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NONEXISTENCE OF NONTRIVIAL PERIODIC SOLUTIONS FOR SEMILINEAR WAVE EQUATIONS

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Abstract. The semilinear wave equation

$$\begin{cases} \Box u + g(t,x,u) = \lambda u, & \text{in } \Omega \\ u(t,0) = u(t,\pi) = 0, & t \in [0,2\pi] \\ u \text{ is } 2\pi\text{-periodic in } t \end{cases}$$

is considered as an eigenvalue problem with parameter λ. The nonexistence of nontrivial periodic solutions in case $\lambda = 4k+2$ is treated and the other cases are studied under some uniform boundedness conditions on g.

1. Introduction

A large number of papers are devoted to the study of the nonexistence of nontrivial solutions (*i.e.* eigenfunctions) of semilinear eigenvalue problems. The great importance follows from the fact that these are closed related with the theory of bifurcation, in special with finding the bifurcation points and the bifurcation intervals. Such results for semilinear elliptic equations was established for example by Chiappinelli in [1] using critical point theory of Ljusternik Schnirelmann or by Berger in [2] in connection with bifurcation theory.

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We consider here the semilinear wave equation in the form

$$\begin{cases} u_{tt} - u_{xx} + g(t,x,u) = \lambda u, & \text{in } \Omega \\ u(t,0) = u(t,\pi) = 0, & t \in [0,2\pi] \\ u \text{ is } 2\pi\text{-periodic in } t \end{cases} , \tag{1.1}$$

where $\Omega = (0,2\pi) \times (0,\pi)$ and λ is a real parameter.

Let $\widetilde{C}^2$ be the space of twice continuously differentiable functions $v : \overline{\Omega} \to \mathbf{R}$ such that $v(t,0) = v(t,\pi) = 0$ and $v(\cdot,x)$ is 2π-periodic. By $H = L^2(\Omega)$ denote the completion of the space $\widetilde{C}^2$ endowed with the inner product

$$(v,w) = \int_\Omega vw \quad , \quad v,w \in \widetilde{C}^2$$

and the corresponding norm

$$\|v\| = \sqrt{(v,v)} \quad , \quad v \in \widetilde{C}^2.$$

The set $(\psi_{nk})_{(n,k)\in \mathbf{N}\times\mathbf{Z}}$ forming an orthonormal basis in H and consists of eigenfunctions of the linear operator

$$\Box = \partial_t^2 - \partial_x^2$$

is defined by

$$\psi_{nk}(x,t) = \begin{cases} \frac{\sqrt{2}}{\pi} \sin nx \sin kt, & (n,k) \in \mathbf{N} \times \mathbf{N} \\ \frac{1}{\pi} \sin nx, & n \in \mathbf{N},\ k = 0 \\ \frac{\sqrt{2}}{\pi} \sin nx \cos kt, & n \in \mathbf{N},\ -k \in \mathbf{N} \end{cases} .$$

Obviously,

$$\Box \psi_{nk} = (n^2 - k^2)\psi_{nk}.$$

Let $L : D(L) \subset H \to H$ be given by

$$Lu = \sum_{n=1}^{\infty} \sum_{k=-\infty}^{\infty} (n^2 - k^2)(u,\psi_{nk})\psi_{nk} \,,$$

with the domain

$$D(L) = \left\{ u \in H \mid \sum_{n=1}^{\infty} \sum_{k=-\infty}^{\infty} (n^2 - k^2) \left|(u, \psi_{nk})\right|^2 < \infty \right\}.$$

The operator L is densely defined, selfadjoint and with a closed range. Its spectrum

$$\sigma(L) = \left\{ \lambda_{nk} = n^2 - k^2 \mid (n,k) \in \mathbf{N} \times \mathbf{Z} \right\}$$

is unbounded from above and below and any non-zero eigenvalue has a finite algebraic multiplicity. More precisely,

$$\sigma(L) = \mathbf{Z} \setminus \{4k+2 \mid k \in \mathbf{Z}\}$$

which follows easily.

Assume that $g : \Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function and there exists $c > 0$ satisfying

$$|g(t,x,u)| \le c\,|u| \quad , \quad \forall\, (t,x) \in \Omega,\ u \in \mathbf{R}.$$

In consequence,

$$g(t,x,0) = 0 \quad , \qquad \forall\, (t,x) \in \Omega$$

so the problem (1.1) has the trivial solution $u = 0$.

The Nemytskii operator

$$(Su)(t,x) = g(t,x,u(t,x))$$

generated by g is bounded and continuous from $L^2(\Omega)$ into itself. Consequently, the generalized solution of (1.1) is any function $u \in L^2(\Omega)$ such that

$$(u, v_{tt} - v_{xx}) + (Su, v) = \lambda(u,v)\ ,\ \ \forall\, v \in \widetilde{C}^2.$$

From now, we will write this equation in the operator form

$$Lu + S(u) = \lambda u, \tag{1.2}$$

where L inherets the properties of the generalized d'Alembertian with periodic boundary conditions.

2. **The results**

First we give a result in case $\lambda \in \mathbf{Z} \setminus \sigma(L)$, thus $\lambda = 4k+2$, $k \in \mathbf{Z}$.

THEOREM 2.1. *Assume that*

$$|g(t,x,u)| \leq c\,|u| \quad , \quad \forall\,(t,x) \in \Omega,\; u \in \mathbf{R}$$

for some $c \in (0,1)$. *Then the problem*

$$\begin{cases} u_{tt} - u_{xx} + g(t,x,u) = (4k+2)u, & \text{in } \Omega \\ u(t,0) = u(t,\pi) = 0, & t \in [0,2\pi] \\ u \text{ is } 2\pi\text{-periodic in } t \end{cases}$$

has no nontrivial solutions. Consequently, $4k+2$ *are not bifurcation points.*

Proof. Denote $\lambda_k = 4k+2$. The closest eigenvalues from $4k+2$ are $4k+1$ and $4k+3$, so

$$\text{dist}(\lambda_k, \sigma(L)) = 1.$$

Let us suppose by contrary that there exists a nontrivial solution u of the equation

$$Lu + S(u) = \lambda_k u.$$

Since L is selfadjoint, then for $\lambda \notin \sigma(L)$, the resolvent $(L-\lambda I)^{-1}$ of L at λ is bounded, linear map with norm

$$\left|\left|(L-\lambda I)^{-1}\right|\right| = \frac{1}{\text{dist}(\lambda, \sigma(L))},$$

(Kato [3], pp.272). Therefore,

$$(L - \lambda_k I)u = -S(u)$$

or

$$u = -(L - \lambda_k I)^{-1} S(u).$$

By taking the norm, we derive

$$||u|| = \left|\left|(L-\lambda_k I)^{-1} S(u)\right|\right| \leq$$

$$\leq \left|\left|(L-\lambda_k I)^{-1}\right|\right| \cdot ||S(u)|| \leq \frac{1}{\text{dist}(\lambda_k, \sigma(L))} \cdot c\,||u||\,.$$

Now, if divide by $||u|| \neq 0$, we obtain

$$1 \leq \frac{1}{\text{dist}(\lambda_k, \sigma(L))} \cdot c$$

or

$$c \geq \text{dist}(\lambda_k, \sigma(L)) = 1,$$

a contradiction. This shows that $u = 0$. □

Further we will give a more general result. For each real number $\lambda \notin \sigma(L)$, $\lambda \neq 4k+2$, we study cases when $\lambda \in I_k$, where the intervals I_k are given in the next table. If denote by $\mu_k \in \sigma(L)$ the closest eigenvalue from λ, we have the following situation:

I_k	μ_k	$\text{dist}(\lambda, \sigma(L))$
$(4k, 4k+1)$	$4k$ or $4k+1$	$\min\{4k+1-\lambda, \lambda-4k\}$
$(4k+1, 4k+2)$	$4k+1$	$\lambda - 4k - 1$
$(4k+2, 4k+3)$	$4k+3$	$4k+3-\lambda$
$(4k+3, 4k+4)$	$4k+3$ or $4k+4$	$\min\{4k+4-\lambda, \lambda-4k-3\}$

Now we are in position to give the following

Theorem 2.2. *Assume that* $\lambda \in (i, i+1)$, $i \in \mathbf{Z}$ *and*

$$|g(t,x,u)| \leq c\,|u| \quad , \quad \forall\,(t,x) \in \Omega,\ u \in \mathbf{R}$$

for some $0 < c < c(\lambda)$, *where*

$$c(\lambda) = \begin{cases} \lambda - i & , \ if \quad i = 4k+1 \\ i+1-\lambda & , \ if \quad i = 4k+2 \\ \min\{i+1-\lambda, \lambda - i\} & , \ if \quad i = 4k \quad or \quad i = 4k+3 \end{cases} .$$

Then the problem

$$\begin{cases} u_{tt} - u_{xx} + g(t,x,u) = \lambda u, & \text{in } \Omega \\ u(t,0) = u(t,\pi) = 0, & t \in [0, 2\pi] \\ u \text{ is } 2\pi\text{-periodic in } t \end{cases}$$

has no nontrivial solutions. Consequently, these points λ *are not bifurcation points.*

Proof. We can easily see that

$$c(\lambda) = \text{dist}(\lambda, \sigma(L)),$$

so

$$0 < c < c(\lambda).$$

By assuming that there exists a nontrivial solution of the problem (1.2), we obtain as above that

$$u = -(L - \lambda I)^{-1} S(u)$$

and finally

$$1 \leq \frac{1}{\text{dist}(\lambda, \sigma(L))} \cdot c \Leftrightarrow c \geq \text{dist}(\lambda, \sigma(L)) = c(\lambda),$$

contradiction. Consequently, the problem admits only the solution $u = 0$. □

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