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HOMOMORPHS WITH RESPECT TO WHICH ANY HALL π-SUBGROUP OF A FINITE π-SOLVABLE GROUP IS A PROJECTOR

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Dedicated to Professor Ştefan Cobzaş at his 60^{th} *anniversary*

Abstract. Let π be a set of primes. The paper studies some special homomorphs of finite π-solvable groups, proving that some of them are Schunck classes. These homomorphs are used to give conditions on an arbitrary homomorph $\underline{X}$, such that any Hall π-subgroup of a finite π-solvable group G to be an $\underline{X}$-projector of G. Particularly, for π the set of all primes, one obtain the converse of a result given by W. Gaschütz in [8].

1. **Preliminaries**

In [4] we gave conditions with respect to which any $\underline{X}$-projector H of a finite π-solvable group G in a Hall π-subgroup of G, where $\underline{X}$ is a π-closed Schunck class with the P property. It is the aim of this paper to solve the converse problem: to give conditions on an arbitrary homomorph $\underline{X}$, such that any Hall π-subgroup H of a finite π-solvable group G to be an $\underline{X}$-projector of G. This problem leads us to the study of some special homomorphs, some of them being Schunck classes.

All groups considered in this paper are finite. Let π be a set of primes, π' the complement to π in the set of all primes and $O_{\pi'}(G)$ the largest normal π'-subgroup of a group G.

We first remind some useful definitions and theorems.

Definition 1.1. ([8], [11]) a) A class $\underline{X}$ of groups is a *homomorph* if $\underline{X}$ is epimorphically closed, i.e. if $G \in \underline{X}$ and N is a normal subgroup of G, then $G/N \in \underline{X}$.

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b) A group G is *primitive* if G has a *stabilizer*, i.e. a maximal subgroup H of G with $core_G H = \{1\}$, where $core_G H = \cap\{H^g / g \in G\}$.

c) A homomorph $\underline{X}$ is a *Schunck class* if $\underline{X}$ is *primitively closed*, i.e. if any group G, all of whose primitive factor groups are in $\underline{X}$, is itself in $\underline{X}$.

Definition 1.2. a) A positive integer n is said to be a π*-number* if for any prime divisor p of n we have $p \in \pi$.

b) A finite group G is a *p-group* if $|G|$ is a π-number.

Definition 1.3. ([6]) A group G is π*-solvable* if every chief factor of G is either a solvable π-group or a π'-group. For π the set of all primes, we obtain the notion of *solvable group*.

Definition 1.4. A class $\underline{X}$ of groups is said to be π*-closed* if

$$G/O_{\pi'}(G) \in \underline{X} \ \Rightarrow \ G \in \underline{X}.$$

A π-closed homomorph, respectively a π-closed Schunck class is called π*-homomorph*, respectively π*-Schunck class*.

Definition 1.5. ([7], [8]) Let $\underline{X}$ be a class of groups, G a group and H a subgroup of G.

a) H is an $\underline{X}$*-maximal subgroup* of G if:

i) $H \in \underline{X}$;

ii) $H \leq H^* \leq G$, $H^* \in \underline{X}$ imply $H = H^*$.

b) H is an $\underline{X}$*-projector* of G if, for any normal subgroup N of G, HN/N is $\underline{X}$-maximal in G/N.

c) H is an $\underline{X}$*-covering subgroup* of G if:

i) $H \in \underline{X}$;

ii) $H \leq K \leq G$, $K_0 \lhd K$, $K/K_0 \in \underline{X}$ imply $K = HK_0$.

Definition 1.6. ([3]) Let $\underline{X}$ be a class of groups. We say that $\underline{X}$ has the P *property* if, for any π-solvable group G and for any minimal normal subgroup M of G such that M is a π'-group, we have $G/M \in \underline{X}$.

Theorem 1.7. ([1]) *A solvable minimal normal subgroup of a finite group is abelian.*

Theorem 1.8. ([8]) *Let $\underline{X}$ be a class of groups, G a group and H a subgroup of G. H is an $\underline{X}$-projector of G if and only if:*

a) H is an $\underline{X}$-maximal subgroup of G;

b) HM/M is an $\underline{X}$-projector of G/M, for all minimal normal subgroups M of G.

Theorem 1.9. ([2]) *a) Let $\underline{X}$ be a class of groups, G a group and H a subgroup of G. If H is an $\underline{X}$-covering subgroup of G or H is an $\underline{X}$-projector of G, then H is $\underline{X}$-maximal in G.*

b) If $\underline{X}$ is a homomorph and G is a group, then a subgroup H of G is an $\underline{X}$-covering subgroup of G if and only if H is an $\underline{X}$-projector in any subgroup K of G with $H \subseteq K$.

Theorem 1.10. *Let $\underline{X}$ be a homomorph.*

a) ([7]) If H is an $\underline{X}$-covering subgroup of a group G and N is a normal subgroup of G, then HN/N is an $\underline{X}$-covering subgroup of G/N.

b) ([8]) If H is an $\underline{X}$-projector of a group G and N is a normal subgroup of G, then HN/N is an $\underline{X}$-projector of G/N.

c) ([7]) If H is an $\underline{X}$-covering subgroup of G and $H \leq K \leq G$, then H is an $\underline{X}$-covering subgroup of K.

Theorem 1.11. ([5]) *Let $\underline{X}$ be a π-homomorph. The following conditions are equivalent:*

(1) $\underline{X}$ is a Schunck class;

(2) any π-solvable group has $\underline{X}$-covering subgroups;

(3) any $\underline{X}$-solvable group has $\underline{X}$-projectors.

2. Some properties of the Hall π-subgroups in finite π-solvable groups

The Hall subgroups were introduced in [9], where Ph. Hall studied them in finite solvable groups. In [6], S. A. Čunihin extended this study to finite π-solvable groups.

Definition 2.1. Let G be a group and H a subgroup of G.

a) H is a *π-subgroup* of G if H is a π-group.

b) H is a *Hall π-subgroup* of G if:

i) H is a π-subgroup of G;

ii) $(|H|, |G : H|) = 1$.

We shall use some properties of the Hall π-subgroups, which we give below.

Theorem 2.2. ([10]) *Let G be a group and G a Hall π-subgroup of G.*

a) If $H \leq K \leq G$, then H is a Hall π-subgroup of K.

b) If N is a normal subgroup of G, then HN/N is a Hall π-subgroup of G/N.

Theorem 2.3. (Ph. Hall, S. A. Čunihin) ([10]) *If G is a π-solvable group, then:*

a) G has Hall π-subgroup and G has Hall π'-subgroups;

b) any two Hall π-subgroups of G are conjugate in G; any two Hall π'-subgroups of G are conjugate in G too.

We now prove a consequence of theorems 2.2 and 2.3.

Theorem 2.4. *Let G be a π-solvable group. If H is a Hall π-subgroup of G and H^* is a π-subgroup of G such that $H \subseteq H^*$, then $H = H^*$.*

Proof. By 2.2.a), H is a Hall π-subgroup of H^*. But H^* being a π-group and $|H^*|$ and $|H^* : H^*| = 1$ being coprime, it follows that H^* is a Hall π-subgroup of H^*. Applying now 2.3.b), we obtain that H and H^* are conjugate in H^*, i.e. there is an element $x \in H^*$ such that $H = (H^*)^x = H^*$. □

Finally we give a result proved in [4]:

Theorem 2.5. ([4]) *Let G be a π-solvable group, H a subgroup of G and N a normal subgroup of G. If HN/N is a Hall π-subgroup of G/N and H is a Hall π-subgroup of HN, then H is a Hall π-subgroup of G.*

3. **Some useful homomorphs**

Let π be an arbitrary set of primes. Of special interest for our considerations will be the following classes of finite π-solvable groups:

Notations 3.1.

$\underline{W}_\pi = \{G / G \text{ finite } \pi - \text{solvable group}\}$;

$\underline{G}_\pi = \{G \in \underline{W}_\pi / G\pi - \text{group}\}$;

$\underline{G}_{\pi'} = \{G / G\pi' - \text{group}\}$;

$\underline{K}_\pi = \{G \in \underline{W}_\pi / O_{\pi'}(G) \neq 1\}$;

$\underline{M}_\pi = \underline{W}_\pi \setminus \underline{K}_\pi = \{G \in \underline{W}_\pi / O_{\pi'}(G) = 1\}$.

Remark 3.2. $\underline{G}_\pi \subseteq \underline{M}_\pi \subseteq \underline{W}_\pi$.

We now give some properties of the above classes.

Theorem 3.3. *$\underline{W}_\pi$ is a π-Schunck class.*

Proof. $\underline{W}_\pi$ is a homomorph. Indeed, if G is a π-solvable group and N is a normal subgroup of G, then G/N is a π-solvable group.

$\underline{W}_\pi$ is π-closed, since if $G/O_{\pi'}$ is a π-solvable group, then, observing that $O_{\pi'}(G)$ is π-solvable, we deduce that G is π-solvable.

In order to prove that the π-homomorph $\underline{W}_\pi$ is a Schunck class, it suffices to notice that any π-solvable group G is its own $\underline{W}_\pi$-covering subgroup. Applying 1.11, we obtain that $\underline{W}_\pi$ is a Schunck class. □

Theorem 3.4. *$\underline{G}_\pi$ is a homomorph.*

Proof. Let $G \in \underline{G}_\pi$ and let N be a normal subgroup of G. Then G/N is π-solvable and, $|G/N|$ being a divisor of $|G|$, G/N is a π-group. So $G/N \in \underline{G}_\pi$. □

Theorem 3.5. *a) $\underline{K}_\pi$ consists of all π-solvable groups G for which there is a minimal normal subgroup M of G, such that M is a π'-group.*

b) $\underline{K}_\pi$ is a homomorph.

Proof. a) Let $G \in \underline{K}_\pi$. It follows that G is π-solvable and $O_{\pi'}(G) \neq 1$. Hence there is a minimal normal subgroup M of G, such that $M \subseteq O_{\pi'}(G)$. So M is a π'-group.

Conversely, if G is a π-solvable group and there is a minimal normal subgroup M of G, such that M is a π'-group, then $M \subseteq O_{\pi'}(G)$ and so $O_{\pi'}(G) \neq 1$.

b) Let $G \in \underline{K}_\pi$ and let L be a normal subgroup of G. Then, G being π-solvable, G/L is also π-solvable. Let us prove that $O_{\pi'}(G/L) \neq 1$. Indeed, we notice that $O_{\pi'}(G)L$ is normal in G and so $O_{\pi'}(G)L/L$ is normal in G/L. But $O_{\pi'}(G)L/L \cong O_{\pi'}(G)/(O_{\pi'}(G) \cap L)$ is a π'-group. It follows that $O_{\pi'}(G)L/L \subseteq O_{\pi'}(G/L)$. From $O_{\pi'}(G) \neq 1$ we deduce that $O_{\pi'}(G)L/L \neq 1$ and so $O_{\pi'}(G/L) \neq 1$. □

Theorem 3.6. *$\underline{M}_\pi$ consists of all π-solvable groups G for which any minimal normal subgroup M of G is a solvable π-group.*

Proof. Let $G \in \underline{M}_\pi$. Then G is a π-solvable group and $O_{\pi'}(G) = 1$. Let M be a minimal normal subgroup of G. G being π-solvable, M is either a solvable

π-group or a π'-group. But π'-group implies $M \subseteq O_{\pi'}(G) = 1$, hence $M = 1$, which is a contradiction with the fact that M is a minimal normal subgroup of G. It follows that M is a solvable π-group.

Conversely, let G be a π-solvable group, such that any minimal normal subgroup M of G is a solvable π-group. This means that G has not minimal normal subgroups which are π'-groups. We must prove that $O_{\pi'}(G) = 1$. Suppose that $O_{\pi'}(G) \neq 1$. It follows that there is a minimal normal subgroup M of G, such that $M \subseteq O_{\pi'}(G)$. So M is a π'-group, in contradiction with the above. □

Theorem 3.7. *a)* $\underline{G}_{\pi'} \subseteq \underline{W}_{\pi}$*;*

b) $\underline{G}_{\pi'}$ *is a* π*-Schunck class. Furthermore, for any finite* π*-solvable group* G*,* H *is an* $\underline{G}_{\pi'}$*-covering subgroup of* G *if and only if* H *is a Hall* π'*-subgroup of* G*.*

Proof. Let G be a π'-group. Then any chief factor M/N of G is a π'-group. Hence G is π-solvable.

b) We prove that $\underline{G}_{\pi'}$ is a Schunck class using theorem 1.11. In [5], we proved that $\underline{G}_{\pi'}$ is a π-homomorph and that a subgroup H of a π-solvable group G is an $\underline{G}_{\pi'}$-covering subgroup of G if and only if H is a Hall π'-subgroup of G. So, by 1.11, $\underline{G}_{\pi'}$ is a π-Schunck class.

As a new fact, by using the properties given in 2.2 and 2.5, we give here a new proof of the following result: If G is a π-solvable group and H is an $\underline{G}_{\pi'}$-covering subgroup of G, then H is a Hall π'-subgroup of G.

Let G be a π-solvable group and H an $\underline{G}_{\pi'}$-covering subgroup of G. We prove, by induction on $|G|$, that H is a Hall π'-subgroup of G. Two cases are possible:

1) $H = G$. Then the result is obvious.

2) $H \neq G$. Let M be a minimal normal subgroup of G. By 1.10.a), HM/M is an $\underline{G}_{\pi'}$-covering subgroup of G/M, hence, by induction, HM/M is a Hall π'-subgroup of G/M. By 1.10.c), H is an $\underline{G}_{\pi'}$-covering subgroup of HM. We now consider two cases:

a) $HM \neq G$. By the induction, H is a Hall π'-subgroup of HM. Then, by 2.5, H is a Hall π'-subgroup of HM. Then, by 2.5, H is a Hall π'-subgroup of G.

b) $HM = G$. Then $HM/M = G/M$. But HM/M being a Hall π'-subgroup of G/M, we obtain that G/M is a π'-group. We prove that $|G : H|$ is a π-number. For the minimal normal subgroup M of the π-solvable group G we have two possibilities:

b_1) M is a solvable π-group. Then $|G : H| = |HM : H| = |M : M \cap H|$ divides $|M|$ and so $|G : H|$ is a π-number.

b_2) M is a π'-group. Then $|G| = |G/M||M|$ is a π'-number. So $G \in \underline{G}_{\pi'}$. But H being an $\underline{G}_{\pi'}$-covering subgroup of G, it follows that H is $\underline{G}_{\pi'}$-maximal in G. Then $H = G$, in contradiction with our assumption. □

The last results of this section refer to the connection of the classes $\underline{K}_\pi$ and $\underline{M}_\pi$ to the π-homomorphs with the P property studied in [3].

Theorem 3.8. *If $\underline{X}$ is a π-homomorph with the P property, then $\underline{K}_\pi \subseteq \underline{X}$.*

Proof. Let $G \in \underline{K}_\pi$. By 3.5.a), G is π-solvable and there is a minimal normal subgroup M of G, such that M is a π'-group. Then $M \subseteq O_{\pi'}(G)$, hence

$$G/O_{\pi'}(G) \cong (G/M)(O_{\pi'}(G)/M). \tag{1}$$

But $\underline{X}$ has the P property and so $G/M \in \underline{X}$ and $\underline{X}$ being a homomorph we deduce from (1) that $G/O_{\pi'}(G) \in \underline{X}$. By the π-closure of $\underline{X}$, $G \in \underline{X}$. So $\underline{K}_\pi \subseteq \underline{X}$. □

Theorem 3.9. *If $\underline{X}$ is a π-homomorph, such that $\underline{X} \subseteq \underline{M}_\pi$, then $\underline{X}$ has not the P property.*

Proof. Suppose that $\underline{X}$ has the P property. Then, by 3.8, we have $\underline{K}_\pi \subseteq \underline{X}$. But $\underline{X} \subseteq \underline{M}_\pi$. We obtain the contradiction $\underline{K}_\pi \subseteq \underline{M}_\pi$, where $\underline{M}_\pi = \underline{W}_\pi \setminus \underline{K}_\pi$. □

4. **When are the Hall π-subgroups projectors in finite π-solvable groups**

In [4], we gave conditions with respect to which an $\underline{X}$-projector H of a finite π-solvable G is a Hall π-subgroup of G, where $\underline{X}$ is a π-closed Schunck class with the P property.

Here we study the converse problem: to find conditions on the Schunck class $\underline{X}$, such that any Hall π-subgroup H of a finite π-solvable group G to be an $\underline{X}$-projector of G.

The main result is the following:

Theorem 4.1. *Let $\underline{X}$ be a homomorph, such that $\underline{G}_\pi \subseteq \underline{X} \subset \underline{M}_\pi$. If G is a finite π-solvable group and H is a Hall π-subgroup of G, then H is an $\underline{X}$-projector of G.*

Proof. By induction on $|G|$. Let G be a finite π-solvable group and H a Hall π-subgroup of G (H exists by 2.3.a)). We shall prove that H is an $\underline{X}$-projector of G, by verifying conditions (a) and (b) from theorem 1.8.

a) *H is $\underline{X}$-maximal in G.* Indeed, we shall prove below (i) and (ii) from 1.5.a).

i) $H \in \underline{X}$, since H being a Hall π-subgroup of G we have $H \in \underline{G}_\pi \subseteq \underline{X}$.

ii) $H \leq H^* \leq G$, $H^* \in \underline{X}$ imply $H = H^*$. In order to show this, we consider two cases:

α) $H^* \neq G$. In this case, $|H^*| < |G|$ and H being by 2.2.a) a Hall π-subgroup of H^*, we may apply the induction and obtain that H is an $\underline{X}$-projector of H^*, hence, by 1.9.a), H is $\underline{X}$-maximal in H^*. But $H^* \in \underline{X}$. So $H = H^*$.

β) $H^* = G$. Then $G \in \underline{X} \subset \underline{M}_\pi$. So we distinguish two cases:

β_1) There is a minimal normal subgroup M of G, such that $M \subseteq H$. By 2.2.b), H/M is a Hall π-subgroup of G/M. We notice that $|G/M| < |G|$. It follows by the induction that H/M is an $\underline{X}$-projector of G/M, hence, by 1.9.a), H/M is $\underline{X}$-maximal in G/M. But, $\underline{X}$ being a homomorph, $G \in \underline{X}$ implies $G/M \in \underline{X}$. So $H/M = G/M$, hence $H = G = H^*$.

β_2) For any minimal normal subgroup N of G, we suppose that N is not included in H. Since $G \in \underline{M}_\pi$, there is a minimal normal subgroup M of G, such that M is a solvable π-group. Then, by 1.7, M is abelian. We also have that M is not included in H.

By 2.2.b), HM/M is a Hall π-subgroup of G/M. By the induction, HM/M is an $\underline{X}$-projector of G/M, hence HM/M is $\underline{X}$-maximal in G/M. But, $\underline{X}$ being a homomorph, $G \in \underline{X}$ implies $G/M \in \underline{X}$. So $HM/M = G/M$, hence $HM = G$.

Let us prove that $H \cap M$ is normal in G. Let $g \in G$ and $x \in H \cap M$. Since $HM = G$, we have that $g = hm$, where $h \in H$, $m \in M$. Then

$$g^{-1}xg = (hm)^{-1}x(hm) = (m^{-1}h^{-1})x(hm) = m^{-1}(h^{-1}xh)m$$

$$= m^{-1}m(h^{-1}xh) = h^{-1}xh \in H \cap M,$$

where we applied that $H \cap M$ is normal in H and that M is abelian. So $H \cap M$ is normal in G. Furthermore, since M is not included in H, we have $H \cap M \neq M$ and M being a minimal normal subgroup of G, it follows that $H \cap M = 1$.

Finally we have

$$G/M = HM/M \cong H/M \cap M = H/1 \cong H,$$

which implies that $|G/M| = |H|$ and so G/M is a π-group. But M is a π-group too. So G is a π-group. But H is a Hall π-subgroup of G. Then, by 2.4, $H = G = H^*$. Condition a) is proved.

b) *HN/M is an $\underline{X}$-projector of G/M, for all minimal normal subgroups M of G.* Indeed, if M is a minimal normal subgroup of G, then by applying the induction for the π-solvable group G/M, with $|G/M| < |G|$, and for its Hall π-subgroup HM/M (see 2.2.b)), we obtain that HM/M is an $\underline{X}$-projector of G/M. □

Remark. Particularly, for π the set of all primes, theorem 4.1 represents the converse of a result given by W. Gaschütz in [8].

From the proof of theorem 4.1 we notice that this theorem can also be given in the following form:

Theorem 4.2. *Let $\underline{X}$ be a homomorph, such that $\underline{X} \subset \underline{M}_\pi$. If G is a finite π-solvable group and H is a Hall π-subgroup of G, such that we have $H \in \underline{X}$, then H is an $\underline{X}$-projector of G.*

Theorem 4.1 has the following important consequence:

Theorem 4.3. *Let $\underline{X}$ be a homomorph, such that $\underline{G}_\pi \subseteq \underline{X} \subset \underline{M}_\pi$. If G is a finite π-solvable group and H is a Hall π-subgroup of G, then H is an $\underline{X}$-covering subgroup of G.*

Proof. We use theorem 1.9.b). Let K be a subgroup of G, such that $H \subseteq K$. We prove that H is an $\underline{X}$-projector of K. By 2.2.a), H is a Hall π-subgroup of K. As a subgroup of the π-solvable group G, K is also a π-solvable group. Applying now theorem 4.1, H is an $\underline{X}$-projector of K. □

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