STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LI**, Number 2, June 2006

PFAFFIAN TRANSFORMATIONS

MARIA TERESA CALAPSO, FILIP DEFEVER, AND RADU ROŞCA

Abstract. Geometrical and structural properties are proved for manifolds possessing a particular locally conformal almost cosymplectic structure.

1. Introduction

Let $M(g, \Omega, \phi, \eta, \xi)$ be an $2m + 1$-dimensional Riemannian manifold with metric tensor g and associated Levi-Civita connection ∇. The quadruple $(\Omega, \phi, \xi, \eta)$ consists of a structure 2-form Ω of rank $2m$, an endomorphism ϕ of the tangent bundle, the Reeb vector field ξ, and its corresponding Reeb covector field η, respectively.

We assume that the 2-form Ω satisfies the relation

$$d\Omega = \lambda\ \eta \wedge \Omega\,, \tag{1}$$

where λ is constant, and that the 1-form η is given by

$$\eta = \lambda df\,, \tag{2}$$

for some scalar function f on M. We may therefore notice that a locally conformal almost cosymplectic structure [7] [10] is defined on the manifold M.

In addition, we assume that the field ϕ of endomorphisms of the tangent spaces defines a quasi-Sasakian structure, thus realizing in particular the identity

$$\phi^2 = -\mathrm{Id} + \eta \otimes \xi\,.$$

Received by the editors: 22.11.2005.

2000 *Mathematics Subject Classification.* 53B20.

Key words and phrases. Pfaffian transformation, almost cosymplectic structure.

Moreover, we will assume the presence on M of a structure vector field X satisfying the property

$$\nabla X = f dp + \lambda \nabla \xi \,. \tag{3}$$

In the present paper various properties involving the above mentioned objects are studied. In particular, for the Lie differential of Ω and η with respect to X, one has

$$\begin{aligned} \mathcal{L}_X \eta &= 0\,, \\ \mathcal{L}_X \Omega &= 0\,, \end{aligned}$$

which shows that η and Ω define Pfaffian transformations [3].

2. **Preliminaries**

Let (M, g) be an n-dimensional Riemannian manifold and let ∇ be the covariant differential operator defined by the metric tensor. We assume in the sequel that M is oriented and that the connection ∇ is symmetric.

Let $\Gamma TM = \Xi(M)$ be the set of sections of the tangent bundle TM, and

$$\flat : TM \xrightarrow{\flat} T^*M \qquad \text{and} \qquad \sharp : TM \xleftarrow{\sharp} T^*M$$

the classical isomorphisms defined by the metric tensor g (i.e. $^\flat$ is the index lowering operator, and $^\sharp$ is the index raising operator).

Following [12], we denote by

$$A^q(M, TM) = \Gamma\mathrm{Hom}(\Lambda^q TM, TM)\,,$$

the set of vector valued q-forms ($q < \dim M$) , and we write for the covariant derivative operator with respect to ∇

$$d^\nabla : A^q(M, TM) \to A^{q+1}(M, TM)\,. \tag{4}$$

It should be noticed that in general $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$, unlike $d^2 = d \circ d = 0$.

Furthermore, we denote by $dp \in A^1(M, TM)$ the canonical vector valued 1-form of M, which is also called the soldering form of M [3]; since ∇ is assumed to be symmetric, we recall that the identity $d^\nabla(dp) = 0$ is valid.

The operator

$$d^\omega = d + e(\omega)\,,$$

acting on ΛM is called the cohomology operator [5]. Here, $e(\omega)$ means the exterior product by the closed 1-form ω, i.e.

$$d^\omega u = du + \omega \wedge u\,,$$

with $u \in \Lambda M$. A form $u \in \Lambda M$ such that

$$d^\omega u = 0\,,$$

is said to be d^ω-closed, and ω is called the cohomology form.

A vector field $X \in \Xi(M)$ which satisfies

$$d^\nabla(\nabla X) = \nabla^2 X = \pi \wedge dp \in A^2(M, TM)\,, \qquad \pi \in \Lambda^1 M\,, \tag{5}$$

and where π is conformal to $X^\flat$, is defined to be an exterior concurrent vector field [14]. In this case, if $\mathcal{R}$ denotes the Ricci tensor field of ∇, one has

$$\mathcal{R}(X, Z) = -2m\lambda^3(\kappa + \eta) \wedge dp\,, \qquad Z \in \Xi(M)$$

3. **Geometrical properties**

In terms of a local field of adapted vectorial frames $\mathcal{O} = \mathrm{vect}\{e_A | A = 0, \cdots 2m\}$ and its associated coframe $\mathcal{O}^* = \mathrm{covect}\{\omega^A | A = 0, \cdots 2m\}$, the soldering form dp can be expressed as

$$dp = \sum_{A=0}^{2m} \omega^A \otimes e_A\,;$$

and we recall that E. Cartan's structure equations can be written as

$$\nabla e_A = \sum_{B=0}^{2m} \theta_A^B \otimes e_B \,, \tag{6}$$

$$d\omega^A = -\sum_{B=0}^{2m} \theta_B^A \wedge \omega^B \,, \tag{7}$$

$$d\theta_B^A = -\sum_{C=0}^{2m} \theta_B^C \wedge \theta_C^A + \Theta_B^A \,. \tag{8}$$

In the above equations θ (respectively Θ) are the local connection forms in the tangent bundle TM (respectively the curvature 2-forms on M).

In terms of the frame fields $\mathcal{O}$ and $\mathcal{O}^*$ with $e_0 = \xi$ and $\omega^0 = \eta$, the structure vector field X and the 2-form Ω can be expressed as

$$X = \sum_{a=1}^{2m} X^a e_a \,, \tag{9}$$

$$\Omega = \sum_{i=1}^{m} \omega^i \wedge \omega^{i^*} \,, \qquad i^* = i + m \,. \tag{10}$$

Taking the Lie differential of Ω and η with respect to X, one calculates

$$\mathcal{L}_X \eta = 0 \,, \tag{11}$$

$$\mathcal{L}_X \Omega = 0 \,. \tag{12}$$

According to [6] the above equations (11) and (12) prove that that η and Ω define a Pfaffian transformation [3].

Next, by (2) one gets that

$$\theta_0^a = \lambda \omega^a \,. \tag{13}$$

Since we also assume that

$$\nabla X = f dp + \lambda \nabla \xi \,, \tag{14}$$

we further also derive that

$$\nabla \xi = \lambda (dp - \eta \otimes \xi) \,. \tag{15}$$

Since the q-th covariant differential $\nabla^q Z$ of a vector field $Z \in \Xi(M)$ is defined inductively, i.e.

$$\nabla^q Z = d^{\nabla}(\nabla^{q-1} Z)\,,$$

this yields

$$\nabla^2 \xi = \lambda^2 \eta \otimes dp\,, \tag{16}$$

$$\nabla^3 \xi = 0\,. \tag{17}$$

Hence, one may say that the 3-covariant Reeb vector field ξ is vanishing.

Next, by (13), one derives that

$$\nabla^2 X = \lambda^3 (df + \eta) \wedge dp = \frac{1+\lambda}{\lambda} \eta \wedge dp\,, \tag{18}$$

and consecutively one gets that

$$\nabla^3 X = 0\,. \tag{19}$$

This shows that both vector fields ξ and X together define a 3-vanishing structure.

Moreover, by reference to [13], it follows from (18) that one may write that

$$\nabla^2 X = -\frac{1}{2m} \mathrm{Ric}(X) - X^{\flat} \wedge dp\,, \tag{20}$$

where Ric is the Ricci tensor.

Reminding that by the definition of the operator ϕ

$$\phi e_i = e_{i^*} \qquad i \in \{1\,, \cdots m\}\,,$$

$$\phi e_{i^*} = -e_i \qquad i^* = i + m\,,$$

one can check that indeed $\phi^2 = -\mathrm{Id}$. Acting with ϕ on the vector field X, one obtains in a first step that

$$\phi X = \sum_{i=1}^{m} X^i e_{i^*} - X^{i^*} e_i \qquad i^* = i + m\,. \tag{21}$$

Calculating the Lie derivative of ϕ w.r.t. ξ, one gets

$$(\mathcal{L}_\xi \phi)X = [\xi\,, \phi X] - \phi[\xi\,, X]\,. \tag{22}$$

Since clearly

$$[\xi\,, \phi X] = 0\,, \tag{23}$$

there follows that

$$(\mathcal{L}_\xi \phi)X = 0\,. \tag{24}$$

Hence, the Jacobi bracket corresponding to the Reeb vector field ξ vanishes.

By reference to the definition of the divergence

$$\text{div } Z = \sum_{A=0}^{2m} \omega^A \left(\nabla_{e_A} Z\right)$$

one obtains in the case under consideration that

$$\text{div } X = 2m(\lambda + f^2)\,, \tag{25}$$

and

$$\text{div } \phi X = 0\,. \tag{26}$$

Calculating the differential of the dual form $X^\flat$ of X, one gets

$$dX^\flat = \sum_{a=1}^{2m} \left(dX^a + \sum_{b=1}^{2m} X^b \theta_b^a \right) \wedge \omega^a\,. \tag{27}$$

Since

$$dX^a + \sum_{b=1}^{2m} X^b \theta_b^a = \lambda \omega^a\,, \tag{28}$$

one has that

$$dX^\flat = 0\,, \tag{29}$$

which means that the Pfaffian $X^\flat$ is closed. This implies that $X^\flat$ is an eigenfunction of the Laplacian Δ, and one can write that

$$\Delta X^\flat = f||X||^2 X^\flat\,.$$

If we set

$$2l = ||X||^2 \,, \tag{30}$$

one also derives by (28) that

$$dl = \lambda X^\flat \,. \tag{31}$$

From (31) it follows that $dX^\flat = 0$ which is indeed in accordance with (29).

Returning to the operator ϕ, one calculates that

$$\nabla(\phi X) = \lambda\phi dp - \sum_{i=1}^{m}\left(\sum_{a=1}^{2m}(X^a\theta^i_a)\otimes e_{i^*} + \sum_{a=1}^{2m}(X^a\theta^{i^*}_a)\otimes e_i\right) \,. \tag{32}$$

Hence there follows that

$$[\xi\,, X] \quad = \quad \rho\xi - \phi C \,, \tag{33}$$

$$[\xi\,, \phi X] \quad = \quad ((C^0)^2 + C^0(1-\lambda))\xi \,, \tag{34}$$

$$[X\,, \phi X] \quad = \quad \nabla_\xi \phi C = C^0\xi - C \tag{35}$$

which shows that the triple $\{X\,, \xi\,, \phi X\}$ defines a 3-distribution on M.

It is also interesting to draw the attention on the fact that X possesses the following property. From (14) and (15) one derives that

$$\nabla_X X = fX \,, \tag{36}$$

which means that X is an affine geodesic vector field.

Finally, if we denote by Σ the exterior differential system which defines X, it follows by Cartan's test [1] that the characteristic numbers are

$$r = 3\,, \qquad s_0 = 1\,, \qquad s_1 = 2\,.$$

Since $r = s_0 + s_1$, it follows that Σ is in involution and the existence of X depends on an arbitrary function of 1 argument.

Summarizing, we can organize our results into the following

Theorem 3.1. *Let M be a $2m+1$-dimensional Riemannian manifold and let ∇ be the Levi-Civita connection and ξ be the Reeb vector field and η the Reeb covector field on M. On has the following properties:*

- **(i):** *ξ and X define a 3-vanishing structure;*
- **(ii):** *the Jacobi bracket corresponding to ξ vanishes;*
- **(iii):** *the harmonic operator acting on $X^\flat$ gives*

$$\Delta X^\flat = f||X||^2 X^\flat \,,$$

  which proves that $X^\flat$ is an eigenfunction of Δ, having $f||X||^2$ as eigenvalue;
- **(iv):** *the 2-form Ω and the Reeb covector η define a Pfaffian transformation, i.e.*

$$\begin{aligned} \mathcal{L}_X\Omega &= 0\,, \\ \mathcal{L}_X\eta &= 0\,; \end{aligned}$$

- **(v):** *the Ricci tensor is determined by $\nabla^2 X$;*
- **(vi):** *one has*

$$\nabla_X X = fX\,, \qquad f = scalar\,,$$

  which shows that X is an affine geodesic;
- **(vii):** *the triple $\{X, \xi, \phi X\}$ is a 3-distribution on M and is in involution in the sense of Cartan.*

4. The structure 2-form Ω

In the present section, we derive some properties of the structure 2-form Ω. First, we recall that one has

$$d\Omega = \lambda\eta \wedge \Omega\,, \qquad \lambda = \text{constant}\,. \tag{37}$$

By Lie differentiation with respect to X, one gets

$$\mathcal{L}_X\Omega = 0\,. \tag{38}$$

Further, since $i_\xi \Omega = 0$, one calculates that

$$\begin{aligned} \mathcal{L}_\xi \Omega &= \lambda \Omega\,, \\ d\,(\mathcal{L}_\xi \Omega) &= \lambda^2 \eta \wedge \Omega\,. \end{aligned}$$

Moreover, by the Lie bracket [,] one also has that

$$i_{[X,\xi]}\Omega = 0\,. \tag{39}$$

Next, we consider the vector field ϕX. By (32), one calculates that

$$\mathcal{L}_{\phi X}\Omega = -2\lambda\eta \wedge X^\flat\,, \qquad \lambda = \text{constant}\,. \tag{40}$$

Since $X^\flat$ is closed, this yields

$$d\,(\mathcal{L}_{\phi X}\Omega) = 0\,. \tag{41}$$

This shows that ϕX defines a relative conformal transformation [15] [8] of Ω. In addition, one also derives that

$$\mathcal{L}_{[X,\xi]}\Omega = \mathcal{L}_X \mathcal{L}_\xi \Omega - \mathcal{L}_\xi \mathcal{L}_X \Omega = \mathcal{L}_X \mathcal{L}_\xi \Omega$$

and

$$\mathcal{L}_{fX}\Omega = f\mathcal{L}_X\Omega + df \wedge i_X\Omega = df \wedge i_X \Omega$$

Theorem 4.1. *The structure 2-form Ω satisfies the following relations*

(i):

$$d\Omega = \lambda\eta \wedge \Omega$$

(ii):

$$\begin{aligned} \mathcal{L}_\xi \Omega &= \lambda \Omega \\ d\,(\mathcal{L}_\xi \Omega) &= \lambda^2 \eta \wedge \Omega \end{aligned}$$

(iii):

$$i_{[X,\xi]}\Omega = 0$$

(iv):

$$d\,(\mathcal{L}_{\phi X}\Omega) = 0$$

(v):

$$\mathcal{L}_{[X,\xi]}\Omega = \mathcal{L}_X \mathcal{L}_\xi \Omega$$

(vi):

$$\mathcal{L}_{fX}\Omega = df \wedge i_X \Omega$$

References

[1] Cartan, E., *Systemes differentiels extérieurs et leurs applications géometriques*, Hermann, Paris (1945).

[2] Defever, F., Rosca, R., *On a class of even-dimensional manifolds structured by a $\mathcal{T}$-parallel connection*, Tsukuba Math. J. **25** (2001) 359-369.

[3] Dieudonné, J., *Treatise on Analysis*, Vol. 4, Academic Press, New York (1974).

[4] Godbillon, C., *Géometrie différentielle et mécanique analytique*, Herman, Paris (1969).

[5] Guedira, F., Lichnerowicz, A., *Geometrie des algebres de Lie locales de Kirilov*, J. Math. Pures Appl. **63**(1984). 407-494.

[6] Lichnerowicz, A., *Les relations intégrales d'invariance et leurs applications a la dynamique*, Bull. Sci. Math. **70**(1946), 82-95.

[7] Matsumoto, K., Mihai, I., Rosca, R., *A certain locally conformal almost cosymplectic manifold and its submanifolds*, Tensor N.S. **51**(1992), 91-102.

[8] Matsumoto, K., Mihai, I., Rosca, R., *On gradient almost torse forming vector fields*, Math. J. Toyama Univ. **19**(1996), 149-157.

[9] Mihai, I., Rosca, R., Verstraelen, L., *Some aspects of the differential geometry of vector fields*, Padge, K. U. Brussel **2**(1996).

[10] Olszak, Z., Rosca, R., *Normal locally conformal almost cosymplectic manifolds*, Publ. Math. Debrecen **39**(1991), 3-4.

[11] Petrovic, M., Rosca, R., Verstraelen, L., *Exterior concurrent vector fields on Riemannian manifolds,* Soochow J. Math. **15**(1989), 179-187.

[12] Poor, W.A., *Differential Geometric Structures*, Mc Graw Hill, New York (1981).

[13] Rosca, R., *On parallel conformal connections*, Kodai Math. J. **2**(1979), 1-9.

[14] Rosca, R., *Exterior concurrent vectorfields on a conformal cosymplectic manifold admitting a Sasakian structure*, Libertas Math. (Univ. Arlington, Texas) **6**(1986), 167-174.

[15] Rosca, R., *On conformal cosymplectic quasi-Sasakian manifolds*, Giornate di Geometria, Univ. Messina (1988).

Seminario Matematico, Universita di Messina, C. Da Papardo - Salita Sperone 31, 98165 S. Agata - Messina, Italy

Departement Industriële Wetenschappen en Technologie, Katholieke Hogeschool Brugge-Oostende, Zeedijk 101, 8400 Oostende, Belgium

59 Avenue Emile Zola, 75015 Paris, France