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FUNCTIONAL-DIFFERENTIAL EQUATIONS OF MIXED TYPE, VIA WEAKLY PICARD OPERATORS

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Abstract. In this paper we apply the weakly Picard operators technique to study the following second order functional differential equations of mixed type

$$-x''(t) = f(t, x(t), \int_{t-h}^{t} x(s)ds, \int_{t}^{t+h} x(s)ds), \quad t \in [a, b], h > 0.$$

1. Introduction

The purpose of this paper is to study,the following boundary value problem:

$$-x''(t) = f(t, x(t), \int_{t-h}^{t} x(s)ds, \int_{t}^{t+h} x(s)ds), \quad t \in [a, b], h > 0. \tag{1}$$

$$\begin{cases} x(t) = \varphi(t) & , \quad t \in [a-h, a] \\ x(t) = \psi(t) & , \quad t \in [b, b+h] \end{cases}. \tag{2}$$

Where:

(H_1) $f \in C([a, b] \times \mathbb{R}^3, \mathbb{R})$.

(H_2) There exists $L_f > 0$ such that:

$$|f(t, u_1, u_2, u_3) - f(t, v_1, v_2, v_3)| \leq L_f \sum_{i=1}^{3} |u_i - v_i|,$$

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for all $t \in [a,b], u_i, v_i \in \mathbb{R}, i = \overline{1,3}$.

(H_3) $\varphi \in C([a-h,a]), \psi \in C([b,b+h])$.

Let G be the Green function of the following problem:

$$\begin{cases} -x'' = \lambda \\ x(a) = 0 \\ x(b) = 0 \end{cases}.$$

From the definition of the Green function we have that,the problem (1)+(2), $x \in C([a-h,b+h]) \cap C^2([a,b])$, is equivalent with the fixed point equation:

$$x(t) = \begin{cases} \varphi(t), & t \in [a-h,a] \\ w(\varphi,\psi)(t) + \int\limits_a^b G(t,s) f(s, x(s), \int\limits_{s-h}^{s} x(u)du, \int\limits_{s}^{s+h} x(u)du))ds, & t \in [a,b] \\ \psi(t), & t \in [b,b+h] \end{cases} \tag{3}$$

$x \in C([a-h,b+h])$, where:

$$w(\varphi,\psi)(t) := \frac{t-a}{b-a} \cdot \psi(b) + \frac{b-t}{b-a} \cdot \varphi(a).$$

The equation (1) is equivalent with:

$$x(t) = \begin{cases} x(t), & t \in [a-h,a] \\ w(x|_{[a-h,a]}, x|_{[b,b+h]}) + \\ + \int\limits_a^b G(t,s) f(s, x(s), \int\limits_{s-h}^{s} x(u)du, \int\limits_{s}^{s+h} x(u)du)ds, & t \in [a,b] \\ x(t), & t \in [b,b+h] \end{cases}. \tag{4}$$

We consider the following operators:

$$B_f, E_f : C([a-h,b+h]) \to C([a-h,b+h])$$

where:

$B_f(x)(t) :=$second part of (3)

and

$E_f(x)(t)$:=second part of (4).

We denote by $X := C([a-h, b+h])$.

Let be

$$X_{\varphi,\psi} := \{x \in X \mid x \mid_{[a-h,\, a]} = \varphi, x \mid_{[b,\, b+h]} = \psi\}.$$

Then

$$X = \bigcup_{\substack{\varphi \in C([a-h,a]) \\ \psi \in C([b,b+h])}} X_{\varphi,\psi}$$

is a partition of X.

2. **Weakly Picard operators**

Led (X,d) be a metric space and $A : X \longrightarrow X$ an operator. We shall use the following notations:

$F_A := \{x \in X \mid A(x) = x\}$- the fixed point set of A.

$I(A) := \{Y \subset X \mid A(Y) \subset, Y \neq \emptyset\}$-the family of the nonempty invariant subsets of A.

$$A^{n+1} := A \circ A^n \; A^0 = 1_X,\; A^1 = A,\; n \in \mathbb{N}.$$

Definition 2.1. [1],[2] *An operator A is weakly Picard operator (WPO) if the sequence*

$$(A^n(x))_{n \in N}$$

converges , for all $x \in X$ and the limit (which depend on x) is a fixed point of A.

Definition 2.2. [1],[2] *If the operator A is WPO and $F_A = \{x^*\}$ then by definition A is Picard operator.*

Definition 2.3. [1],[2] *If A is WPO, then we consider the operator*

$$A^\infty : X \to X, A^\infty(x) = \lim_{n\to\infty} A^n(x).$$

We remark that $A^\infty(X) = F_A$.

Definition 2.4. [1],[2] *Let be A an WPO and $c > 0$. The operator A is c-WPO if*

$$d(x, A^\infty(x)) \leq c \cdot d(x, A(x)).$$

We have the following characterization of the WPOs:

Theorem 2.1. [1],[2]*Let (X, d) be a metric space and $A : X \to X$ an operator. The operator A is WPO (c-WPO) if and only if there exists a partition of X,*

$$X = \bigcup_{\lambda \in \Lambda} X_\lambda$$

such that

(a) $X_\lambda \in I(A)$

(b) $A \mid X_\lambda : X_\lambda \to X_\lambda$ *is a Picard (c-Picard) operator, for all $\lambda \in \Lambda$.*

For the class of c-WPOs we have the following data dependence result:

Theorem 2.2. [1],[2] *Let (X, d) be a metric space and $A_i : X \to X, i = \overline{1,2}$ an operator. We suppose that :*

(i) *the operator A_i is $c_i - WPO$, $i = \overline{1,2}$.*

(ii) *there exists $\eta > o$ such that*

$$d(A_1(x), A_2(x)) \leq \eta, (\forall)x \in X.$$

Then

$$H(F_{A_1}, F_{A_2}) \leq \eta \; max\{c_1, c_2\}.$$

Here stands for Hausdorff-Pompeiu functional.

We have:

Lemma 2.1. [1],[2] *Let$(X, d, \leq)$ be an ordered metric space and $A : X \to X$ an operator such that:*

a)A is monotone increasing.

b)A is WPO.

Then the operator A^∞ is monotone increasing.

Lemma 2.2. [1],[2] *Let* $(X, d, \leq)$ *be an ordered metric space and* $A, B, C : X \to X$ *such that:*

(i)$A \leq B \leq C$.

(ii) the operators A,B,C are W.P.Os.

(iii) the operator B is monotone increasing.

Then

$$x \leq y \leq z \Longrightarrow A^{\infty}(x) \leq B^{\infty}(y) \leq C^{\infty}(z).$$

Lemma 2.3. [1],[2]*Let* $(X, d, \leq)$ *be an ordered metric space,* $A : X \longrightarrow X$ *an operator and* $x, y \in X$ *such that*

$$x < y, x \leq A(x), y \geq A(y).$$

We suppose that

(i) *A is W.P.O*;

(ii) *A is monotone increasing.*

Then

(a) $x \leq A^{\infty}(x) \leq A^{\infty}(y) \leq y$;

(b) $A^{\infty}(x)$ *is the minimal fixed point of A in* $[x, y]$ *and* $A^{\infty}(y)$ *is the maximal fixed point of A in* $[x, y]$

3. Boundary value problem

We consider the problem (1)+(2)

Theorem 3.1. *We suppose that*

(a) *The conditions* $(H_1) - (H_3)$ *are satisfied.*

(b) $\frac{1}{8}L_f(b-a)^2(1+2h) < 1$

Then the problem (1)+(2) has a unique solution in X.

Proof. The problem (1)+(2) is equivalent with the fixed point equation

$$B_f(x) = x, x \in X.$$

From the condition (H_2) we have

$$|B_f(x)(t) - B_f(y)(t)| \leq$$

$$\leq \int_a^b G(t,s)|f(s,x(s),\int_{s-h}^{s} x(u)du,\int_{s}^{s+h} x(u)du) - f(s,y(s),\int_{s-h}^{s} y(u)du,\int_{s}^{s+h} y(u)du)|ds \leq$$

$$\leq L_f \int_a^b G(t,s)[|x(s)-y(s)| + \int_{s-h}^{s} |x(u)-y(u)|du + \int_{s}^{s+h} |x(u)-y(u)|du]ds \leq$$

$$\leq \frac{L_f}{8}(b-a)^2\|x-y\|_C(1+2h),$$

for all $x, y \in X_{\varphi,\psi}$.

Then B_f is Picard operator on $X_{\varphi,\psi}$.

From this we have the conclusion.

Remark 3.1. *From the Theorem 3.1,using the Theorem 2.1, we have that the operator* E_f *is W.P.O and* $F_{E_f} \bigcap X_{\varphi,\psi} = \{x^\star_{\varphi,\psi}\}$ *where* $x^\star_{\varphi,\psi}$ *is the unique solution of (1)+(2).*

4. Inequalities of Čaplygin type

We have

Theorem 4.1. *We suppose that*

(a) *The conditions* $(H_1) - (H_3)$ *are satisfied;*

(b) $\frac{L_f}{8}(b-a)^2(1+2h) < 1$;

(c) *the operator* $f(t,\cdot,\cdot,\cdot) : \mathbb{R}^3 \longrightarrow \mathbb{R}$ *is monotone increasing for all* $t \in [a,b]$;

Let x *be a solution of the corresponding equation (1) and* y *a solution of the inequality*

$$-y''(t) \leq f(t,y(t),\int_{t-h}^{t} y(s)ds,\int_{t}^{t+h} y(s)ds).$$

Then

$$y(t) \leq x(t), (\forall)t \in [a-h,a] \cup [b,b+h] \Longrightarrow y \leq x$$

Proof. In the terms of the operator E_f we have that

$x = E_f(x)$,

$y \leq E_f(y)$

$w(y\mid_{[a-h,a]}, y\mid_{[b,b+h]}) \leq w(x\mid_{[a-h,h]}, x\mid_{[b,b+h]})$.

On the other hand, from the condition (c), using Lemma 2.1 we have that the operator E_f^{∞} is monotone increasing.

From this using Lemma 2.3 we have that

$$y \leq E_f^{\infty}(y) = E_f^{\infty}(\widetilde{w}(y)) \leq E_f^{\infty}(\widetilde{w}(x)) = x,$$

where, for $z \in X$,

$$\widetilde{w}(z) = \begin{cases} z(t) & , \quad t \in [a-h,a] \\ w(z\mid_{[a-h,h]}, z\mid_{[b,b+h]}) & , \quad t \in [a,b] \\ z(t) & , \quad t \in [b,b+h] \end{cases}$$

5. **Data dependence: Monotony**

Now we shall study the monotony of the solutions of the equation (1) with respect to initial conditions.We have

Theorem 5.1. *Let $f_i \in C([a,b]\times\mathbb{R}^3,\mathbb{R})$,$i=\overline{1,3}$ be as in the Theorem 3.1.We suppose that*

(a) $f_2(t,\cdot,\cdot,\cdot):\mathbb{R}^3 \longrightarrow \mathbb{R}$ *is monotone increasing;*

(b) $f_1 \leq f_2 \leq f_3$;

Let x_i, be a solution of the equation

$$-x''(t) = f_i(t, x(t), \int_{t-h}^{t} x(s)ds, \int_{t}^{t+h} x(s)ds), \quad i=\overline{1,3}.$$

If

$$x_1(t) \leq x_2(t) \leq x_3(t), (\forall) t \in [a-h,a] \cup [b,b+h]$$

then

$$x_1 \leq x_2 \leq x_3.$$

Proof. The operators E_{f_i} are *W.P.O.s*. From the condition (a) the operator E_{f_2} is monotone increasing. From (b) it follows that $E_{f_1} \leq E_{f_2} \leq E_{f_3}$.We remark that $x_i = E_{f_i}^{\infty}(\widetilde{w}(x_i))$, $i = \overline{1,3}$.

Now the proof follows from Lemma 2.2.

Theorem 5.2. *We consider the equation (1) under conditions of the Theorem 3.1. Let x, y be two solutions of the equations (1).We suppose that f is monotone increasing. If*

$$x(t) \leq y(t), \quad (\forall) t \in [a-h, a] \cup [b, b+h],$$

then

$$x \leq y,$$

on $[a-h, b+h]$.

Proof. The operator E_f is *W.P.O*. Because f is monotone increasing we obtain that E_f is monotone increasing. From Lemma 2.1 we have that E_f^{∞} is increasing. It follows that $E_f^{\infty}(\widetilde{w}(x)) \leq E_f^{\infty}(\widetilde{w}(y))$ and $x \leq y$.

6. **Data dependence: continuity**

Next, for $i = \overline{1,2}$, we consider the equations:

$$-x''(t) = f_i(t, x(t), \int_{t-h}^{t} x(s)ds, \int_{t}^{t+h} x(s)ds). \tag{5}$$

Theorem 6.1. *Let f_1 and f_2 be as in the Theorem 3.1.Let S_i be the solutions set of the equation (5) corresponding to $f_i, i = \overline{1,2}$.*

If $\eta > 0$ is such that

$$|f_1(t, u, v, w) - f_2(t, u, v, w)| \leq \eta,$$

for all $t \in [a, b], u, v, w \in \mathbb{R}$,

then

$$H(S_1, S_2) \leq \frac{\eta(b-a)^2}{8 - L(b-a)^2(1+2h)}$$

where $L := \max\{L_{f_1}, L_{f_2}\}$.

Proof. In the conditions of the Theorem 3.1 the operators $E_{f_i}, i = \overline{1,2}$ are $c_i - W.P.O.s$, with

$$c_i = (1-\alpha_i)^{-1}$$

where,

$$\alpha_i = \frac{1}{8} \cdot L_{f_i}(b-a)^2(1+2h).$$

From

$$|E_{f_1}(x)(t) - E_{f_2}(x)(t)| \leq$$

$$\leq \int_a^b G(t,s)|f_1(s,x(s),\int_{s-h}^{s} x(u)du,\int_{s}^{s+h} x(u)du) - f_2(s,x(s),\int_{s-h}^{s} x(u)du,\int_{s}^{s+h} x(u)du)|ds \leq$$

$$\leq \eta \int_a^b G(t,s)ds \leq \eta \frac{(b-a)^2}{8},$$

using the Theorem 2.2, we have the conclusions.

7. Smooth dependence on parameters

Consider the following boundary value problem with parameter

$$-x''(t) = f(t,x(t),\int_{t-h}^{t} x(s)ds,\int_{t}^{t+h} x(s)ds;\lambda), t\in[a,b], \lambda \in J \qquad (6)$$

$$\begin{cases} x(t) = \varphi(t) \quad , \quad t \in [a-h,a] \\ x(t) = \psi(t) \quad , \quad t\in[b,b+h] \end{cases}. \qquad (7)$$

We suppose that

(C_1) $J \subseteq \mathbb{R}$, a compact interval;

(C_2) $f \in C^1([a,b]\times\mathbb{R}^3\times J,\mathbb{R})$;

(C_3) There exists $L_f > 0$ such that:

$$|\frac{\partial f}{\partial u_i}(t,u_1,u_2,u_3;\lambda)| \leq L_f,$$

for all $t\in[a,b], u_i \in \mathbb{R}, i = \overline{1,3}$.

(C_4) $\varphi \in C([a-h,a]), \psi \in C([b,b+h])$.

(C_5) $\frac{1}{8}L_f(b-a)^2 < 1$

In the above conditions from Theorem 3.1 we have that the problem (6)+(7) has a unique solution, $x^\star(\cdot;\lambda)$.

Now we prove that $x^\star(t,\cdot) \in C^1(J)$.For this we consider the equation

$$-x''(t,\lambda) = f(t, x(t,\lambda), \int_{t-h}^{t} x(s,\lambda)ds, \int_{t}^{t+h} x(s,\lambda)ds; \lambda), \tag{8}$$

for all $t \in [a,b], \lambda \in J, x \in C([a-h,b+h] \times J)$.

The problem (8)+(7) is equivalent with

$$x(t,\lambda) = \begin{cases} \varphi(t), & t \in [a-h,a], \lambda \in J \\ w(\varphi,\psi)(t) + \int_a^b G(t,s) f(s, x(s,\lambda), & \\ \int_{s-h}^{s} x(u,\lambda)du, \int_{s}^{s+h} x(u,\lambda)du)ds, & t \in [a,b], \lambda \in J \\ x(t), & t \in [b,b+h], \lambda \in J \end{cases} . \tag{9}$$

We consider the operator

$$B : C([a-h,b+h] \times J) \longrightarrow C([a-h,b+h] \times J),$$

where

$B(x)(t)$ =second part of (9).

Let $X := C([a-h,b+h] \times J)$ and let, $\|\cdot\|$, be the Chebyshev norm on X. It is clear that in the condition $(C_1) - (C_5)$ the operator B is Picard operator.

Let $x^\star$ be the unique fixed point of B.We suppose that there exists $\dfrac{\partial x^\star}{\partial \lambda}$.Then for (9) we have that

$$\frac{\partial x^\star}{\partial \lambda}(t,\lambda) =$$

$$\begin{cases} 0, & t\in[a-h,a],\lambda\in J\\ \int\limits_a^b G(t,s)\frac{\partial f}{\partial u_1}(s,x^\star(s,\lambda),\int\limits_{s-h}^{s}x^\star(u,\lambda)du,\int\limits_{s}^{s+h}x^\star(u,\lambda)du)\cdot & \\ \cdot\frac{\partial x^\star}{\partial\lambda}(s,\lambda)ds+ & \\ +\int\limits_a^b G(t,s)\frac{\partial f}{\partial u_2}(s,x^\star(s,\lambda),\int\limits_{s-h}^{s}x^\star(u,\lambda)du,\int\limits_{s}^{s+h}x^\star(u,\lambda)du)\cdot & \\ \cdot\int\limits_{s-h}^{s}\frac{\partial x^\star}{\partial\lambda}(u,\lambda)duds+ & \\ +\int\limits_a^b G(t,s)\frac{\partial f}{\partial u_3}(s,x^\star(s,\lambda),\int\limits_{s-h}^{s}x^\star(u,\lambda)du,\int\limits_{s}^{s+h}x^\star(u,\lambda)du)\cdot & \\ \cdot\int\limits_{s}^{s+h}\frac{\partial x^\star}{\partial\lambda}(u,\lambda)duds+ & \\ +\int\limits_a^b G(t,s)\frac{\partial f}{\partial\lambda}(s,x^\star(s,\lambda),\int\limits_{s-h}^{s}x^\star(u,\lambda)du,\int\limits_{s}^{s+h}x^\star(u,\lambda)du)ds, & t\in[a,b],\lambda\in J\\ 0, & t\in[b,b+h] \end{cases}$$

This relation suggest us to consider the following operator

$$C : X\times X\longrightarrow X$$

$$(x,y)\longrightarrow C(x,y),$$

where

$$C(x,y)(t,\lambda)=$$

$$\begin{cases} 0, & t\in[a-h,a],\lambda\in J\\ \int\limits_a^b G(t,s)\frac{\partial f}{\partial u_1}(s,x(s,\lambda),\int\limits_{s-h}^{s}x(u,\lambda)du,\int\limits_{s}^{s+h}x(u,\lambda)du)\cdot & \\ \cdot y(s,\lambda)ds+ & \\ +\int\limits_a^b G(t,s)\frac{\partial f}{\partial u_2}(s,x(s,\lambda),\int\limits_{s-h}^{s}x(u,\lambda)du,\int\limits_{s}^{s+h}x(u,\lambda)du)\cdot & \\ \cdot\int\limits_{s-h}^{s}y(u,\lambda)duds+ & \\ +\int\limits_a^b G(t,s)\frac{\partial f}{\partial u_3}(s,x(s,\lambda),\int\limits_{s-h}^{s}x(u,\lambda)du,\int\limits_{s}^{s+h}x(u,\lambda)du)\cdot & \\ \cdot\int\limits_{s}^{s+h}y(u,\lambda)duds+ & \\ +\int\limits_a^b G(t,s)\frac{\partial f}{\partial\lambda}(s,x(s,\lambda),\int\limits_{s-h}^{s}x(u,\lambda)du,\int\limits_{s}^{s+h}x(u,\lambda)du)ds, & t\in[a,b],\lambda\in J\\ 0, & t\in[b,b+h],\lambda\in J \end{cases}$$

In this way we have that the operator

$$A : X \times X \longrightarrow X \times X$$

$$(x, y) \longrightarrow (B(x), C(x, y)),$$

where B is Picard operator and $C(x, \cdot) : X \longrightarrow X$ is a $\alpha-$ contraction, with

$$\alpha = L_f(1 + 2h)\frac{(b-a)^2}{8}.$$

From the theorem of fibre contraction(see [1],[5]) we have that the operator A is a Picard operator.So the sequences

$$x_{n+1} = B(x_n),$$

$$y_{n+1} = C(x_n, y_n),$$

converges uniformly (with respect to $t \in [a-h, b+h], \lambda \in J$) to $(x^\star, y^\star) \in F_A$, for all $x_0, y_0 \in C([a-h, b+h] \times J)$.

If we take, $x_0 = 0$, $y_0 = \dfrac{\partial x_0}{\partial \lambda} = 0$, then, $y_1 = \frac{\partial x_1}{\partial \lambda}$.

By induction,we prove that

$$y_n = \frac{\partial x_n}{\partial \lambda}, (\forall) n \in \mathbb{N}$$

Thus

$$x_n \longrightarrow x^\star,\ as\ n \longrightarrow \infty,\quad uniformly,$$

$$\frac{\partial x_n}{\partial \lambda} \longrightarrow y^\star\ as\ n \longrightarrow \infty,\quad uniformly.$$

These imply that there exists $\frac{\partial x^\star}{\partial \lambda}$ and, $\frac{\partial x^\star}{\partial \lambda} = y^\star$.

From the above consideration, we have that

Theorem 7.1. *Consider the problem (7)+(8) in the conditions* $(C_1) - (C_5)$*.Then*

(a) *The problem, (7)+(8), has in* $C([a-h, b+h])$ *a unique solution* $x^\star$.

(b) $x^\star(t, \cdot) \in C^1(J), (\forall) t \in [a-h, b+h]$.

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