STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LI**, Number 2, June 2006

BOUNDARY VALUE PROBLEMS FOR ITERATIVE FUNCTIONAL-DIFFERENTIAL EQUATIONS

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Abstract. We consider the following boundary value problem

$$-x''(t) = f(t, x(t), x(x(t))), \quad t \in [a, b];$$
$$x(t) = \alpha(t), \quad a_1 \le t \le a,$$
$$x(t) = \beta(t), \quad b \le t \le b_1.$$

Using the weakly Picard operators technique we establish an existence and uniqueness theorem and some data dependence results.

1. Introduction

By an iterative functional-differential equation we understand an equation of the following type (see [1]–[5], [7], [9], [12]–[14])

$$x'(t) = f(t, x(t), \ldots, x^m(t)), \ t \in J \subset \mathbb{R}$$

or (see [6], [8])

$$x''(t) = f(t, x(t), \ldots, x^m(t)), \ t \in J \subset \mathbb{R}$$

where $x^k(t) := (x \circ x \circ \cdots \circ x)(t), \ k \in \mathbb{N}$.

The purpose of this paper is to study the following boundary value problem

$$-x''(t) = f(t, x(t), x(x(t))), \quad t \in [a, b]; \tag{1.1}$$

$$\begin{cases} x(t) = \alpha(t) & t \in [a_1, a], \\ x(t) = \beta(t) & t \in [b, b_1], \end{cases} \tag{1.2}$$

Received by the editors: 23.01.2006.

2000 *Mathematics Subject Classification.* 47H10, 34K10, 34K20.

Key words and phrases. iterative functional-differential equation, boundary value problem, weakly Picard operator, fibre Picard operator, data dependence.

where

(C_1) $a_1 \le a < b \le b_1$;

(C_2) $f \in C([a,b] \times [a_1,b_1]^2)$;

(C_3) $\alpha \in C([a_1,a],[a_1,b_1])$ and $\beta \in C([b,b_1],[a_1,b_1])$;

(C_4) there exists $L_f > 0$ such that:

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \le L_f \left(|u_1 - v_1| + |u_2 - v_2|\right),$$

for all $t \in [a,b]$, $u_i, v_i \in [a_1,b_1]$, $i = 1,2$.

By a solution of the problem (1.1)–(1.2) we understand a function $x \in C^2([a,b],[a_1,b_1]) \cap C([a_1,b_1],[a_1,b_1])$ which satisfies (1.1)–(1.2).

The problem (1.1)–(1.2) is equivalent with the following fixed point equation

$$x(t) = \begin{cases} \alpha(t), \quad t \in [a_1,a], \\ w(\alpha,\beta)(t) + \int_a^b G(t,s) f(s,x(s),x(x(s)))\,\mathrm{d}s, \quad t \in [a,b], \\ \beta(t), \quad t \in [b,b_1], \end{cases} \tag{1.3}$$

and $x \in C([a_1,b_1],[a_1,b_1])$, where

$$w(\alpha,\beta)(t) := \frac{t-a}{b-a}\beta(b) + \frac{b-t}{b-a}\alpha(a),$$

and G is the Green function of the problem

$$-x'' = \chi,\ x \in C[a,b]\ x(a) = 0,\ x(b) = 0.$$

On the other hand, the equation (1.1) is equivalent with

$$x(t) = \begin{cases} x(t), \quad t \in [a_1,a], \\ w(x|_{[a_1,a]}, x|_{[b,b_1]})(t) + \int_a^b G(t,s) f(s,x(s),x(x(s)))\,\mathrm{d}s, \quad t \in [a,b], \\ x(t), \quad t \in [b,b_1], \end{cases} \tag{1.4}$$

and $x \in C([a_1,b_1],[a_1,b_1])$.

In this paper we apply the weakly Picard operators technique to study the equations (1.3) and (1.4).

2. Weakly Picard operators

In this paper we need some notions and results from the weakly Picard operator theory (for more details see I. A. Rus [10] and [11]).

Let (X, d) be a metric space and $A : X \to X$ an operator. We shall use the following notations:

$F_A := \{x \in X \mid A(x) = x\}$- the fixed point set of A;

$I(A) := \{Y \subset X \mid A(Y) \subset Y, Y \neq \emptyset\}$- the family of the nonempty invariant subsets of A;

$A^{n+1} := A \circ A^n, \quad A^1 = A, \quad A^0 = 1_X, \quad n \in \mathbb{N}$;

$P(X) := \{Y \subset X | Y \neq \emptyset\}$;

$H(Y, Z) := \max\left\{\sup\limits_{y\in Y} \inf\limits_{z\in Z} d(y, z), \sup\limits_{z\in Z} \inf\limits_{y\in Y} d(y, z)\right\}$ -the Pompeiu–Hausdorff functional on $P(X) \times P(X)$.

Definition 2.1. *Let (X, d) be a metric space. An operator $A : X \to X$ is a Picard operator (PO) if there exists $x^* \in X$ such that:*

(i) $F_A = \{x^*\}$*;*

(ii) *the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.*

Theorem 2.1 (Contraction principle). *Let (X, d) be a complete metric space and $A : X \to X$ a γ-contraction. Then*

(i) $F_A = \{x^*\}$,

(ii) $(A^n(x_0))_{n\in\mathbb{N}}$ *converges to x^* for all $x_0 \in X$,*

(iii) $d(x^*, A^n(x_0)) \leq \dfrac{\gamma^n}{1-\gamma} d(x_0, A(x_0))$, *for all $n \in \mathbb{N}$.*

Remark 2.1. *Accordingly to the definition, the contraction principle insures that, if $A : X \to X$ is a γ-contraction on the complet metric space X, then it is a Picard operator.*

Theorem 2.2. *Let (X, d) be a complete metric space and $A, B : X \to X$ two operators. We suppose that*

(i) *the operator A is a γ-contraction;*

(ii) $F_B \neq \emptyset$;

(iii) *there exists* $\eta > 0$ *such that*

$$d(A(x), B(x)) \leq \eta, \ \forall \ x \in X.$$

Then if $F_A = \{x_A^*\}$ *and* $x_B^* \in F_B$, *we have*

$$d(x_A^*, x_B^*) \leq \frac{\eta}{1-\gamma}.$$

Definition 2.2. *Let* (X, d) *be a metric space. An operator* A *is a weakly Picard operator (WPO) if the sequence* $(A^n(x))_{n\in\mathbb{N}}$ *converges for all* $x \in X$, *and its limit (which may depend on* x*) is a fixed point of* A.

Theorem 2.3. *Let* (X, d) *be a metric space and* $A : X \to X$ *an operator. The operator* A *is weakly Picard operator if and only if there exists a partition of* X,

$$X = \bigcup_{\lambda\in\Lambda} X_\lambda,$$

where Λ *is the indices' set of partition, such that*

(a) $X_\lambda \in I(A)$, *for all* $\lambda \in \Lambda$;

(b) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ *is a Picard operator for all* $\lambda \in \Lambda$.

Definition 2.3. *If* A *is weakly Picard operator then we consider the operator* A^∞ *defined by*

$$A^\infty : X \to X, \quad A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

It is clear that

$$A^\infty(X) = F_A \text{ and } \omega_A(x) = \{A^\infty(x)\},$$

where $\omega_A(x)$ is the ω-limit point set of A.

Definition 2.4. *Let* A *be a weakly Picard operator and* $c > 0$. *The operator* A *is* c*–weakly Picard operator if*

$$d(x, A^\infty(x)) \leq c\, d(x, A(x)), \ \forall \ x \in X.$$

Example 2.1. *Let* (X, d) *be a complete metric space and* $A : X \to X$ *a continuous operator. We suppose that there exists* $\gamma \in [0, 1)$ *such that*

$$d(A^2(x), A(x)) \leq \gamma\, d(x, A(x)), \ \forall \ x \in X.$$

Then A is c–weakly Picard operator with $c = \dfrac{1}{1-\gamma}$.

Theorem 2.4. *Let (X,d) be a metric space and $A_i : X \to X,\ i = 1,2$. Suppose that*

(i) *the operator A_i is c_i–weakly Picard operator, $i = 1,2$;*

(ii) *there exists $\eta > 0$ such that*

$$d(A_1(x), A_2(x)) \le \eta,\ \forall\ x \in X.$$

Then

$$H(F_{A_1}, F_{A_2}) \le \eta \max(c_1, c_2).$$

Theorem 2.5 (Fibre contraction principle). *Let (X,d) and (Y,ρ) be two metric spaces and $A : X \times X \to X \times Y$, $A = (B,C)$, $(B : X \to X, C : X \times Y \to Y)$ a triangular operator. We suppose that*

(i) *(Y,ρ) is a complete metric space;*

(ii) *the operator B is PO;*

(iii) *there exists $l \in [0,1)$ such that $C(x,\cdot) : Y \to Y$ is l-contraction, for all $x \in X$;*

(iv) *if $(x^*, y^*) \in F_A$, then $C(\cdot, y^*)$ is continuous in x^*.*

Then the operator A is PO.

3. Boundary value problem

In what follows we consider the fixed point equation (1.3). Let

$$B_f : C([a_1,b_1],[a_1,b_1]) \to C([a_1,b_1],\mathbb{R}),$$

where $B_f(x)(t) :=$ the right hand side of (1.3). Let $L > 0$ and

$$C_L([a_1,b_1],[a_1,b_1]) := \{x \in C([a_1,b_1],[a_1,b_1]) | |x(t_1) - x(t_2)| \le L|t_1 - t_2|,$$

$\forall\ t_1, t_2 \in [a_1,b_1]\}$.

It is clear that $C_L([a_1,b_1],[a_1,b_1])$ is a complete metric space with respect to the metric,

$$d(x_1,x_2) := \max_{a_1 \le t \le b_1} |x_1(t) - x_2(t)|.$$

We have

Theorem 3.1. *We suppose that*

(i) *the conditions* $(C_1) - (C_4)$ *are satisfied;*

(ii) $\alpha \in C_L([a_1, a], [a_1, b_1])$, $\beta \in C_L([b, b_1], [a_1, b_1])$;

(iii) m_f *and* $M_f \in \mathbb{R}$ *are such that* $m_f \le f(t, u_1, u_2) \le M_f$, $\forall\ t \in [a, b]$, $u_i \in [a_1, b_1]$, $i = 1, 2$, *and moreover,*

$$\begin{aligned} a_1 &\le \min(\alpha(a), \beta(b)) + m_f \frac{(b-a)^2}{8}, \text{ for } m_f < 0,\\ a_1 &\le \min(\alpha(a), \beta(b)), \qquad \text{for } m_f \ge 0,\\ b_1 &\ge \max(\alpha(a), \beta(b)), \qquad \text{for } M_f \le 0,\\ b_1 &\ge \max(\alpha(a), \beta(b)) + M_f \frac{(b-a)^2}{8}, \text{ for } M_f > 0, \end{aligned}$$

and

$$\frac{|\beta(b) - \alpha(a)|}{b-a} + |M_f| \frac{a^2 + b^2 - 6ab}{2(b-a)} \le L;$$

(iv) $\dfrac{(b-a)^2}{8} L_f(L+2) < 1.$

Then the boundary value problem (1.1)–(1.2) *has, in* $C_L([a_1, b_1], [a_1, b_1])$, *a unique solution. Moreover, the operator*

$$B_f : C_L([a_1, b_1], [a_1, b_1]) \to C_L([a_1, b_1], C_L([a_1, b_1], [a_1, b_1]))$$

is a c–Picard operator with $c = \dfrac{8}{8 - (b-a)^2 L_f(L+2)}$.

Proof. First of all we remark that the condition (iii) implies that $C_L([a_1, b_1], [a_1, b_1])$ is an invariant subset for B_f. Indeed, we have $a_1 \le B_f(x)(t) \le b_1$, $x(t) \in [a_1, b_1]$ for all $t \in [a, b]$. Actually, using the positivity of the Green function, for m_f and $M_f \in \mathbb{R}$ such that

$$m_f \le f(t, u_1, u_2) \le M_f,\ \forall\ t \in [a, b],\ u_i \in [a_1, b_1],\ i = 1, 2,$$

we have

$$G(t, s) m_f \le G(t, s) f(s, x(s), x(x(s))) \le G(t, s) M_f,\ \forall\ t \in [a, b].$$

This implies that

$$\int_a^b G(t,s)m_f\,\mathrm{d}s \leq \int_a^b G(t,s)f(s,x(s),x(x(s)))\,\mathrm{d}s \leq \int_a^b G(t,s)M_f\,\mathrm{d}s,\ \forall\ t\in[a,b],$$

that is,

$$w(\alpha,\beta)(t)+m_f\int_a^b G(t,s)\,\mathrm{d}s \leq B_f(x)(t) \leq w(\alpha,\beta)(t)+M_f\int_a^b G(t,s)\,\mathrm{d}s,\ \forall\ t\in[a,b].$$

It is easy to see that,

$$\min_{t\in[a,b]}\int_a^b G(t,s)\,\mathrm{d}s = \min_{t\in[a,b]}\frac{(t-a)(b-t)}{2} = 0$$

and

$$\max_{t\in[a,b]}\int_a^b G(t,s)\,\mathrm{d}s = \max_{t\in[a,b]}\frac{(t-a)(b-t)}{2} = \frac{(b-a)^2}{8}.$$

Therefore, if condition (iii) holds, we have satisfied the invariance property for the operator B_f in $C([a_1,b_1],[a_1,b_1])$.

Now, consider $t_1,t_2\in[a_1,a]$. Then,

$$|B_f(x)(t_1)-B_f(x)(t_2)| = |\alpha(t_1)-\alpha(t_2)| \leq L|t_1-t_2|,$$

because of $\alpha\in C_L([a_1,a],[a_1,b_1])$.

Similarly, for $t_1,t_2\in[b,b_1]$

$$|B_f(x)(t_1)-B_f(x)(t_2)| = |\beta(t_1)-\beta(t_2)| \leq L|t_1-t_2|,$$

that follows from (ii), too.

On the other hand, if $t_1,t_2\in[a,b]$, we have,

$$\begin{aligned}
&|B_f(x)(t_1)-B_f(x)(t_2)| = \\
&= \left|w(\alpha,\beta)(t_1)-w(\alpha,\beta)(t_2)+\int_a^b [G(t_1,s)-G(t_2,s)]f(s,x(s),x(x(s)))\mathrm{d}s\right| = \\
&= \left|\frac{t_1-t_2}{b-a}\,(\beta(b)-\alpha(a))+\int_a^b [G(t_1,s)-G(t_2,s)]f(s,x(s),x(x(s)))\mathrm{d}s\right| \leq \\
&\leq \left|\frac{\beta(b)-\alpha(a)}{b-a}(t_1-t_2)\right| + \left|\int_a^b [G(t_1,s)-G(t_2,s)]f(s,x(s),x(x(s)))\mathrm{d}s\right| \leq \\
&\leq \left|\frac{\beta(b)-\alpha(a)}{b-a}\right||t_1-t_2| + |M_f|\left|\int_a^b [G(t_1,s)-G(t_2,s)]\mathrm{d}s\right|.
\end{aligned}$$

But,

$$\int_a^b [G(t_1,s)-G(t_2,s)]\mathrm{d}s = \int_a^{t_1}\left[\frac{(s-a)(b-t_1)}{b-a}-\frac{(s-a)(b-t_2)}{b-a}\right]\mathrm{d}s+$$
$$+\int_{t_1}^{t_2}\left[\frac{(t_1-a)(b-s)}{b-a}-\frac{(s-a)(b-t_2)}{b-a}\right]\mathrm{d}s+$$
$$+\int_{t_2}^{b}\left[\frac{(t_1-a)(b-s)}{b-a}-\frac{(t_2-a)(b-s)}{b-a}\right]\mathrm{d}s.$$

After some calculation we obtain,

$$\int_a^b [G(t_1,s)-G(t_2,s)]\mathrm{d}s = \left[(a-b)(t_1+t_2)-a^2-4ab+b^2\right]\frac{t_1-t_2}{2(b-a)}.$$

Thus,

$$\left|\int_a^b [G(t_1,s)-G(t_2,s)]\mathrm{d}s\right| \leq \frac{a^2+b^2-6ab}{2(b-a)}|t_1-t_2|.$$

So, we can affirm that

$$|B_f(x)(t_1)-B_f(x)(t_2)| \leq \left[\frac{|\beta(b)-\alpha(a)|}{b-a}+|M_f|\frac{a^2+b^2-6ab}{2(b-a)}\right]|t_1-t_2|,$$

$\forall\ t_1,t_2\in[a,b],\ t_1\leq t_2$, and due to (iii), $B_f(x)$ is L–Lipschitz.

Thus, according to the above, we have $C_L([a_1,b_1],[a_1,b_1])\in I(B_f)$.

From the condition (iv) it follows that, B_f is an L_{B_f}–contraction, with

$$L_{B_f} := \frac{(b-a)^2}{8}L_f(L+2).$$

Indeed, for all $t\in[a_1,a]\cup[b,b_1]$, we have $|B_f(x_1)(t)-B_f(x_2)(t)|=0$.

Otherwise, for $t \in [a, b]$

$$|B_f(x_1)(t) - B_f(x_2)(t)| =$$
$$= \left| \int_a^b G(t,s) \left[f(s, x_1(s), x_1(x_1(s))) - f(s, x_2(s), x_2(x_2(s))) \right] \mathrm{ds} \right| \leq$$
$$\leq \max_{x \in [a,b]} \left| \int_a^b G(t,s) \mathrm{ds} \right| L_f \left(|x_1(s) - x_2(s)| + |x_1(x_1(s)) - x_2(x_2(s))| \right) \leq$$
$$\leq \frac{(b-a)^2}{8} L_f \left(\|x_1 - x_2\|_C + |x_1(x_1(s)) - x_1(x_2(s))| + |x_1(x_2(s)) - x_2(x_2(s))| \right) \leq$$
$$\leq \frac{(b-a)^2}{8} L_f \left(\|x_1 - x_2\|_C + L|x_1(s) - x_2(s)| + \|x_1 - x_2\|_C \right) \leq$$
$$\leq \frac{(b-a)^2}{8} L_f (L+2) \|x_1 - x_2\|_C \,.$$

So, B_f is a c–Picard operator, with $c = \dfrac{1}{1 - L_{B_f}}$. □

In what follows, consider the following operator

$$E_f : C_L([a_1, b_1], [a_1, b_1]) \to C_L([a_1, b_1], [a_1, b_1]),$$

where

$$E_f(x)(t) := \text{the right hand side of (1.4).}$$

Theorem 3.2. *In the conditions of the Theorem 3.1, the operator*

$$E_f : C_L([a_1, b_1], [a_1, b_1]) \to C_L([a_1, b_1], [a_1, b_1])$$

is WPO.

Proof. The operator E_f is a continuous operator but it is not a contraction operator.

Let take the following notation:

$$X_{\alpha,\beta} := \{x \in C([a_1, b_1], [a_1, b_1]) \,|\, x|_{[a_1,a]} = \alpha,\, x|_{[b,b_1]} = \beta\}.$$

Then we can write

$$C_L([a_1, b_1], [a_1, b_1]) = \bigcup_{\substack{\alpha \in C_L([a_1,a],[a_1,b_1]) \\ \beta \in C_L([b,b_1],[a_1,b_1])}} X_{\alpha,\beta}. \tag{3.5}$$

We have that $X_{\alpha,\beta} \in I(E_f)$ and $E_f|_{X_{\alpha,\beta}}$ is a Picard operator, because it is the operator which appears in the proof of the Theorem 3.1.

By applying the Theorem 2.3, we obtain that E_f is WPO. □

4. **Increasing solutions of** (1.1)

4.1. **Inequalities of Čaplygin type.** We have

Theorem 4.1. *We suppose that*

(a) *the conditions of the Theorem 3.1 are satisfied;*

(b) $u_i, v_i \in [a_1, b_1],\ u_i \leq v_i,\ i = 1, 2,$ *imply that*

$$f(t, u_1, u_2) \leq f(t, v_1, v_2),$$

for all $t \in [a, b]$.

Let x *be a increasing solution of the equation* (1.1) *and* y *an increasing solution of the inequality*

$$-y''(t) \leq f(t, y(t), y(y(t))), \quad t \in [a, b].$$

Then

$$y(t) \leq x(t),\ \forall\, t \in [a_1, a] \cup [b, b_1] \Rightarrow y \leq x.$$

Proof. In the terms of the operator E_f, we have

$$x = E_f(x) \text{ and } y \leq E_f(y),$$

and

$$w(y|_{[a_1,a]}, y|_{[b,b_1]}) \leq w(x|_{[a_1,a]}, x|_{[b,b_1]}).$$

However, from the condition (b), we have that the operator E_f^∞ is increasing (see Lemma 7.1 in [11]), we have

$$y \leq E_f^\infty(y) = E_f^\infty(\widetilde{w}(y)) \leq E_f^\infty(\widetilde{w}(x)) = x,$$

thus $y \le x$. Here, for $z \in C[a,b]$, we used the notation

$$\widetilde{w}(z)(t) := \begin{cases} z(z), t \in [a_1, a], \\ w(z|_{[a_1,a]}, z|_{[b,b_1]})(t)\, t \in [a,b], \\ z(b), t \in [b, b_1]. \end{cases}$$

□

4.2. **Comparison theorem.** In what follows we want to study the monotony of the solution of the problem (1.1)–(1.2), with respect to α, β and f. We will use the result below:

Lemma 4.1 (Abstract comparison lemma). *Let $(X, d, \le)$ be an ordered metric space and $A, B, C : X \to X$ be such that:*

(i) $A \le B \le C$;
(ii) *the operators A, B, C are weakly Picard operators;*
(iii) *the operator B is increasing.*

Then

$$x \le y \le z \Rightarrow A^{\infty}(x) \le B^{\infty}(y) \le C^{\infty}(z).$$

In this case we can establish the theorem

Theorem 4.2. *Let $f_i \in C([a,b] \times [a_1, b_1]^2)$, $i = 1, 2, 3$. We suppose that*

(a) $f_2(t, \cdot, \cdot) : [a_1, b_1]^2 \to [a_1, b_1]^2$ *is increasing;*
(b) $f_1 \le f_2 \le f_3$.

Let x_i be a increasing solution of the equation

$$-x'' = f_i(t, x(t), x(x(t))), \quad t \in [a,b].$$

If

$$x_1(t) \le x_2(t) \le x_3(t), \ \forall\, t \in [a_1, a] \cap [b, b_1],$$

then

$$x_1 \le x_2 \le x_3.$$

Proof. The operators E_{f_i}, $i = 1, 2$ are weakly Picard operators. Taking into consideration the condition (a) the operator E_{f_2} is increasing. From (b) we have that

$$E_{f_1} \le E_{f_2} \le E_{f_3}.$$

We note that $x_i = E_{f_i}^{\infty}(\widetilde{w}(x_i))$, $i = 1, 2$. Now, using the Abstract comparison lemma, the proof is complete.

□

5. **Data dependence: continuity**

Consider the boundary value problem (1.1)–(1.2) and suppose the conditions of the Theorem 3.1 are satisfied. Denote by $x(\cdot; \alpha, \beta, f)$ the solution of this problem. We can state the following result:

Theorem 5.1. *Let α_i, β_i, f_i, $i = 1, 2$, be as in the Theorem 3.1. Furthermore, we suppose that*

(i) *there exists $\eta_1 > 0$, such that*

$$|\alpha_1(t) - \alpha_2(t)| \le \eta_1, \ \forall\, t \in [a_1, a],$$

and

$$|\beta_1(t) - \beta_2(t)| \le \eta_1, \ \forall t \in [b, b_1];$$

(ii) *there exists $\eta_2 > 0$ such that*

$$|f_1(t, u_1, u_2) - f_2(t, u_1, u_2)| \le \eta_2, \ \forall\, t \in [a, b], \ \forall\, u_i \in [a_1, b_1], \, i = 1, 2.$$

Then

$$|x(t; \alpha_1, \beta_1, f_1) - x(t; \alpha_2, \beta_2, f_2)| \le \frac{8\eta_1 + \eta_2(b-a)^2}{8 - L_f(L+2)(b-a)^2}$$

where $L_f = \max(L_{f_1}, L_{f_2})$.

Proof. Consider the operators $B_{\alpha_i, \beta_i, f_i}$, $i = 1, 2$. From Theorem 3.1 these operators are contractions. Additionally,

$$||B_{\alpha_1,\beta_1,f_1}(x) - B_{\alpha_2,\beta_2,f_2}(x)||_C =$$

$$= \left| [w(\alpha_1, \beta_1)(t) - w(\alpha_2, \beta_2)(t)] + \int_a^b G(t, s) \left[f_1(s, x(s), x(x(s))) - f_2(s, x(s), x(x(s))) \right] \mathrm{d}s \right| \le$$

$$\leq \left| \frac{t-a}{b-a} \left[\beta_1(b) - \beta_2(b)\right] + \frac{b-t}{b-a} \left[\alpha_1(a) - \alpha_2(a)\right] \right| + \max_{t\in[a,b]} \left| \int_a^b G(t,s) \mathrm{d}s \right| \eta_2$$

$$\leq \eta_1 + \eta_2 \frac{(b-a)^2}{8},$$

$\forall\ x \in C_L([a_1, b_1], [a_1, b_1])$.

Now, the proof follows from the Theorem 2.2, with

$$A := B_{\alpha_1,\beta_1,f_1}, \quad B := B_{\alpha_2,\beta_2,f_2}, \quad \eta := \eta_1 + \eta_2 \frac{(b-a)^2}{8}$$

and

$$\gamma := L_A = \frac{(b-a)^2}{8} L_f (L+2).$$

□

From the theorem above we have

Theorem 5.2. *Let $\alpha_i, \beta_i, f_i,\ i \in \mathbb{N}$ and α, β, f be as in the Theorem 3.1. We suppose that*

$$\alpha_i \overset{univ.}{\longrightarrow} \alpha \text{ as } i \to \infty,$$

$$\beta_i \overset{univ.}{\longrightarrow} \beta \text{ as } i \to \infty,$$

$$f_i \overset{univ.}{\to} f \text{ as } i \to \infty.$$

Then

$$x(\cdot, \alpha_i, \beta_i, f_i) \overset{univ.}{\longrightarrow} x(\cdot, \alpha, \beta, f), \text{ as } i \to \infty.$$

Theorem 5.3. *Let f_1 and f_2 be as in the Theorem 3.1. Let $F_{E_{f_i}}$ be the solution set of equation* (1.1) *corresponding to $f_i,\ i = 1, 2$. Suppose that there exists $\eta > 0$ such that*

$$|f_1(t, u_1, u_2) - f_2(t, u_1, u_2)| \leq \eta, \tag{5.6}$$

for all $t \in [a, b],\ u_i \in [a_1, b_1],\ i = 1, 2$. Then

$$H_{||\cdot||_C}(F_{E_{f_1}}, F_{E_{f_2}}) \leq \frac{\eta(b-a)^2}{8 - L_f(L+2)(b-a)^2}$$

where $L_f := \max(L_{f_1}, L_{f_2})$ and $H_{||\cdot||_C}$ denotes the Pompeiu–Hausdorff functional with respect to $||\cdot||_C$ on $C_L([a_1, b_1], [a_1, b_1])$.

Proof. We will look for those c_i, for which in condition of the Theorem 3.1 the operators $E_{f_i},\ i=1,2$, are c_i- weakly Picard operators.

Let $X_{\alpha,\beta} := \{x \in C_L([a_1,b_1],[a_1,b_1])\,|\,x|_{[a_1,a]} = \alpha, x|_{[b,b_1]} = \beta\}$

It is clear that $E_{f_i}|_{X_{\alpha,\beta}} = B_{f_i}$. So, from Theorem 2.3 and Theorem 3.1 we have

$$||E^2_{f_i}(x) - E_{f_i}(x)||_C \leq L_{f_i}(L+2)\frac{(b-a)^2}{8}||E_{f_i}(x) - x||_C$$

for all $x \in C_L([a_1,b_1],[a_1,b_1]),\ i=1,2$.

Now, choosing $\lambda_i = \dfrac{(b-a)^2}{8}L_{f_i}(L+2)$, we get that E_{f_i} are c_i- weakly Picard operators, with $c_i = (1-\lambda_i)^{-1}$.

From (5.6) we obtain that

$$||E_{f_1}(x) - E_{f_2}(x)||_C \leq \eta\frac{(b-a)^2}{8},\ \text{for all } x \in C_L([a_1,b_1],[a_1,b_1]).$$

Applying Theorem 2.4 we have that

$$H_{||\cdot||_C}(F_{E_{f_1}}, F_{E_{f_2}}) \leq \frac{\eta(b-a)^2}{8 - L_f(L+2)(b-a)^2}.$$

□

6. Data dependence: differentiability

Consider the following boundary value problem with parameter

$$-x''(t) = f(t, x(t), x(x(t)); \lambda), \quad t \in [a,b]; \tag{6.7}$$

$$\begin{cases} x(t) = \alpha(t) & t \in [a_1,a], \\ x(t) = \beta(t) & t \in [b,b_1]. \end{cases} \tag{6.8}$$

Suppose that we have satisfied the following conditions:

(P_1) $a_1 \leq a < b \leq b_1$; $J \subset \mathbb{R}$, a compact interval;

(P_2) $\alpha \in C^1_L([a_1,a],[a_1,b_1])$ and $\beta \in C^1_L([b,b_1],[a_1,b_1])$;

(P_3) $f \in C^1([a,b]\times[a_1,b_1]^2 \times J)$;

(P_4) there exists $L_f > 0$ such that

$$\left|\frac{\partial f(t,u_1,u_2;\lambda)}{\partial u_i}\right| \leq L_f,$$

for all $t \in [a,b]$, $u_i \in [a_1,b_1]$, $i=1,2,\ \lambda \in J$;

(P_5) m_f and $M_f \in \mathbb{R}$ are such that $m_f \leq f(t, u_1, u_2) \leq M_f,\ \forall\, t \in [a, b],\ u_i \in [a_1, b_1],\ i = 1, 2$, moreover we have

$$\begin{aligned} a_1 &\leq \min(\alpha(a), \beta(b)) + m_f \frac{(b-a)^2}{8},\ \textit{for } m_f < 0,\\ a_1 &\leq \min(\alpha(a), \beta(b)), \qquad\qquad \textit{for } m_f \geq 0,\\ b_1 &\geq \max(\alpha(a), \beta(b)), \qquad\qquad \textit{for } M_f \leq 0,\\ b_1 &\geq \max(\alpha(a), \beta(b)) + M_f \frac{(b-a)^2}{8},\ \textit{for } M_f > 0, \end{aligned}$$

and

$$\frac{|\beta(b) - \alpha(a)|}{b-a} + |M_f| \frac{a^2 + b^2 - 6ab}{2(b-a)} \leq L;$$

(P_6) $\dfrac{(b-a)^2}{8} L_f(L+2) < 1$.

Then, from the Theorem 3.1, we have that the problem (6.7)–(6.8) has a unique solution, $x^*(\cdot, \lambda)$.

We will prove that $x^*(t, \cdot) \in C^1(J)$, for all $t \in [a_1, b_1]$.

For this, we consider the equation

$$\begin{aligned} -x''(t;\lambda) =& f(t, x(t;\lambda), x(x(t;\lambda);\lambda);\lambda),\ t \in [a,b],\ \lambda \in J,\\ & x \in C([a_1, b_1] \times J, [a_1, b_1] \times J) \cap C^2([a,b] \times J, [a_1, b_1] \times J). \end{aligned} \tag{6.9}$$

The problem (6.9)–(6.8) is equivalent with the following functional-integral equation

$$x(t;\lambda) = \begin{cases} \alpha(t), t \in [a_1, a], \quad \lambda \in J,\\ w(\alpha, \beta)(t) + \int_a^b G(t,s) f(s, x(s;\lambda), x(x(s;\lambda);\lambda);\lambda)\,\mathrm{d}s,\ t \in [a,b],\ \lambda \in J\\ \beta(t), t \in [b, b_1], \quad \lambda \in J. \end{cases} \tag{6.10}$$

Now, let take the operator

$$B : C_L([a_1, b_1] \times J, [a_1, b_1] \times J) \to C_L([a_1, b_1] \times J, [a_1, b_1] \times J),$$

where $B(x)(t;\lambda) :=$ the right hand side of (6.10).

Let $X := C_L([a_1, b_1] \times J, [a_1, b_1])$. It is clear from the proof of the Theorem 3.1 that in the conditions $(P_1) - (P_6)$, the operator $B : (X, \|\cdot\|_C) \to (X, \|\cdot\|_C)$

is a PO. Let x^* be the unique fixed point of B. We consider the subset $X_1 \subset X$, $X_1 := \left\{ x \in X | \ \frac{\partial x}{\partial t} \in C[a_1, b_1] \right\}$. We remark that $x^* \in X_1$, $B(X_1) \subset X_1$ and $B : (X_1, \| \cdot \|_C) \to (X_1, \| \cdot \|_C)$ is PO. Let $Y := C([a_1, b_1] \times J)$.

Supposing that there exists $\frac{\partial x^*}{\partial \lambda}$, from (6.10) we have that

$$\frac{\partial x^*(t;\lambda)}{\partial \lambda} = \int_a^b G(t,s) \frac{\partial f(s, x^*(s;\lambda), x^*(x^*(s;\lambda);\lambda);\lambda)}{\partial u_1} \cdot \frac{\partial x^*(s;\lambda)}{\partial \lambda} \, \mathrm{d}s +$$
$$+ \int_a^b G(t,s) \frac{\partial f(s, x^*(s;\lambda), x^*(x^*(s;\lambda);\lambda);\lambda)}{\partial u_2} \cdot$$
$$\cdot \left[\frac{\partial x^*(x^*(s;\lambda);\lambda)}{\partial u_1} \cdot \frac{\partial x^*(s;\lambda)}{\partial \lambda} + \frac{\partial x^*(x^*(s;\lambda);\lambda)}{\partial \lambda} \right] \mathrm{d}s +$$
$$+ \int_a^b G(t,s) \frac{\partial f(s, x^*(s;\lambda), x^*(x^*(s;\lambda);\lambda);\lambda)}{\partial \lambda} \, \mathrm{d}s, \quad t \in [a,b], \ \lambda \in J.$$

This relation suggest us to consider the following operator

$$C : X_1 \times Y \to Y$$

$$(x, y) \mapsto C(x, y)$$

with

$$C(x,y)(t;\lambda) := \int_a^b G(t,s) \frac{\partial f(s, x(s;\lambda), x(x(s;\lambda);\lambda);\lambda)}{\partial u_1} \cdot y(s;\lambda) \, \mathrm{d}s +$$
$$+ \int_a^b G(t,s) \frac{\partial f(s, x(s;\lambda), x(x(s;\lambda);\lambda);\lambda)}{\partial u_2} \cdot$$
$$\cdot \left[\frac{\partial x(x(s;\lambda);\lambda)}{\partial u_1} \cdot y(s;\lambda) + \frac{\partial x(x(s;\lambda);\lambda)}{\partial \lambda} \right] \mathrm{d}s +$$
$$+ \int_a^b G(t,s) \frac{\partial f(s, x(s;\lambda), x(x(s;\lambda);\lambda);\lambda)}{\partial \lambda} \, \mathrm{d}s, \quad t \in [a,b], \ \lambda \in J$$

and

$$C(x,y)(t,\lambda) := 0, \text{ for } t \in [a_1, a] \cup [b, b_1], \ \lambda \in J.$$

In this way we have the triangular operator

$$A : X_1 \times Y \to X_1 \times Y$$

$$(x, y) \mapsto (B(x), C(x, y)),$$

where B is a Picard operator and $C(x,\cdot) : Y \to Y$ is an L_C- contraction, with $L_C = \frac{(b-a)^2}{8}\widetilde{L}_f(L+2)$, where $\widetilde{L}_f = max(L_f, LL_f)$.

From the fibre contraction theorem we have that the operator A is Picard operator, i.e. the sequences

$$x_{n+1} := B(x_n),$$
$$y_{n+1} := C(x_n, y_n), \quad n \in \mathbb{N}$$

converges uniformly, with respect to $t \in [a_1, b_1]$, $\lambda \in J$, to $(x^*, y^*) \in F_A$, for all $x_0 \in X_1$, $y_0 \in Y$.

If we take $x_0 = 0$, $y_0 = \frac{\partial x_0}{\partial \lambda} = 0$, then $y_1 = \frac{\partial x_1}{\partial \lambda}$.

By induction we prove that $y_n = \frac{\partial x_n}{\partial \lambda}$, $\forall\, n \in \mathbb{N}$.

So,

$$x_n \overset{unif.}{\longrightarrow} x^* \text{ as } n \to \infty,$$
$$\frac{\partial x_n}{\partial \lambda} \to y^* \text{ as } n \to \infty.$$

From these we have that there exists $\frac{\partial x^*}{\partial \lambda}$ and $\frac{\partial x^*}{\partial \lambda} = y^*$.

Taking into consideration the above, we can formulate the theorem

Theorem 6.1. *Consider the problem* (6.9)–(6.8)*, and suppose the conditions* $(P_1) - (P_6)$ *holds. Then,*

(i) (6.9)–(6.8) *has a unique solution,* x^**, in* $C([a_1, b_1] \times J, [a_1, b_1])$,

(ii) $x^*(t, \cdot) \in C^1(J)$, $\forall\, t \in [a_1, b_1]$.

Remark 6.1. *By the same arguments we have that, if* $f(t, \cdot, \cdot) \in C^k$*, then* $x^*(t, \cdot) \in C^k(J)$, $\forall\, t \in [a_1, b_1]$.

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