(h) \right\|_{Y_{i_j}} \right)$$

From the differentiability of the functions $f_1,\ldots,f_m$ we deduce clearly that:

$$\lim_{h\to\theta_X} \mathbf{C}^{(k)}_{i_1,\ldots,i_k}(x,h) = \prod_{j\in\{1,\ldots,m\}\setminus\{i_1,\ldots,i_k\}} \|f_j(x)\|_{Y_j} \times \prod_{j=1}^{k} \left\| f'_{i_j}(x) \right\|. \tag{5}$$

We have:

$$\|Q(x,h)\|_Z \leq \|L\| \cdot \|h\|_X \sum_{k=2}^{m} \|h\|^{k-1}_X \sum_{1\leq i_1<\cdots<i_k\leq m} \mathbf{C}^{(k)}_{i_1,\ldots,i_k}(x,h)$$

and from this relation we deduce for any $h \neq \theta_X$ the inequalities:

$$\begin{aligned} &0 \leq \|R_x(h)\|_Z \leq \\ &\leq \|P(x,h)\|_Z + \|L\| \cdot \|h\|_X \sum_{k=2}^{m} \|h\|^{k-1}_X \sum_{1\leq i_1<\cdots<i_k\leq m} \mathbf{C}^{(k)}_{i_1,\ldots,i_k}(x,h) \end{aligned} \tag{6}$$

From the relations $(4)-(6)$ we deduce that:

$$\lim_{h\to\theta_X} \|R_x(h)\|_Z = 0. \tag{7}$$

The relations (3) and (7) indicate that the function (1) has a differential at the point $x \in int(D)$ and its value is given through the formula (2).

The lemma is proved. □

In order to pass to the expression of the differential of an order $n \in \mathbb{N}$ it is necessary to make certain specifications and to adopt certain notations.

To begin with, let be the set:

$$\mathbb{A}_{m,n} = \{\alpha/\,\alpha = (\alpha_1, \ldots, \alpha_m) \in (\mathbb{N} \cup \{0\})^m\,,\ \alpha_1 + \cdots + \alpha_m = n\}\,.$$

In certain cases we can use the notation $|\alpha|$ for $\alpha_1 + \cdots + \alpha_m$.

Considering a finite set $K \subseteq \mathbb{N}$, for a number $p \in \mathbb{N}$ we can consider the set:

$$\mathcal{C}_p\,(K) = \{\,i/\,i = (i_1, \ldots, i_p) \in \mathbb{K}^p,\ i_1 < \cdots < i_p\}\,,$$

evidently $\mathcal{C}_p\,(K)$ represents the set of all subsets with p elements of the set K.

Evidently in the case in which the set K has q elements and $p \leq q$, then the set $\mathcal{C}_p\,(K)$ has $\binom{q}{p} = \frac{q!}{p!(q-p)!}$ elements, and if $p > q$ the set $\mathcal{C}_p\,(K)$ is a void set.

In the special case in which $K = \{1, 2, \ldots, n\}$, we will use the notation $\mathcal{C}_{n,k}$ for $\mathcal{C}_k\,(K)$, with $k \leq n$ and evidently this set has $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ elements.

Let us consider now the finite set $K \subseteq \mathbb{N}$ having n elements and we will build the sets $J_0, J_1, \ldots, J_m \subseteq K$ considering $J_0 = K$. Let us also consider for $m \in \mathbb{N}$ a system $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{A}_{m,n}$.

Starting from these elements let us make the following construction.

To start with, we choose a system $\left(i_1^{(1)}, \ldots, i_{\alpha_1}^{(1)}\right) \in \mathcal{C}_{\alpha_1}\,(J_0)\,.$

Let be now the set $J_1 = J_0 \diagdown \left\{i_1^{(1)}, \ldots, i_{\alpha_1}^{(1)}\right\}$that has $n - \alpha_1$ elements. We choose a new system:

$$\left(i_1^{(2)}, \ldots, i_{\alpha_2}^{(2)}\right) \in \mathcal{C}_{\alpha_2}\,(J_1)\,.$$

So there exist $\binom{n-\alpha_1}{\alpha_2} = \frac{(n-\alpha_1)!}{\alpha_2!(n-\alpha_1-\alpha_2)!}$ possibilities for the choice of this new system.

Further on, for the systems $\left(i_1^{(1)}, \ldots, i_{\alpha_1}^{(1)}\right)$ and $\left(i_1^{(2)}, \ldots, i_{\alpha_2}^{(2)}\right)$ that are chosen above and are fixed we consider the set $J_2 = J_1 \diagdown \left\{i_1^{(2)}, \ldots, i_{\alpha_2}^{(2)}\right\}$with $n - \alpha_1 - \alpha_2$ elements, then we choose a new system:

$$\left(i_1^{(3)}, \ldots, i_{\alpha_3}^{(3)}\right) \in \mathcal{C}_{\alpha_3}\,(J_2)\,,$$

existing $\binom{n-\alpha_1-\alpha_2}{\alpha_3} = \frac{(n-\alpha-\alpha_2)!}{\alpha_3!(n-\alpha_1-\alpha_2-\alpha_3)!}$ possibilities for the choice of this new system.

We continue in this manner using mathematical induction.

Thus for the systems $\left(i_1^{(1)},\ldots,i_{\alpha_1}^{(1)}\right)$, $\ldots$, $\left(i_1^{(k-1)},\ldots,i_{\alpha_{k-1}}^{(k-1)}\right)$ already chosen and fixed, we consider the set:

$$J_{k-1}=J_{k-2}\diagdown\left\{i_1^{(k-1)},\ldots,i_{\alpha_{k-1}}^{(k-1)}\right\}=$$
$$=\{1,2,\ldots,n\}\diagdown\left\{i_1^{(1)},\ldots,i_{\alpha_1}^{(1)},\ldots,i_1^{(k-1)},\ldots,i_{\alpha_{k-1}}^{(k-1)}\right\},$$

that has $n-\alpha_1-\cdots-\alpha_{k-1}$ elements and we choose the new system $\left\{i_1^{(k)},\ldots,i_{\alpha_k}^{(k)}\right\}\in\mathcal{C}_{\alpha_k}(J_{k-1})$ existing $\binom{n-\alpha_1-\cdots-\alpha_{k-1}}{\alpha_k}=\frac{(n-\alpha_1-\cdots-\alpha_{k-1})!}{\alpha_k!(n-\alpha_1-\cdots-\alpha_{k-1}-\alpha_k)!}$ possibilities for the choice of the new system.

At the end of this process we have already chosen and fixed the systems:

$$\left(i_1^{(1)},\ldots,i_{\alpha_1}^{(1)}\right),\ \ldots\ ,\ \left(i_1^{(m-1)},\ldots,i_{\alpha_{m-1}}^{(m-1)}\right)$$

we consider the set:

$$J_{m-1}=J_{m-2}\diagdown\left\{i_1^{(m-1)},\ldots,i_{\alpha_{m-1}}^{(m-1)}\right\}=$$
$$=\{1,2,\ldots,n\}\diagdown\left\{i_1^{(1)},\ldots,i_{\alpha_1}^{(1)},\ldots,i_1^{(m-1)},\ldots,i_{\alpha_{m-1}}^{(m-1)}\right\},$$

and we choose the new system $\left(i_1^{(m)},\ldots,i_{\alpha_m}^{(m)}\right)\in\mathcal{C}_{\alpha_m}(J_{m-1})$ existing

$$\binom{n-\alpha_1-\cdots-\alpha_{m-1}}{\alpha_m}=\frac{(n-\alpha_1-\cdots-\alpha_{m-1})!}{\alpha_m!\,(n-\alpha_1-\cdots-\alpha_{m-1}-\alpha_m)!}$$

possibilities for the choice of the new system.

If we consider:

$$J_m=J_{m-1}\diagdown\left\{i_1^{(m)},\ldots,i_{\alpha_m}^{(m)}\right\}$$

this set has $n-\alpha_1-\cdots-\alpha_{m-1}-\alpha_m=0$ elements, therefore $J_m=\varnothing$ and so the process is finished.

We denote by

$$I=\left(\left(i_1^{(1)},\ldots,i_{\alpha_1}^{(1)}\right),\ \ldots\ ,\ \left(i_1^{(m)},\ldots,i_{\alpha_m}^{(m)}\right)\right)$$

a system composed of systems obtained through the process already presented.

For the numbers $m,n\in\mathbb{N}$ and $\alpha\in\mathbb{A}_{m,n}$ fixed, let us denote through $\mathcal{A}_{m,n}^{[\alpha]}(K)$ the set of all systems built in the manner already indicated.

It is clear that the number of elements of the set $\mathcal{A}_{m,n}^{[\alpha]}(K)$ is $\frac{n!}{\alpha_1!\,\ldots\,\alpha_m!}$.

In the case in which $K = \{1, 2, \ldots, n\}$, we will use the notation $\mathcal{A}_{m,n}^{[\alpha]}$ for $\mathcal{A}_{m,n}^{[\alpha]}(\{1, 2, \ldots, n\})$.

We can now enunciate the following:

Remark 2.2. *With the hypotheses of the* ***lemma 2.1*** *the relation concerning the value of $F'(x)h$ can be written under the form:*

$$F'(x)h = \sum_{\alpha \in \mathbb{A}_{m,1}} \sum_{I \in \mathcal{A}_{m,1}^{[\alpha]}} L\left(f_1^{(\alpha_1)}(x) h_{i_1^{(1)}} \ldots h_{i_{\alpha_1}^{(1)}}, \ \ldots \ , \ f_m^{(\alpha_m)}(x) h_{i_1^{(m)}} \ldots h_{i_{\alpha_m}^{(m)}}\right)$$

with $h_1 = h$.

Indeed, the fact that $\alpha \in \mathbb{A}_{m,1}$ means that $\alpha \in (\alpha_1, \ldots, \alpha_m) \in (\mathbb{N} \cup \{0\})^m$ (therefore $\alpha_i \in \mathbb{N} \cup \{0\}$ for any $i = \overline{1,m}$) with $|\alpha| = \alpha_1 + \cdots + \alpha_m = 1$, so we deduce that there exists a number $k \in \{1, 2, \ldots, m\}$, so that:

$$\alpha_i = \begin{cases} 0 & \text{for} \quad i \neq k, \\ 1 & \text{for} \quad i = k, \end{cases}$$

so the only possibility for the choice of

$$I = \left(\left(i_1^{(1)}, \ldots, i_{\alpha_1}^{(1)}\right), \ \ldots \ , \ \left(i_1^{(m)}, \ldots, i_{\alpha_m}^{(m)}\right)\right) = \left(i_1^{(k)}, \ldots, i_{\alpha_k}^{(k)}\right) = i_1^{(k)} \in \mathcal{A}_{m,1}^{[\alpha]}$$

is $i_1^{(k)})1$ and because $h_1 = h$, it is clear that:

$$L\left(f_1^{(\alpha_1)}(x) h_{i_1^{(1)}} \ldots h_{i_{\alpha_1}^{(1)}}, \ \ldots \ , \ f_m^{(\alpha_m)}(x) h_{i_1^{(m)}} \ldots h_{i_{\alpha_m}^{(m)}}\right) =$$
$$= L\left(f_1(x), \ldots, f_{k-1}(x), f_k'(x)h, f_{k+1}(x), \ldots, f_m(x)\right),$$

which justifies the proposition from this remark.

Taking into account the **remark 2.2** as well, we are now able to establish the theorem concerning the values of the differential of the order n of the non-linear mapping (1).

Thus we have:

Theorem 2.3. *If for $n \in \mathbb{N}$ the non-linear mappings $f_i : D \to Y_i, \ i = \overline{1,m}$ admit a differential of the order n at the point $x \in int(D)$, then the non-linear mapping* (1) *also admits a differential of the order n at the same point x and:*

$$F^{(n)}(x)\,h_1\dots h_n =$$
$$= \sum_{\alpha\in\mathbb{A}_{m,n}} \sum_{I\in\mathcal{A}^{[\alpha]}_{m,n}} L\left(f_1^{(\alpha_1)}(x)\,h_{i_1^{(1)}}\dots h_{i_{\alpha_1}^{(1)}},\ \dots\ ,\ f_m^{(\alpha_m)}(x)\,h_{i_1^{(m)}}\dots h_{i_{\alpha_m}^{(m)}}\right).$$

Proof. We will proceed through mathematical induction after $n \in \mathbb{N}$.

For $n = 1$ the proposition is true on account of the **lemma 2.1** and of the **remark 2.2**.

We suppose therefore that the property in discussion is true for a number $n \in \mathbb{N}$. We will prove that this property is true for n substituted by $n + 1$.

Therefore we consider that the non-linear mappings $f_i : D \to Y_i,\ i = \overline{1,m}$ admit at the point $x \in int\,(D)$ differentials with the order $n + 1$. On the basis of the definition there exists a neighbourhood V of the point x, so that the functions $f_i : D \to Y_i,\ i = \overline{1,m}$ admit differentials of the order n at every point $u \in V \cap D$.

On the basis of the hypothesis of the induction we deduce that the function $F : D \to Z$ defined through (1) also admits a differential of the order n at the point $u \in V \cap D$ and the equality in the conclusion of the theorem takes place with x replaced by u.

Choosing therefore $h_1 \in X$ so that $x + h_1 \in V \cap D$ and arbitrarily $h_2, \dots, h_n, h_{n+1} \in X$ the equality in the conclusion of the theorem will be true for $h_1, \dots, h_n$ replaced by $h_2, \dots, h_{n+1}$ and $\mathcal{A}^{[\alpha]}_{m,n}$ by $\mathcal{A}^{[\alpha]}_{m,n}(\{2, \dots, n+1\})$ and there will be another similar equality but with x replaced by $x + h_1$.

Subtracting these equalities member by member we obtain:

$$\left[F^{(n)}(x+h_1) - F^{(n)}(x)\right] h_2 \dots h_{n+1} =$$
$$= \sum_{\alpha\in\mathbb{A}_{m,n}} \sum_{I\in\mathcal{A}^{[\alpha]}_{m,n}(\{2,\dots,n+1\})} \mathcal{L}^{(I)}_{\alpha}(x; h_1, h_2, \dots, h_{n+1}),$$

where:

$$\mathcal{L}_{\alpha}^{(I)}(x;h_1,h_2,\ldots,h_{n+1}) =$$
$$= L\left(f_1^{(\alpha_1)}(x+h_1)\,h_{i_1^{(1)}}\ldots h_{i_{\alpha_1}^{(1)}},\ \ldots\ ,\ f_m^{(\alpha_m)}(x+h_1)\,h_{i_1^{(m)}}\ldots h_{i_{\alpha_m}^{(m)}}\right) -$$
$$-L\left(f_1^{(\alpha_1)}(x)\,h_{i_1^{(1)}}\ldots h_{i_{\alpha_1}^{(1)}},\ \ldots\ ,\ f_m^{(\alpha_m)}(x)\,h_{i_1^{(m)}}\ldots h_{i_{\alpha_m}^{(m)}}\right),$$

in the last expression $\alpha = (\alpha_1,\ldots,\alpha_m) \in \mathbb{A}_{m,n}$ and:

$$I = \left(\left(i_1^{(1)},\ldots,i_{\alpha_1}^{(1)}\right),\ \ldots\ ,\ \left(i_1^{(m)},\ldots,i_{\alpha_m}^{(m)}\right)\right) \in \mathcal{A}_{m,n}^{[\alpha]}(\{2,\ldots,n+1\}). \tag{8}$$

Let be the number $i \in \{1,2,\ldots,m\}$. From the existence of the Fréchet differential of the order $n+1$ of the function $f_i : D \to Y_i$ at the point $x \in int(D)$ we deduce the existence of these differentials for every $k \leq n+1$.

From this fact we deduce that for any $k \leq n$ and $h_1 \in X$ there exists $R_x^{(k,i)}$: $X \to \left(X^{(k)}, Y_i\right)^*$ with $\lim_{h_1 \to \theta_X} \left\|R_x^{(k,i)}(h_1)\right\| = 0$ so that:

$$f_i^{(k)}(x+h_1) = f_i^{(k)}(x) + f_i^{(k+1)}(x)\,h_1 + \|h_1\|_X\,R_x^{(k,i)}(h_1). \tag{9}$$

From $\alpha \in \mathbb{A}_{m,n}$ we deduce that $\alpha = (\alpha_1,\ldots,\alpha_m) \in (\mathbb{N}\cup\{0\})^m$ and $|\alpha| = \alpha_1+\cdots+\alpha_m = n$, therefore $\alpha_i \in \{0,1,\ldots,n\}$. So the relation (9) is true for $k = \alpha_i$.

Using a similar process with that from the **lemma 2.1** and taking into account the **remark 2.2**, we obtain for $\alpha \in \mathbb{A}_{m,n}$ and $I \in \mathcal{A}_{m,n}^{[\alpha]}$ the equality:

$$\begin{aligned}\mathcal{L}_{\alpha}^{(I)}(x;h_1,h_2,\ldots,h_{n+1}) &= \\ = \sum_{\beta\in\mathbb{A}_{m,1}} \sum_{J\in\mathcal{A}_{m,1}^{[\beta]}} L\left(T_1^{(\alpha,\beta;I,J)},\ \ldots\ ,\ T_m^{(\alpha,\beta;I,J)}\right) &+ \\ + \|h_1\|_X\,\mathbf{R}_{\alpha}^{(I)}(x;h_1,h_2,\ldots,h_{n+1})&,\end{aligned} \tag{10}$$

with $\beta \in \mathbb{A}_{m,1}$ (therefore $\beta = (\beta_1,\ldots,\beta_m) \in (\mathbb{N}\cup\{0\})^m$ and $|\beta| = \beta_1+\cdots+\beta_m = 1$), while:

$$J = \left(\left(j_1^{(1)},\ldots,j_{\beta_1}^{(1)}\right),\ \ldots\ ,\ \left(j_1^{(m)},\ldots,j_{\beta_m}^{(m)}\right)\right) \in \mathcal{A}_{m,1}^{[\beta]} = \mathcal{A}_{m,1}^{[\beta]}(\{1\}), \tag{11}$$

where for $k=\overline{1,m}$ we have denoted:

$$T_k^{(\alpha,\beta;I,J)} = \left(f_k^{(\alpha_k)}\right)^{(\beta_k)}(x)\, h_{j_1^{(k)}} \dots h_{j_{\beta_k}^{(k)}} h_{i_1^{(k)}} \dots h_{i_{\alpha_k}^{(k)}} =$$
$$= f_k^{(\alpha_k+\beta_k)}(x)\, h_{j_1^{(k)}} \dots h_{j_{\beta_k}^{(k)}} h_{i_1^{(k)}} \dots h_{i_{\alpha_k}^{(k)}}.$$

The element $\mathbf{R}_\alpha^{(I)}(x; h_1, h_2, \dots, h_{n+1}) \in Z$ has the value θ_Z in the case in which $h_1 = \theta_X$ and the value that is deductible from (10) for $h_1 \neq \theta_X$.

So:

$$\left[F^{(n)}(x+h_1) - F^{(n)}(x)\right] h_2 \dots h_{n+1} =$$
$$= \mathcal{E}(x; h_1, h_2, \dots, h_n, h_{n+1}) + \|h_1\|_X \mathcal{R}(x; h_1, h_2, \dots, h_n, h_{n+1}), \tag{12}$$

where:

$$\mathcal{E}(x; h_1, h_2, \dots, h_n, h_{n+1}) =$$
$$= \sum_{\alpha\in\mathbb{A}_{m,n}} \sum_{I\in\mathcal{A}_{m,n}^{[\alpha]}(\{2,\dots,n+1\})} \sum_{\beta\in\mathbb{A}_{m,1}} \sum_{J\in\mathcal{A}_{m,1}^{[\beta]}} L\left(T_1^{(\alpha,\beta;I,J)},\ \dots\ ,\ T_m^{(\alpha,\beta;I,J)}\right), \tag{13}$$

while:

$$\mathcal{R}(x; h_1, h_2, \dots, h_n, h_{n+1}) =$$
$$= \begin{cases} \theta_Z, & \text{for } h_1 = \theta_X, \\ \dfrac{\left[F^{(n)}(x+h_1) - F^{(n)}(x)\right] h_2 \dots h_{n+1} - \mathcal{E}(x; h_1, \dots, h_{n+1})}{\|h_1\|_X} & \text{for } h_1 \neq \theta_X. \end{cases}$$

It is clear that for $h_1 \neq \theta_X$ we have:

$$\mathcal{R}(x; h_1, h_2, \dots, h_n, h_{n+1}) =$$
$$= \sum_{\alpha\in\mathbb{A}_{m,n}} \sum_{I\in\mathcal{A}_{m,n}^{[\alpha]}(\{2,\dots,n+1\})} \mathbf{R}_\alpha^{(I)}(x; h_1, h_2, \dots, h_n, h_{n+1}). \tag{14}$$

Now let be:

$$\gamma = (\gamma_1, \dots, \gamma_m) = \alpha + \beta = (\alpha_1+\beta_1, \dots, \alpha_m+\beta_m) \in (\mathbb{N}\cup\{0\})^m.$$

Because $|\alpha| = n$ and $|\beta| = 1$ we deduce that:

$$|\gamma| = \gamma_1 + \dots + \gamma_m = (\alpha_1 + \dots + \alpha_m) + (\beta_1 + \dots + \beta_m) = |\alpha| + |\beta| = n+1,$$

therefore $\gamma \in \mathbb{A}_{m,n+1}$.

For the system I which verifies (8) and the system J which verifies (11), let us introduce:

$$\left(s_1^{(k)}, \ldots, s_{\gamma_k}^{(k)}\right) = \left(j_1^{(k)}, \ldots, j_{\beta_k}^{(k)}, i_1^{(k)}, \ldots, i_{\alpha_k}^{(k)}\right); \quad k = \overline{1, m}$$

and:

$$S = \left(\left(s_1^{(1)}, \ldots, s_{\gamma_1}^{(1)}\right), \ldots, \left(s_1^{(m)}, \ldots, s_{\gamma_m}^{(m)}\right)\right). \tag{15}$$

Because $\beta \in \mathbb{A}_{m,1}$ we deduce that there exists a number $r \in \{1, \ldots, m\}$ so that:

$$\beta_i = \begin{cases} 0 & \text{for} \quad i \neq r, \\ 1 & \text{for} \quad i = r, \end{cases}$$

so the only possibility for the choosing of the index system:

$$J = \left(\left(j_1^{(1)}, \ldots, j_{\beta_1}^{(1)}\right), \ \ldots \ , \ \left(j_1^{(m)}, \ldots, j_{\beta_m}^{(m)}\right)\right) = \left(j_1^{(r)}, \ldots, j_{\beta_r}^{(r)}\right) =$$
$$= \left(j_1^{(r)}\right) \in \mathcal{A}_{m,1}^{[\beta]}\left(\{1\}\right),$$

is $j_1^{(k)} = 1$.

Form here we deduce that the systems from S are identical with a system $I \in \mathcal{A}_{m,n}^{[\alpha]}\left(\{2, \ldots, n+1\}\right)$ (having the form (11)) except the subsystem situated on the position r. To this subsystem we add the element 1 on its first position. This indicates that $S \in \mathcal{A}_{m,n+1}^{[\gamma]}$.

Through the aforementioned process starting with the elements $\alpha \in \mathbb{A}_{m,n}$, $\beta \in \mathbb{A}_{m,1}$, $I \in \mathcal{A}_{m,n}^{[\alpha]}\left(\{2, \ldots, n+1\}\right)$ and $J \in \mathcal{A}_{m,1}^{[\beta]}\left(\{1\}\right)$ we obtain a $\gamma \in \mathbb{A}_{m,n+1}$ together with $S \in \mathcal{A}_{m,n+1}^{[\gamma]}$.

The inverted process starting from $\gamma \in \mathbb{A}_{m,n+1}$ together with $S \in \mathcal{A}_{m,n+1}^{[\gamma]}$ exists with a unique determination, a $\alpha \in \mathbb{A}_{m,n}$ together with $I \in \mathcal{A}_{m,n}^{[\alpha]}\left(\{2, \ldots, n+1\}\right)$ and $J \in \mathcal{A}_{m,1}^{[\beta]}\left(\{1\}\right)$, so that we obtain the systems from S through (15), the systems I and J having the forms (8) and (11) respectively.

So it is clear that for any $k = \overline{1, m}$ we have:

$$T_k^{(\alpha,\beta;I,J)} = f_k^{(\gamma_k)}(x)\, h_{s_1^{(k)}} \ldots h_{s_{\gamma_k}^{(k)}}$$

and from (13) we deduce that:

$$\mathcal{E}(x; h_1, h_2, \ldots, h_n, h_{n+1}) = \sum_{\gamma \in \mathbb{A}_{m,n+1}} \sum_{S \in \mathcal{A}_{m,n+1}^{[\gamma]}} \mathbf{L}_{\gamma}^{(S)}, \tag{16}$$

where:

$$\mathbf{L}_{\gamma}^{(S)} = L\left(f_1^{(\gamma_1)}(x)\, h_{s_1^{(1)}} \ldots h_{s_{\gamma_1}^{(1)}}, \ldots,\ f_m^{(\gamma_m)}(x)\, h_{s_1^{(m)}} \ldots h_{s_{\gamma_m}^{(m)}}\right). \tag{17}$$

Let us denote:

$$\mathcal{H}_{n,X} = \{ h/\, h = (h_1, \ldots, h_n) \in X^n,\ \|h_1\|_X = \cdots = \|h_n\|_X = 1\}$$

and let us now evaluate $\|\mathcal{R}(x; h_1, h_2, \ldots, h_n, h_{n+1})\|_Z$ supposing that $(h_2, \ldots, h_{n+1}) \in \mathcal{H}_{n,X}$ which means that $\|h_2\|_X = \cdots = \|h_{n+1}\|_X = 1$.

First we notice that for any $h_1 \neq \theta_X,\ \alpha \in \mathbb{A}_{m,n}$ and $I \in \mathcal{A}_{m,n}^{[\alpha]}$ we have:

$$\begin{aligned} &\mathbf{R}_{\alpha}^{(I)}(x; h_1, h_2, \ldots, h_{n+1}) = \\ &= \sum_{j=1}^{m} \mathbf{G}_{j,\alpha}^{(I)}(x; h_1, h_2, \ldots, h_{n+1}) + \\ &+ \frac{1}{\|h_1\|_X} \sum_{k=2}^{m} \sum_{1 \le r_1 < \cdots < r_k \le m} \mathbf{E}_{r_1,\ldots,r_k}^{(k,\alpha,I)}(x; h_1, h_2, \ldots, h_{n+1}). \end{aligned} \tag{18}$$

In (18) $\mathbf{G}_{j,\alpha}^{(I)}(x; h_1, h_2, \ldots, h_{n+1}) \in Z$ for $j \in \{1, 2, \ldots, m\}$ represents the value of the mapping $L \in (Y_1, \ldots, Y_m; Z)^*$ with the arguments:

$$f_q^{(\alpha_q)}(x)\, h_{i_1^{(q)}} \ldots h_{i_{\alpha_q}^{(q)}} \in Y_q;\ q = \overline{1, m},$$

except the argument of the rank j, this argument being:

$$R_x^{(\alpha_j, j)}(h_1)\, h_{i_1^{(j)}} \ldots h_{i_{\alpha_j}^{(j)}}.$$

So:

$$\left\| \mathbf{G}_{j,\alpha}^{(I)}(x; h_1, h_2, \ldots, h_{n+1}) \right\|_Z \le \|L\| \cdot \left\| R_x^{(\alpha_j, j)}(h_1) \right\| \prod_{q=\overline{1,m}\ q \neq k} \left\| f_q^{(\alpha_q)}(x) \right\|,$$

here we take into account that $I \in \mathcal{A}_{m,n}^{[\alpha]}(\{2, \ldots, n+1\})$, therefore:

$$\prod_{q=1}^{m} \left(\left\| h_{i_1^{(q)}} \right\|_X \ \ldots\ \left\| h_{i_{\alpha_q}^{(q)}} \right\|_X \right) = \|h_2\|_X\ \ldots\ \|h_{n+1}\|_X = 1.$$

In the same relation (18) for $k \in \{2, \dots, n+1\}$ and $r_1, \dots, r_k \in \mathbb{N}$ with $1 \le r_1 < \cdots < r_k \le m$ the expression $\mathbf{E}^{(k,\alpha,I)}_{r_1,\dots,r_k}(x; h_1, h_2, \dots, h_{n+1})$ is the value of the mapping $L \in (Y_1, \dots, Y_m; Z)^*$ with the arguments $f_q^{(\alpha_q)}(x)\, h_{i_1^{(q)}} \dots h_{i_{\alpha_q}^{(q)}} \in \in Y_q;\ q = \overline{1,m}$ except the arguments situated in the position $r_1, \dots, r_k$ where the arguments:

$$\left[f_p^{(\alpha_p+1)}(x)\, h_1 + \|h_1\|_X\, R_x^{(\alpha_p,p)}(h_1)\right] h_{i_1^{(p)}} \dots h_{i_{\alpha_p}^{(p)}}; \quad p \in \{r_1, \dots, r_k\}$$

appear.

So:

$$\left\|\mathbf{E}^{(k,\alpha,I)}_{r_1,\dots,r_k}(x; h_1, h_2, \dots, h_{n+1})\right\| \le \|h_1\|_X^k \times \|L\| \times$$
$$\times \prod_{q=1}^{k} \left(\left\|f_{r_q}^{(\alpha_{r_q}+1)}(x)\right\| + \left\|R_x^{(\alpha_{r_q},r_q)}(h_1)\right\|\right) \times \prod_{q\in\{1,\dots,m\}\setminus\{r_1,\dots,r_k\}} \left\|f_q^{(\alpha_q)}(x)\right\|.$$

Therefore we can write that:

$$\left\|\mathbf{R}^{(I)}_{\alpha}(x; h_1, h_2, \dots, h_{n+1})\right\|_Z \le \|L\|\, \mathbf{C}^{(I)}_{\alpha}(x, h_1) \tag{19}$$

where:

$$\mathbf{C}^{(I)}_{\alpha}(x, h_1) = \sum_{k=1}^{m} \left(\left\|R_x^{(\alpha_{r_k},r_k)}(h_1)\right\| \times \prod_{q=\overline{1,m}\ q\ne k} \left\|f_q^{(\alpha_q)}(x)\right\|\right) +$$
$$+\sum_{k=2}^{m} \|h_1\|_X^{k-1} \times \sum_{1\le r_1<\cdots<r_k\le m} \mathbf{D}^{(k,\alpha,I)}_{r_1,\dots,r_k}(x, h_1),$$

while for $k \in \{2, \dots, m\}$ and $r_1, \dots, r_k \in \mathbb{N}$ with $1 \le r_1 < \cdots < r_k \le m$ we have:

$$\mathbf{D}^{(k,\alpha,I)}_{r_1,\dots,r_k}(x, h_1) =$$
$$= \prod_{q=1}^{k} \left(\left\|f_{r_q}^{(\alpha_{r_q}+1)}(x)\right\| + \left\|R_x^{(\alpha_{r_q},r_q)}(h_1)\right\|\right) \times \prod_{q\in\{1,\dots,m\}\setminus\{r_1,\dots,r_k\}} \left\|f_q^{(\alpha_q)}(x)\right\|.$$

Thus, it is clear from the hypotheses on account of which for the specified values of k and of $r_1, \dots, r_k$ we have:

$$\lim_{h_1\to\theta_X} \mathbf{D}^{(k,\alpha,I)}_{r_1,\dots,r_k}(x, h_1) = \prod_{q=1}^{k} \left\|f_{r_q}^{(\alpha_{r_q}+1)}(x)\right\| \cdot \prod_{q\in\{1,\dots,m\}\setminus\{r_1,\dots,r_k\}} \left\|f_q^{(\alpha_q)}(x)\right\|,$$

that for any $\alpha \in \mathbb{A}_{m,n}$ and $I \in \mathcal{A}_{m,n}^{[\alpha]}$ we have:

$$\lim_{h \to \theta_X} \mathbf{C}_\alpha^{(I)}(x, h_1) = 0$$

and so, in the same situation as in (19) we deduce that:

$$\lim_{h_1 \to \theta_X} \sup_{(h_2,\dots,h_{n+1}) \in \mathcal{H}_{n,X}} \|\mathcal{R}(x; h_1, h_2, \dots, h_{n+1})\|_Z = 0. \tag{20}$$

We define the mapping:

$$\begin{gathered} F^{(n+1)}(x) \in \left(X^{(n+1)}, Z\right)^*, \\ F^{(n+1)}(x)\, h_1 h_2 \dots h_{n+1} = \mathcal{E}(x; h_1, h_2, \dots, h_{n+1}), \end{gathered} \tag{21}$$

and it is clear that if we define $R_x(h_1) \in \left(X^{(n)}, Z\right)^*$ through:

$$R_x(h_1) = \begin{cases} \Theta_n & ;\ h_1 = \theta_X \\ \dfrac{F^{(n)}(x+h_1) - F^{(n)}(x) - F^{(n+1)}(x)\, h_1}{\|h_1\|_X} & ;\ h_1 \neq \theta_X \end{cases}$$

we have in $\left(X^{(n)}, Z\right)^*$ the equality:

$$F^{(n)}(x+h_1) - F^{(n)}(x) = F^{(n+1)}(x)\, h_1 + \|h_1\|_X R_x(h_1). \tag{22}$$

In the same time for $h_1 \neq \theta_X$ we have:

$$\begin{gathered} \|R_x(h_1)\, h_2 \dots h_{n+1}\|_Z \leq \\ \leq \frac{\left\|\left[F^{(n)}(x+h_1) - F^{(n)}(x)\right] h_2 \dots h_{n+1} - F^{(n+1)}(x)\, h_1 h_2 \dots h_{n+1}\right\|_Z}{\|h_1\|_X} = \\ = \frac{\left\|\left[F^{(n)}(x+h_1) - F^{(n)}(x)\right] h_2 \dots h_{n+1} - \mathcal{E}(x; h_1, h_2, \dots, h_{n+1})\right\|_Z}{\|h_1\|_X} = \\ = \|\mathcal{R}(x; h_1, h_2, \dots, h_{n+1})\|_Z, \end{gathered}$$

therefore:

$$\begin{gathered} 0 \leq \|R_x(h_1)\| = \sup_{(h_2,\dots,h_{n+1}) \in \mathcal{H}_{n,X}} \|R_x(h_1)\, h_2 \dots h_{n+1}\|_Z \leq \\ \leq \sup_{(h_2,\dots,h_{n+1}) \in \mathcal{H}_{n,X}} \|\mathcal{R}(x; h_1, h_2, \dots, h_{n+1})\|_Z. \end{gathered}$$

From here, also using the relation (20), we deduce that:

$$\lim_{h\to\theta_X} \|R_x(h_1)\| = 0. \tag{23}$$

From the relations (22) and (23) we deduce that the mapping $F : D \to Z$ has a Fréchet differential of the order $n+1$ at the point $x \in int(D)$, the expression of the mapping $F^{(n+1)}(x) \in \left(X^{(n+1)}, Z\right)^*$ being specified through (21), therefore on account of the obtained expression (16) for $\mathcal{E}(x; h_1, h_2, \ldots, h_{n+1})$ we have:

$$F^{(n+1)}(x)\, h_1 \ldots h_{n+1} =$$

$$= \sum_{\gamma\in\mathbb{A}_{m,n+1}} \sum_{S\in\mathcal{A}^{[\gamma]}_{m,n+1}} L\left(f_1^{(\gamma_1)}(x)\, h_{s_1^{(1)}} \ldots h_{s_{\gamma_1}^{(1)}}, \ldots,\ f_m^{(\gamma_m)}(x)\, h_{s_1^{(m)}} \ldots h_{s_{\gamma_m}^{(m)}}\right).$$

The aforementioned assertion together with its corresponding equality indicates that the property expressed through this theorem is true for any $n \in \mathbb{N}$ replaced by $n+1$.

On account of the principle of mathematical induction this property is true for any $n \in \mathbb{N}$.

The theorem is proved. □

Remark 2.4. *In the case of* $m = 2$, *case in which* $L \in (L_1, L_2; Z)^*$, $f : D \to Y_1$, $g : D \to Y_2$ *where* $D \subseteq X$ *and* $x \in int(D)$, *in the hypothesis of the existence of the differentials with the order* n *of the considered functions at the point* x, *it results the existence of the differential with the order* n *of the function* $F : D \to Z$, $F(x) = L(f(x), g(x))$ *together with the equality:*

$$F^{(n)}(x)\, h_1 \ldots h_n = \sum_{k=0}^{n} \sum_{i\in\mathcal{C}_{m,k}} L\left(f^{(k)}(x)\, h_{i_1} \ldots h_{i_k}, g^{(n-k)}(x)\, h_{j_1} \ldots h_{j_{n-k}}\right) \tag{24}$$

where we have denoted $i = (i_1, \ldots, i_k) \in \mathcal{C}_{m,k}$ *and:*

$$\{j_1, \ldots, j_{n-k}\} \in \{1, 2, \ldots, n\} \diagdown \{i_1, \ldots, i_k\}$$

with $j_1 < \cdots < j_{n-k}$.

Indeed, in this case:

$$\mathbb{A}_{2,1} = \left\{ \alpha / \alpha = (\alpha_1, \alpha_2) \in (\mathbb{N} \cup \{0\})^2 ,\ \alpha_1 + \alpha_2 = n \right\} =$$

$$= \left\{ (k, n-k) / k = \overline{0, n} \right\}$$

and the set $\mathcal{A}_{2,n}^{[\alpha]} = \mathcal{A}_{2,n}^{(k,n-k)}$ is made of a pair of disjunct systems, the first system has k elements and the second $n-k$ elements. If we put together the elements from these systems we obtain the set $\{1, 2, \ldots, n\}$.

If we denote this pair of systems from $\mathcal{A}_{2,n}^{(k,n-k)}$ by:

$$(i, j) = ((i_1, \ldots, i_k), (j_1, \ldots, j_{n-k}))$$

because $1 \leq i_1 < \cdots < i_k \leq n$ and the system $(j_1, \ldots, j_{n-k})$ is obtained in the aforementioned manner, then $i \in \mathcal{C}_{n,k}$ and we also obtain the equality (24).

Remark 2.5. *In the case when $h_1 = \cdots = h_n = h \in X$ we have for the equality from the conclusion of the **theorem 2.3** the form:*

$$F^{(n)}(x) h^n = \sum_{\alpha \in \mathbb{A}_{m,n}} \frac{n!}{\alpha_1! \ldots \alpha_m!} L\left(f_1^{(\alpha_1)}(x) h^{\alpha_1}, \ldots, f_m^{(\alpha_m)}(x) h^{\alpha_m} \right) \tag{25}$$

and for the equality (24) *we have the form:*

$$F^{(n)}(x) h^n = \sum_{k=0}^{n} \binom{n}{k} L\left(f^{(k)}(x) h^k, g^{(n-k)}(x) h^{n-k} \right). \tag{26}$$

These relations are evident because the number of elements of the set $\mathcal{A}_{m,n}^{[\alpha]}$ is $\frac{n!}{\alpha_1! \ldots \alpha_m!}$, while the number of elements of the set $\mathcal{C}_{n,k}$ is $\frac{n!}{k!(n-k)!} = \binom{n}{k}$.

In the aforementioned writings it is clear that $f^{(k)}(x) h^k$ means:

$$f^{(k)}(x) (\underbrace{h, \ldots, h}_{k \text{ times}}).$$

3. **An application to the differential of certain composed functions**

Let us consider the number $m \in \mathbb{N}$, the linear normed spaces:

$$(X, \|\cdot\|_X),\ (Y_1, \|\cdot\|_{Y_1}),\ \ldots\ ,\ (Y_m, \|\cdot\|_{Y_m}),\ (Z, \|\cdot\|_Z),$$

the set $D \subseteq X$ and the mappings:

$$U_i : D \to Y_i,\ i = \overline{1,m}; \qquad W : D \to (Y_1, \dots, Y_m; Z)^* .$$

Using the aforemationed mappings we consider the composed mapping:

$$G : D \to Z,\ \ G(x) = [W(x)](U_1(x), \dots, U_m(x)) . \tag{27}$$

Concerning the mapping (27), we have the following proposition:

Proposition 3.1. *If for an $n \in \mathbb{N}$ the mappings $W : D \to (Y_1, \dots, Y_m; Z)^*$ and $U_i : D \to Y_i;\ i = \overline{1,m}$ admit Fréchet differentials with the order n at the point $x \in int(D)$, then the mapping $G : D \to Z$ defined through (27) also admits a Fréchet differential of the order n at the same point x, and for any $h_1, \dots, h_n \in X$ we have the equality:*

$$\begin{gathered} G^{(n)}(x)\, h_1 \dots h_n = \\ = \sum_{k=0}^{n} \sum_{\alpha \in \mathbb{A}_{m,n-k}} \sum_{S \in \mathcal{C}_{n,k}} \sum_{I \in \mathcal{A}^{[\alpha]}_{m,n-k}(M_{n,k}(S))} E^{(S,I)}_{k,\alpha}(W, U; x; h_1, \dots, h_n) \end{gathered} \tag{28}$$

where $U = (U_1, \dots, U_m)$ and $E^{(S,I)}_{k,\alpha}(W, U; x; h_1, \dots, h_n)$ is

$$\left[W^{(k)}(x)\, h_{s_1} \dots h_{s_k}\right] \left(U_1^{(\alpha_1)}(x)\, h_{i_1^{(1)}} \dots h_{i_{\alpha_1}^{(1)}}, \dots, U_m^{(\alpha_m)}(x)\, h_{i_1^{(m)}} \dots h_{i_{\alpha_m}^{(m)}}\right) \tag{29}$$

where for $S = (s_1, \dots, s_k) \in \mathcal{C}_{n,k}$ we have denoted:

$$M_{n,k}(S) = \{1, \dots, n\} \setminus \{s_1, \dots, s_k\} .$$

Proof. We will consider the mapping:

$$L : (Y_1, \dots, Y_m; Z)^* \times Y_1 \times \dots \times Y_m \to Z,\ L(T; y_1, \dots, y_m) = T(y_1, \dots, y_m)$$

where $y_i \in Y_i$ with $i = \overline{1,m}$ while $T \in (Y_1, \dots, Y_m; Z)^*$.

From the definition of the operations in the set of mappings we deduce the linearity of the mapping L after the first argument, while from the linearity of the mapping $T : Y_1 \times \dots \times Y_m \to Z$ we deduce the linearity of the mapping L after the last m arguments.

It is also clear that:

$$\|L(T; y_1, \ldots, y_m)\|_Z = \|T(y_1, \ldots, y_m)\|_Z \leq \|T\| \cdot \|y_1\|_{Y_1} \cdot \cdots \cdot \|y_m\|_{Y_m},$$

therefore:

$$L \in \left((Y_1, \ldots, Y_m; Z)^*, Y_1, \ldots, Y_m; Z\right)^*$$

and:

$$G(x) = L(W(x), U_1(x), \ldots, U_m(x))$$

as well.

In this way for the existence and the calculation of the differential with the order n of the non-linear mapping defined by (27) it is possible to use the theorem 2.3, therefore as the mappings $U_i : D \to Y_i;\ i = \overline{1, m}$ and $W : D \to (Y_1, \ldots, Y_m; Z)^*$ have the Fréchet differentials with the order n, at the point $x \in int(D)$, the same fact can be said about the mapping $G : X \to Z$, and for any $h_1, \ldots, h_n \in X$ we have:

$$G^{(n)}(x) h_1 \ldots h_n = \sum_{\gamma \in \mathbb{A}_{m+1,n}} \sum_{J \in \mathcal{A}_{m+1,n}^{[\gamma]}} \mathcal{G}_{\gamma,J}(x; h_1, \ldots, h_n),$$

where $\mathcal{G}_{\gamma,J}(x; h_1, \ldots, h_n)$ has the value:

$$L\left(W^{(\gamma_1)}(x) h_{j_1^{(1)}} \ldots h_{j_{\gamma_1}^{(1)}}, U_1^{(\gamma_2)}(x) h_{j_1^{(2)}} \ldots h_{j_{\gamma_2}^{(2)}}, \ldots, U_m^{(\gamma_{m+1})}(x) h_{j_1^{(m+1)}} \ldots h_{j_{\gamma_{m+1}}^{(m+1)}}\right)$$

for $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{\bar{m}+1}) \in \mathbb{A}_{m+1,n}$ and

$$J = \left(\left(j_1^{(1)}, \ldots, j_{\gamma_1}^{(1)}\right), \left(j_1^{(2)}, \ldots, j_{\gamma_2}^{(2)}\right), \ldots, \left(j_1^{(m+1)}, \ldots, j_{\gamma_{m+1}}^{(m+1)}\right)\right) \in \mathcal{A}_{m+1,n}^{[\gamma]}.$$

The fact that $\gamma \in \mathbb{A}_{m+1,n}$ means that $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{m+1}) \in (\mathbb{N} \cup \{0\})^{m+1}$ with $|\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_{m+1} = n$.

We place:

$$k = \gamma_1,\ \alpha_1 = \gamma_2,\ \ldots,\ \alpha_m = \gamma_{m+1}$$

and we deduce that in fact $k \in \{0, 1, \ldots, n\}$ and $\alpha = (\alpha_1, \ldots, \alpha_m) \in (\mathbb{N} \cup \{0\})^m$ with $|\alpha| = \alpha_1 + \cdots + \alpha_m = n - \gamma_1 = n - k$, therefore $\alpha \in \mathbb{A}_{m,n-k}$.

We then place:

$$S = (s_1, \ldots, s_k) = \left(j_1^{(1)}, \ldots, j_k^{(1)}\right) = \left(j_1^{(1)}, \ldots, j_{\gamma_1}^{(1)}\right)$$

and:

$$I=\left(\left(j_1^{(2)},\ldots,j_{\gamma_2}^{(2)}\right),\left(j_1^{(3)},\ldots,j_{\gamma_3}^{(3)}\right),\ldots,\left(j_1^{(m+1)},\ldots,j_{\gamma_{m+1}}^{(m+1)}\right)\right).$$

Thus it is evident that $J=(S,I)\in\mathcal{A}_{m+1,n}^{[k]}$ if and only if $S\in\mathcal{C}_{n,k}$ and:

$$I\in\mathcal{A}_{m,n-k}^{[\alpha]}\left(\{1,2,\ldots,n\}\setminus\{s_1,\ldots,s_k\}\right)=\mathcal{A}_{m,n-k}^{[\alpha]}\left(M_{n,k}\left(S\right)\right),$$

this fact results from the manner in which we have obtained the systems $J\in\mathcal{A}_{m+1,n}^{[\gamma]}$.

Thus the relations (28) and (29) are clear.

The proposition is proved. □

We have the following:

Remark 3.2. *In the case where $h_1=\cdots=h_n=h\in X$ in the hypotheses of the proposition 3.1 we have the equality:*

$$G^{(n)}\left(x\right)h^n= \\ =\sum_{k=0}^{n}\frac{n!}{k!}\sum_{\alpha\in\mathbb{A}_{m,n-k}}\frac{\left[W^{(k)}\left(x\right)h^k\right]\left(U_1^{(\alpha_1)}\left(x\right)h^{\alpha_1},\ldots,U_m^{(\alpha_m)}\left(x\right)h_m\right)}{\alpha_1!\ldots\alpha_m!}. \tag{30}$$

For $n=1$ we have:

Remark 3.3. *If the mappings $W:D\to(Y_1,\ldots,Y_m;Z)^*$ and $U_i:D\to Y_i;\ i=\overline{1,m}$ are Fréchet differentiable at the point $x\in int\left(D\right)$, then the mapping $G:D\to Z$ defined through* (27) *is also differentiable at the same point x, and for any $h_1,\ldots,h_n\in X$ we have the equality:*

$$G'\left(x\right)h=\left[W'\left(x\right)h\right]\left(U_1\left(x\right),\ldots,U_m\left(x\right)\right)+ \\ +\sum_{j=1}^{m}\left[W\left(x\right)\right]\left(U_1\left(x\right),\ldots,U_{j-1}\left(x\right),U_j'\left(x\right)h,U_{j+1}\left(x\right),\ldots,U_m\left(x\right)\right). \tag{31}$$

For $n\in\mathbb{N}$ arbitrary and $m=1$ we have:

Remark 3.4. *If the linear normed spaces $\left(X,\left\|\cdot\right\|_X\right),\ \left(Y,\left\|\cdot\right\|_Y\right),\ \left(Z,\left\|\cdot\right\|_Z\right)$ and the functions $f:D\to(Y,Z)^*,\ g:D\to Y$, that admit Fréchet differentials with the order n at a point $x\in int\left(D\right)$ are given, then the function:*

$$G:D\to Z;\ G\left(x\right)=\left[f\left(x\right)\right]g\left(x\right)$$

also admits a Fréchet differential with the order n *at the same point* x, *and for any* $h_1, h_2, \ldots, h_n \in X$ *we have:*

$$\begin{aligned}&([f(x)]\, g(x))^{(n)}\, h_1 \ldots h_n = \\ &= \sum_{k=0}^{n} \sum_{i \in \mathcal{C}_{n,k}} \left[f^{(k)}(x)\, h_{i_1} \ldots h_{i_k} \right] g^{(n-k)}(x)\, h_{j_1} \ldots h_{j_{n-k}}\end{aligned} \tag{32}$$

where $i = (i_1, \ldots, i_k) \in \mathcal{C}_{n,k}$ *and* $\{j_1, \ldots, j_{n-k}\} = \{1, 2, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$ *with* $j_1 < \cdots < j_{n-k}$.

The **remarks 3.2-3.4** are evident.

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