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ON THE ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS OF A CERTAIN FOURTH-ORDER DIFFERENTIAL EQUATION

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Abstract. The main purpose of this paper is to establish sufficient conditions under which any solution of (1.1) is uniformly bounded and tend to zero as $t \to \infty$.

1. Introduction and Statement of the Result

As we know from the relevant literature, up to now, many results have been obtained on the asymptotic behaviour of solutions of certain non-linear differential equations of the fourth- order (see, e.g., Hara [2-4], Abou-el-Ela, A.M.A and Sadek, A.I. [1] , Sadek and Elaiw [7] and Tunç, C. and Tunç, E. [5], Tunç [9-10].

In this paper we investigate the asymptotic behaviour of solutions of the real non-linear ordinary differential equation of fourth order:

$$x^{(4)} + a(t)f_1(x,\dot{x},\ddot{x},\dddot{x}) + b(t)f_2(x,\dot{x},\ddot{x}) + c(t)f_3(x,\dot{x}) + d(t)f_4(x) = p(t,x,\dot{x},\ddot{x},\dddot{x}), \tag{1.1}$$

in which the functions $a, b, c, d, f_1, f_2, f_3, f_4,$ and p are continuous for all values of their respective arguments. We assume that the functions a, b, c, d are positive definite and differentiable in $R^+ = [0, \infty)$, and that the derivatives $\frac{\partial}{\partial y}f_2(x,y,z)$, $\frac{\partial}{\partial x}f_3(x,y)$, $\frac{\partial}{\partial y}f_3(x,y)$, $\frac{\partial}{\partial x}f_2(x,y,z)$ and $f_4'(x)$ exist and are continuous for all x, y, z and w. The dots indicate differentiation with respect to t.

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The main purpose of this work is to prove the following

Theorem. *In addition to the basic assumptions on the functions* $a, b, c, d,$ $f_1, f_2, f_3, f_4,$ *and* p, *suppose that*

(i) $A \geq a(t) \geq a_0 > 0, B \geq b(t) \geq b_0 > 0, C \geq c(t) \geq c_0 > 0, D \geq d(t) \geq d_0 > 0$ *for* $t \in R^+$;

(ii) $0 < \left[\frac{f_1(x,y,z,w)}{w} - \alpha_1\right] \leq \min\left\{\frac{c_0\alpha_3}{2\sqrt{3}\alpha_4 DA}\sqrt{(\varepsilon-\varepsilon_0)c_0\alpha_3\varepsilon a_0\alpha_1}, \frac{\sqrt{6}}{3A}\sqrt{\frac{\delta_0\varepsilon}{c_0\alpha_3}}\right\}$ *for all* x, y, z, w; $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_3 > 0$, $\alpha_4 > 0$;

(iii) $f_3(x, 0) = 0$ *and* $\frac{\partial}{\partial y} f_3(x, y) \geq \alpha_3 > 0$ *for all* x *and* y;

(iv) *There is a finite constant* $\delta_0 > 0$ *such that*

$$a_0 b_0 c_0 \alpha_1 \alpha_2 \alpha_3 - C^2 \alpha_3 \frac{\partial}{\partial y} f_3(x, y) - A^2 D \alpha_1^2 \alpha_4 \geq \delta_0$$

for all x, y *and* z;

(v) $0 \leq \frac{\partial}{\partial y} f_3(x,y) - \frac{f_3(x,y)}{y} \leq \delta_1 < \frac{2D\delta_0\alpha_4}{Ca_0\alpha_1 c_0^2\alpha_3^2}$ *for all* x *and* $y \neq 0$,

(vi) $yz\frac{\partial}{\partial x} f_2(x, y, z) \leq 0$ *for all* x, y *and* z

(vii) $f_2(x, y, 0) = 0$, $\frac{\partial}{\partial y} f_2(x,y,z) \leq 0$ *and* $0 \leq \frac{f_2(x,y,z)}{z} - \alpha_2 \leq \frac{\varepsilon_0 c_0^3\alpha_3^3}{BD^2\alpha_4^2}$ $(z \neq 0)$, *where* ε_0 *is a positive constant such that*

$$\varepsilon_0 < \epsilon = \min\left\{\frac{1}{a_0\alpha_1}, \frac{D\alpha_4}{c_0\alpha_3}, \frac{\delta_0}{4a_0c_0\alpha_1\alpha_3\Delta_0}, \frac{Cc_0\alpha_3}{4D\alpha_4\Delta_0}\left(\frac{2D\delta_0\alpha_4}{Ca_0\alpha_1c_0^2\alpha_3^2} - \delta_1\right)\right\} \quad (1.2)$$

with $\Delta_0 = \frac{a_0b_0c_0\alpha_1\alpha_2}{C} + \frac{a_0b_0c_0\alpha_2\alpha_3}{AD\alpha_4}$;

(viii) $\frac{1}{y}\int\limits_0^y \frac{\partial}{\partial x} f_3(x,\zeta)d\zeta \leq \frac{c_0\alpha_3(\varepsilon-\varepsilon_0)}{4C}$ *for all* x *and* $y \neq 0$, *and* $\left\{\frac{\partial}{\partial x} f_3(x,y)\right\}^2 \leq \frac{a_0\delta_0\alpha_1(\varepsilon-\varepsilon_0)}{16C^2}$ *for all* x *and* y;

(ix) $f_4(0) = 0, f_4(x) sgn x > 0$ $(x \neq 0), F_4(x) \equiv \int\limits_0^x f_4(\zeta)d\zeta \to \infty$ *as* $|x| \to \infty$ *and*

$0 \leq \alpha_4 - f_4'(x) \leq \frac{\varepsilon\Delta_0 a_0^2\alpha_1^2}{D}$ *for all* x;

(x) $\int\limits_0^\infty \gamma_0(t)dt < \infty, d'(t) \to 0$ *as* $t \to \infty$, *where* $\gamma_0(t) := |a'(t)| + b'_+(t) + |c'(t)| + |d'(t)|$,

$b'_+(t) = \max\{b'(t), 0\}$;

(xi) $|p(t,x,y,z,w)| \leq p_1(t) + p_2(t)[F_4(x)+y^2+z^2+w^2]^{\delta/2} + \Delta(y^2+z^2+w^2)^{1/2}$, *where δ and Δ are constants such that $0 \leq \delta \leq 1, \Delta \geq 0$ and $p_1(t), p_2(t)$ are nonnegative continuous functions satisfying*

$$\int_0^\infty p_i(t)dt < \infty \qquad (i = 1, 2). \tag{1.3}$$

If Δ is sufficiently small, then every solution $x(t)$ of (1.1) is uniformly bounded and satisfies

$$x(t) \to 0, \dot{x}(t) \to 0, \ddot{x}(t) \to 0, \dddot{x}(t) \to 0, \quad as\ t \to \infty. \tag{1.4}$$

Remark. *Our result includes those of Abou-el-Ela and Sadek [1], Sadek and AL-Elaiw [7].*

2. The function $V_0(t, x, y, z, w)$

In what follows it will be convenient to use the equivalent differential system

$$\dot{x} = y, \dot{y} = z, \dot{z} = w,$$

$$\dot{w} = -a(t)f_1(x, y, z, w) - b(t)f_2(x, y, z) - c(t)f_3(x, y) - d(t)f_4(x) + p(t, x, y, z, w), \tag{2.1}$$

which is obtained from (1.1) by setting $\dot{x} = y, \ddot{x} = z$ and $\dddot{x} = w$.

For the proof of the theorem our main tool is the function $V_0 = V_0(t, x, y, z, w)$ defined as follows:

$$\begin{aligned} 2V_0 = \quad & 2\Delta_2 d(t)\int_0^x f_4(\zeta)d\zeta + 2c(t)\int_0^y f_3(x,\zeta)d\zeta \\ & +[\Delta_2\alpha_2 b(t) - \Delta_1\alpha_4 d(t)]y^2 + a(t)\alpha_1 z^2 + 2\Delta_1 b(t)\int_0^z f_2(x,y,\zeta)d\zeta \\ & -\Delta_2 z^2 + \Delta_1 w^2 + 2d(t)yf_4(x) + 2\Delta_1 d(t)zf_4(x) \\ & +2\Delta_2 a(t)\alpha_1 yz + 2\Delta_1 c(t)zf_3(x,y) + 2\Delta_2 yw + 2zw + k, \end{aligned} \tag{2.2}$$

where

$$\Delta_1 = \frac{1}{a_0\alpha_1} + \varepsilon, \quad \Delta_2 = \frac{\alpha_4 D}{c_0\alpha_3} + \varepsilon \tag{2.3}$$

and k is a positive constant to be determined later in the proof.

Now we will obtain some basic inequalities which will be used in the proof of the result.

By noting (2.3), (i) and (iii) we obtain

$$\Delta_1 - \frac{1}{a(t)\alpha_1} \geq \varepsilon, \quad \text{for all } x, y, z \text{ and all } t \in R^+, \tag{2.4}$$

$$\Delta_2 - \frac{D\alpha_4 y}{c(t) f_3(x,y)} \geq \varepsilon, \quad \text{for all } x, y \neq 0 \text{ and all } t \in R^+. \tag{2.5}$$

In view of (2.3), (i) and (iv) it follows that

$$\begin{aligned}
&\alpha_2 b(t) - \Delta_1 c(t)\tfrac{\partial}{\partial y} f_3(x,y) - \Delta_2 a(t)\alpha_1 \\
&\geq \tfrac{1}{a_0 c_0 \alpha_1 \alpha_3}\left[a_0 b_0 c_0 \alpha_1 \alpha_2 \alpha_3 - C^2 \alpha_3 \tfrac{\partial}{\partial y} f_3(x,y) - A^2 D \alpha_1^2 \alpha_4\right] \\
&- \left[c(t)\tfrac{\partial}{\partial y} f_3(x,y) + a(t)\alpha_1\right]\varepsilon \\
&\geq \tfrac{\delta_0}{a_0 c_0 \alpha_1 \alpha_3} - \left[c(t)\tfrac{\partial}{\partial y} f_3(x,y) + a(t)\alpha_1\right]\varepsilon.
\end{aligned}$$

Also (iv) implies that

$$\frac{\partial}{\partial y} f_3(x,y) < \frac{a_0 b_0 c_0 \alpha_1 \alpha_2}{C^2}, \quad \alpha_1 < \frac{a_0 b_0 c_0 \alpha_2 \alpha_3}{A^2 D \alpha_4}. \tag{2.6}$$

Hence

$$\alpha_2 b(t) - \Delta_1 c(t)\frac{\partial}{\partial y} f_3(x,y) - \Delta_2 a(t)\alpha_1 \geq \frac{\delta_0}{a_0 c_0 \alpha_1 \alpha_3} - \varepsilon \Delta_0, \tag{2.7}$$

for all x, y, z and all $t \in R^+$.

Let Φ_3 be the function defined by

$$\Phi_3(x,y) = \begin{cases} \frac{f_3(x,y)}{y}, & y \neq 0 \\ \\ \frac{\partial}{\partial y} f_3(x,0), & y = 0. \end{cases} \tag{2.8}$$

Then from (iii) and (v) we have

$$\Phi_3(x,y) \geq \alpha_3 \quad \text{for all } x \text{ and } y, \tag{2.9}$$

$$0 \leq \frac{\partial}{\partial y} f_3(x,y) - \Phi_3(x,y) \leq \delta_1 \quad \text{for all } x \text{ and } y. \tag{2.10}$$

From (2.9), (i) and (2.3) we get

$$\Delta_2 - \frac{D\alpha_4}{c(t)\Phi_3(x,y)} \geq \varepsilon, \quad \text{for all } x, y \text{ and all } t \in R^+. \tag{2.11}$$

To prove the present theorem we need the following two lemmas:

Lemma 1. *Subject to the conditions (i)-(ix) of the theorem, there are positive constants D_1 and D_2 such that*

$$D_1[F_4(x) + y^2 + z^2 + w^2 + k] \leq V_0 \leq D_2[F_4(x) + y^2 + z^2 + w^2 + k] \tag{2.12}$$

for all x, y, z and w.

Proof. Since $f_2(x,y,0) = 0$ and $\frac{f_2(x,y,z)}{z} \geq \alpha_2$ $(z \neq 0)$, it is clear that

$$2\Delta_1 b(t) \int_0^z f_2(x,y,\zeta)d\zeta \geq \Delta_1 b(t)\alpha_2 z^2.$$

Therefore it follows from (2.2) that

$$\begin{aligned} 2V_0 \geq \quad & 2\Delta_2 d(t) \int_0^x f_4(\zeta)d\zeta + 2c(t) \int_0^y f_3(x,\zeta)d\zeta + [\Delta_2\alpha_2 b(t) - \Delta_1\alpha_4 d(t)]y^2 \\ & + a(t)\alpha_1 z^2 + \Delta_1 b(t)\alpha_2 z^2 - \Delta_2 z^2 + \Delta_1 w^2 + 2d(t)yf_4(x) + 2\Delta_1 d(t)zf_4(x) \\ & + 2\Delta_2 a(t)\alpha_1 yz + +2\Delta_1 c(t)zf_3(x,y) + 2\Delta_2 yw + 2zw + k. \end{aligned}$$

Rewrite above inequality as follows:

$$2V_0 \geq \frac{c(t)}{\Phi_3(x,y)}\left[\frac{d(t)}{c(t)}f_4(x) + y\Phi_3(x,y) + \Delta_1 z\Phi_3(x,y)\right]^2$$

$$+\frac{a(t)}{\alpha_1}\left[\frac{w}{a(t)} + \alpha_1 z + \Delta_2\alpha_1 y\right]^2 + \left[2\Delta_2 d(t)\int_0^x f_4(\zeta)d\zeta - \frac{d^2(t)f_4^2(x)}{c(t)\Phi_3(x,y)}\right]$$

$$+[\Delta_2 b(t)\alpha_2 - \Delta_1 d(t)\alpha_4 - \Delta_2^2 a(t)\alpha_1]y^2 + 2c(t)\int_0^y f_3(x,\zeta)d\zeta - c(t)\Phi_3(x,y)y^2$$

$$+[\Delta_1\alpha_2 b(t) - \Delta_2 - \Delta_1^2 c(t)\Phi_3(x,y)]z^2 + \left[\Delta_1 - \frac{1}{a(t)\alpha_1}\right]w^2 + k.$$

From (2.4) we get

$$\left[\Delta_1 - \frac{1}{a(t)\alpha_1}\right]w^2 \geq \varepsilon w^2.$$

Then

$$2V_0 \geq V_1 + V_2 + V_3 + \varepsilon w^2 + k, \tag{2.13}$$

where

$$V_1 := 2\Delta_2 d(t)\int_0^x f_4(\zeta)d\zeta - \frac{d^2(t)f_4^2(x)}{c(t)\Phi_3(x,y)},$$

$$V_2 := [\Delta_2\alpha_2 b(t) - \Delta_1\alpha_4 d(t) - \Delta_2^2 a(t)\alpha_1]y^2 + 2c(t)\int_0^y f_3(x,\zeta)d\zeta - c(t)\Phi_3(x,y)y^2,$$

$$V_3 := [\Delta_1\alpha_2 b(t) - \Delta_2 - \Delta_1^2 c(t)\Phi_3(x,y)]z^2.$$

From (2.3), (2.9) and (i) we find

$$\begin{aligned} V_1 &\geq 2\varepsilon d(t)\int_0^x f_4(\zeta)d\zeta + \tfrac{Dd(t)}{c_0\alpha_3}\left[2\alpha_4\int_0^x f_4(\zeta)d\zeta - f_4^2(x)\right] \\ &\geq 2\varepsilon d(t)\int_0^x f_4(\zeta)d\zeta + \tfrac{2Dd(t)}{c_0\alpha_3}\int_0^x [\alpha_4 - f_4'(\zeta)]f_4(\zeta)d\zeta. \end{aligned}$$

Since the second integral on the right hand side is non-negative by (ix), it clear that

$$2\alpha_4\int_0^x f_4(\zeta)d\zeta - f_4^2(x) \geq 0. \tag{2.14}$$

So $V_1 \geq 2\varepsilon d_0 \int_0^x f_4(\zeta)d\zeta$. Also from (2.3), (iii), (i) and (2.7) we obtain

$$\begin{aligned} & \Delta_2\alpha_2 b(t) - \Delta_1\alpha_4 d(t) - \Delta_2^2 a(t)\alpha_1 \\ = \quad & \Delta_2\left[\alpha_2 b(t) - \Delta_1 c(t)\tfrac{\partial}{\partial y}f_3(x,y) - \Delta_2 a(t)\alpha_1\right] \\ & +\Delta_1\left[\Delta_2 c(t)\tfrac{\partial}{\partial y}f_3(x,y) - \alpha_4 d(t)\right] \\ > \quad & \Delta_2\left[\alpha_2 b(t) - \Delta_1 c(t)\tfrac{\partial}{\partial y}f_3(x,y) - \Delta_2 a(t)\alpha_1\right] \\ > \quad & \tfrac{D\alpha_4}{c_0\alpha_3}\left(\tfrac{\delta_0}{a_0c_0\alpha_1\alpha_3} - \varepsilon\Delta_0\right). \end{aligned}$$

Since

$$\begin{aligned} \int_0^y \zeta\tfrac{\partial}{\partial\zeta}f_3(x,\zeta)d\zeta \quad & \equiv y f_3(x,y) - \int_0^y f_3(x,\zeta)d\zeta \\ & = y^2\Phi_3(x,y) - \int_0^y f_3(x,\zeta)d\zeta, \end{aligned}$$

then

$$\begin{aligned} 2c(t)\int_0^y f_3(x,\zeta)d\zeta - c(t)\Phi_3(x,y)y^2 \quad & = c(t)\left[\int_0^y f_3(x,\zeta)d\zeta - \int_0^y \zeta\tfrac{\partial}{\partial\zeta}f_3(x,\zeta)d\zeta\right] \\ & = c(t)\int_0^y\left[\Phi_3(x,y) - \tfrac{\partial}{\partial\zeta}f_3(x,\zeta)\right]\zeta d\zeta \\ & \geq -\tfrac{C\delta_1}{2}y^2, \quad \text{by (2.10)}. \end{aligned}$$

Therefore we have

$$V_2 \geq \left[\frac{D\alpha_4}{c_0\alpha_3}\left(\frac{\delta_0}{a_0c_0\alpha_1\alpha_3} - \varepsilon\Delta_0\right) - \frac{C\delta_1}{2}\right]y^2 \geq \frac{C}{4}\left(\frac{2\alpha_4 D\delta_0}{Ca_0\alpha_1c_0^2\alpha_3^2} - \delta_1\right)y^2, \text{ by (1.2).}$$

Similarly, from (2.3), (i), (2.10) and (2.7) we obtain

$$
\begin{aligned}
&\Delta_1\alpha_2 b(t) - \Delta_2 - \Delta_1^2 c(t)\Phi_3(x,y)\\
=\ &\Delta_1[\alpha_2 b(t) - \Delta_1 c(t)\Phi_3(x,y) - \Delta_2 a(t)\alpha_1] + \Delta_2[\Delta_1 a(t)\alpha_1 - 1]\\
>\ &\Delta_1[\alpha_2 b(t) - \Delta_1 c(t)\tfrac{\partial}{\partial y} f_3(x,y) - \Delta_2 a(t)\alpha_1]\\
>\ &\tfrac{1}{a_0\alpha_1}\left(\tfrac{\delta_0}{a_0 c_0\alpha_1\alpha_3} - \varepsilon\Delta_0\right).
\end{aligned}
$$

Therefore we obtain

$$V_3 \geq \frac{1}{a_0\alpha_1}\left(\frac{\delta_0}{a_0 c_0\alpha_1\alpha_3} - \varepsilon\Delta_0\right) z^2, \text{ by (1.2).}$$

Combining the estimates for V_1, V_2 and V_3 with (2.13) we find

$$2V_0 \geq 2\varepsilon d_0 F_4(x) + \frac{C}{4}\left(\frac{2\alpha_4 D\delta_0}{C a_0\alpha_1 c_0^2\alpha_3^2} - \delta_1\right) y^2 + \left(\frac{3\delta_0}{4a_0^2 c_0\alpha_1^2\alpha_3}\right) z^2 + \varepsilon w^2 + k.$$

Then there exists a positive constant D_1 such that

$$V_0 \geq D_1[F_4(x) + y^2 + z^2 + w^2 + k].$$

Easily,by noting the hypothesis of the theorem, it can be followed that there exists a positive constant D_2 such that

$$V_0 \leq D_2[F_4(x) + y^2 + z^2 + w^2 + k].$$

Therefore (2.12) is verified.

Lemma 2. *Under the conditions of the theorem there exist positive constants D_4, D_5 and D_6 such that*

$$
\begin{aligned}
\dot{V_0}\leq\ & -D_5(y^2+z^2+w^2) + \sqrt{3}D_6(y^2+z^2+w^2)^{1/2}[p_1(t)+p_2(t)]\\
&+\sqrt{3}D_6 p_2(t)[F_4(x)+y^2+z^2+w^2] + D_4\gamma_0 V_0.
\end{aligned}
\tag{2.15}
$$

Proof. An easy calculation from (2.2) and (2.1) yields that

$$\frac{d}{dt}V_0 = \frac{\partial V_0}{\partial w}\ \dot{w} + \frac{\partial V_0}{\partial z}w + \frac{\partial V_0}{\partial y}z + \frac{\partial V_0}{\partial x}y + \frac{\partial V_0}{\partial t}$$

$$= -\Delta_1 a(t)wf_1(x,y,z,w) - \Delta_2 b(t)yf_2(x,y,z) - \Delta_2 c(t)yf_3(x,y) - b(t)zf_2(x,y,z)$$

$$+w^2 + c(t)y\int_0^y \frac{\partial}{\partial x}f_3(x,\zeta)d\zeta + \Delta_1 b(t)z\int_0^z \frac{\partial}{\partial y}f_2(x,y,\zeta)d\zeta + \Delta_1 b(t)y\int_0^z \frac{\partial}{\partial x}f_2(x,y,\zeta)d\zeta$$

$$+\Delta_2 a(t)\alpha_1 z^2 + [\Delta_2\alpha_2 b(t) - \Delta_1\alpha_4 d(t)]yz + \Delta_1 c(t)z^2\frac{\partial}{\partial y}f_3(x,y)$$

$$+\Delta_1 c(t)yz\frac{\partial}{\partial x}f_3(x,y) + d(t)y^2 f_4'(x) + \Delta_1 d(t)yzf_4'(x)$$

$$-\Delta_2 a(t)yf_1(x,y,z,w) + \Delta_2 a(t)\alpha_1 yw$$

$$-a(t)zf_1(x,y,z,w) + a(t)\alpha_1 zw + (\Delta_2 y + z + \Delta_1 w)p(t,x,y,z,w) + \frac{\partial V_0}{\partial t}.$$

Since

$$z\int_0^z \frac{\partial}{\partial y}f_2(x,y,\zeta)d\zeta \le 0, \text{ by (vii) and } \ y\int_0^z \frac{\partial}{\partial x}f_2(x,y,\zeta)d\zeta, \quad \text{by (vi).}$$

Then we find that

$$\begin{aligned}\tfrac{d}{dt}V_0 \quad &= -(V_4+V_5+V_6+V_7+V_8) - \Delta_2 a(t)yf_1(x,y,z,w) + \Delta_2 a(t)\alpha_1 yw \\ &\quad -a(t)zf_1(x,y,z,w) + a(t)\alpha_1 zw + (\Delta_2 y + z + \Delta_1 w)p(t,x,y,z,w) + \tfrac{\partial V_0}{\partial t},\end{aligned} \tag{2.16}$$

where

$$V_4 := \Delta_2 c(t)yf_3(x,y) - \alpha_4 d(t)y^2 - c(t)y\int_0^y \frac{\partial}{\partial x}f_3(x,\zeta)d\zeta - \Delta_1 c(t)yz\frac{\partial}{\partial x}f_3(x,y),$$

$$V_5 := \left[\alpha_2 b(t) - \Delta_1 c(t)\frac{\partial}{\partial y}f_3(x,y) - \Delta_2 a(t)\alpha_1\right]z^2,$$

$$V_6 := [\Delta_1 a(t)\frac{f_1(x,y,z,w)}{w} - 1]w^2,$$

$$V_7 := zb(t)f_2(x,y,z) - \alpha_2 b(t)z^2 + \Delta_2 b(t)yf_2(x,y,z) - \Delta_2\alpha_2 b(t)yz,$$

$$V_8 := \alpha_4 d(t)y^2 - d(t)f_4'(x)y^2 + \Delta_1\alpha_4 d(t)yz - \Delta_1 d(t)f_4'(x)yz.$$

But

$$V_4 = c(t)\Phi_3(x,y)\left[\Delta_2 - \frac{D\alpha_4}{c(t)\Phi_3(x,y)}\right]y^2 - c(t)y\int_0^y \frac{\partial}{\partial x}f_3(x,\zeta)d\zeta - \Delta_1 c(t)yz\frac{\partial}{\partial x}f_3(x,y)$$

$$\geq \varepsilon c_0\alpha_3 y^2 - Cy\int_0^y \frac{\partial}{\partial x}f_3(x,\zeta)d\zeta - \Delta_1 Cyz\frac{\partial}{\partial x}f_3(x,y), \qquad (2.17)$$

by (i), (2.9) and (2.11).

$$\begin{aligned} V_5 &= \left[\alpha_2 b(t) - \Delta_1 c(t)\tfrac{\partial}{\partial y}f_3(x,y) - \Delta_2 a(t)\alpha_1\right]z^2 \\ &\geq \left(\tfrac{\delta_0}{a_0c_0\alpha_1\alpha_3} - \varepsilon\Delta_0\right)z^2, \quad \text{by (2.7)}, \end{aligned} \qquad (2.18)$$

$$V_6 = [\Delta_1 a(t)\frac{f_1(x,y,z,w)}{w} - 1]w^2 \geq \varepsilon\alpha_0\alpha_1 w^2, \qquad (2.19)$$

by (i), (ii) and (2.3).

$$\begin{aligned} V_7 &= b(t)\left[\tfrac{f_2(x,y,z)}{z} - \alpha_2\right](z^2 + \Delta_2 yz), \quad \text{for } z \neq 0 \\ &\geq -\tfrac{\Delta_2^2}{4}b(t)\left[\tfrac{f_2(x,y,z)}{z} - \alpha_2\right]y^2, \quad \text{by (vii)}. \end{aligned}$$

By using (vii) and (2.3) we get for $z \neq 0$

$$\begin{aligned} \frac{\Delta_2^2}{4}b(t)\left[\frac{f_2(x,y,z)}{z} - \alpha_2\right] &\leq \frac{1}{4}b(t)\left(\frac{D\alpha_4}{c_0\alpha_3} + \varepsilon\right)^2 \frac{\varepsilon_0 c_0^3\alpha_3^3}{BD^2\alpha_4^2} \\ &= \frac{1}{4}b(t)\left(1 + \frac{c_0\alpha_3}{D\alpha_4}\varepsilon\right)^2 \frac{\varepsilon_0 c_0\alpha_3}{B} \leq \varepsilon_0 c_0\alpha_3, \end{aligned}$$

since $\varepsilon < \frac{D\alpha_4}{c_0\alpha_3}$ by (1.2). Then

$$V_7 \geq -\varepsilon_0 c_0\alpha_3 y^2 \quad \text{for all } x, y \text{ and } z \neq 0,$$

but $V_7 = 0$ when $z = 0$, so

$$V_7 \geq -\varepsilon_0 c_0\alpha_3 y^2 \quad \text{for all } x, y \text{ and } z. \qquad (2.20)$$

By (ix)

$$V_8 = d(t)[\alpha_4 - f_4'(x)](y^2 + \Delta_1 yz) \geq -\frac{\Delta_1^2}{4}d(t)[\alpha_4 - f_4'(x)]z^2.$$

From (ix) and (2.3) we find

$$\frac{\Delta_1^2}{4}d(t)[\alpha_4 - f_4'(x)] \le \frac{1}{4}d(t)\left(\frac{1}{a_0\alpha_1}+\varepsilon\right)^2 \frac{\varepsilon\Delta_0 a_0^2\alpha_1^2}{D}$$
$$= \frac{1}{4}d(t)\left(1+a_0\alpha_1\varepsilon\right)^2 \frac{\varepsilon_0\Delta_0}{D} \le \varepsilon\Delta_0,$$

since $\varepsilon < \frac{1}{a_0\alpha_1}$ by (1.2). Thus it follows that

$$V_8 \ge -\varepsilon\Delta_0 z^2. \tag{2.21}$$

From (2.17) and (2.20) we have, for $y \neq 0$,

$$\begin{aligned} V_4+V_7 \quad &\ge \left[(\varepsilon-\varepsilon_0)c_0\alpha_3 - \tfrac{C}{y}\int_0^y \tfrac{\partial}{\partial x}f_3(x,\zeta)d\zeta\right]y^2 - \Delta_1 Cyz\tfrac{\partial}{\partial x}f_3(x,y) \\ &\ge \tfrac{3}{4}(\varepsilon-\varepsilon_0)c_0\alpha_3 y^2 - \Delta_1 Cyz\tfrac{\partial}{\partial x}f_3(x,y), \quad \text{by(viii)} \\ &= \tfrac{1}{2}(\varepsilon-\varepsilon_0)c_0\alpha_3 y^2 + \tfrac{1}{4}(\varepsilon-\varepsilon_0)c_0\alpha_3\left[y^2 - \tfrac{4\Delta_1 C}{(\varepsilon-\varepsilon_0)c_0\alpha_3}yz\tfrac{\partial}{\partial x}f_3(x,y)\right] \\ &\ge \tfrac{1}{2}(\varepsilon-\varepsilon_0)c_0\alpha_3 y^2 - \tfrac{\Delta_1^2C^2}{(\varepsilon-\varepsilon_0)c_0\alpha_3}\left[\tfrac{\partial}{\partial x}f_3(x,y)\right]^2 z^2 \\ &\ge \tfrac{1}{2}(\varepsilon-\varepsilon_0)c_0\alpha_3 y^2 - \tfrac{\delta_0}{4a_0\alpha_1c_0\alpha_3}z^2, \end{aligned}$$

by using (vii), (2.3) and (1.2). But $V_4+V_7=0$, when $y=0$,by (2.17) and (2.20); therefore we have

$$V_4+V_7 \ge \frac{1}{2}(\varepsilon-\varepsilon_0)c_0\alpha_3 y^2 - \frac{\delta_0}{4a_0\alpha_1c_0\alpha_3}z^2, \quad \text{for all } y \text{ and } z. \tag{2.22}$$

From the estimates given by (2.18), (2.19), (2.21) and (2.22) we get

$$\begin{gathered} \dot{V_0} \le -\frac{1}{2}(\varepsilon-\varepsilon_0)c_0\alpha_3 y^2 - \left(\frac{3\delta_0}{4a_0c_0\alpha_1\alpha_3} - 2\varepsilon\Delta_0\right)z^2 \\ -\varepsilon a_0\alpha_1 w^2 - a(t)zf_1(x,y,z,w) + a(t)\alpha_1 zw \\ -\Delta_2 a(t)yf_1(x,y,z,w) + \Delta_2 a(t)\alpha_1 yw + (\Delta_2 y + z + \Delta_1 w)p(t,x,y,z,w) + \frac{\partial V_0}{\partial t} \\ \le -\frac{1}{2}(\varepsilon-\varepsilon_0)c_0\alpha_3 y^2 - \frac{1}{4}\frac{\delta_0}{a_0c_0\alpha_1\alpha_3}z^2 - \varepsilon a_0\alpha_1 w^2 - a(t)zf_1(x,y,z,w) + a(t)\alpha_1 zw \\ -\Delta_2 a(t)yf_1(x,y,z,w) + \Delta_2 a(t)\alpha_1 yw + (\Delta_2 y + z + \Delta_1 w)p(t,x,y,z,w) + \frac{\partial V_0}{\partial t}, \end{gathered} \tag{2.23}$$

since $\varepsilon < \frac{\delta_0}{4a_0c_0\alpha_1\alpha_3\Delta_0}$ by (1.2).Consider the expressions

$$W_1 = -\frac{1}{4}(\varepsilon-\varepsilon_0)c_0\alpha_3 y^2 - \frac{1}{4}(\varepsilon-\varepsilon_0)c_0\alpha_3 y^2 - \frac{1}{3}\varepsilon a_0\alpha_1 w^2$$
$$-\Delta_2 a(t)\left[\frac{f_1(x,y,z,w)}{w} - \alpha_1\right] yw$$

and

$$W_2 = -\frac{1}{2}\frac{\delta_0}{a_0c_0\alpha_1\alpha_3}z^2 - \frac{1}{2}\frac{\delta_0}{a_0c_0\alpha_1\alpha_3}z^2 - \frac{1}{3}\varepsilon a_0\alpha_1 w^2 - a(t)\left[\frac{f_1(x,y,z,w)}{w} - \alpha_1\right] zw$$

which is contained in (2.23). Because of the inequalities

$$-W_1 = \tfrac{1}{4}(\varepsilon-\varepsilon_0)c_0\alpha_3 y^2 + \tfrac{1}{4}(\varepsilon-\varepsilon_0)c_0\alpha_3 y^2 + \tfrac{1}{3}\varepsilon a_0\alpha_1 w^2$$

$$+\Delta_2 a(t)\left[\tfrac{f_1(x,y,z,w)}{w} - \alpha_1\right] yw$$

$$\geq \tfrac{1}{4}(\varepsilon-\varepsilon_0)c_0\alpha_3 y^2 + \left[\tfrac{1}{2}\sqrt{(\varepsilon-\varepsilon_0)c_0\alpha_3}\,|y| \pm \sqrt{\tfrac{1}{3}\varepsilon a_0\alpha_1}\,|w|\right]^2$$

$$\geq 0, \quad \text{by (ii)},$$

and

$$-W_2 = \tfrac{1}{2}\tfrac{\delta_0}{a_0c_0\alpha_1\alpha_3}z^2 + \tfrac{1}{2}\tfrac{\delta_0}{a_0c_0\alpha_1\alpha_3}z^2 + \tfrac{1}{3}\varepsilon a_0\alpha_1 w^2 + a(t)\left[\tfrac{f_1(x,y,z,w)}{w} - \alpha_1\right] zw$$

$$\geq \tfrac{1}{2}\tfrac{\delta_0}{a_0c_0\alpha_1\alpha_3}z^2 + \left[\sqrt{\tfrac{1}{2}\tfrac{\delta_0}{a_0c_0\alpha_1\alpha_3}}\,|z| \pm \sqrt{\tfrac{1}{3}\varepsilon a_0\alpha_1}\,|w|\right]^2$$

$$\geq 0, \quad \text{by(ii)},$$

it follows that

$$W_1 \leq -\frac{1}{4}(\varepsilon-\varepsilon_0)c_0\alpha_3 y^2,$$
$$W_2 \leq -\frac{1}{2}\frac{\delta_0}{a_0c_0\alpha_1\alpha_3}z^2.$$

Hence, a combination of the estimates W_1 and W_2 with (2.23) yields that

$$\dot{V_0} \quad \leq -\tfrac{1}{4}(\varepsilon-\varepsilon_0)c_0\alpha_3 y^2 - \tfrac{1}{2}\tfrac{\delta_0}{a_0c_0\alpha_1\alpha_3}z^2 - \tfrac{1}{3}\varepsilon a_0\alpha_1 w^2$$

$$+(\Delta_2 y + z + \Delta_1 w)p(t,x,y,z,w) + \tfrac{\partial V_0}{\partial t}$$

From (2.2) we obtain

$$\begin{aligned}\frac{\partial V_0}{\partial t} = \quad & a'(t)\left[\tfrac{1}{2}\alpha_1 z^2 + \tfrac{1}{2}\Delta_2\alpha_1 yz\right] \\ & +b'(t)\left[\Delta_1\int\limits_0^z f_2(x,y,\zeta)d\zeta + \tfrac{1}{4}\Delta_2\alpha_2 y^2\right] + c'(t)\left[\int\limits_0^y f_3(x,\zeta)d\zeta + \Delta_1 z f_3(x,y)\right] \\ & +d'(t)\left[\Delta_2\int\limits_0^x f_4(\zeta)d\zeta - \tfrac{1}{2}\Delta_1\alpha_4 y^2 + y f_4(x) + \Delta_1 z f_4(x)\right].\end{aligned}$$

From the assumptions in the theorem, (2.6) and (2.14) we have a positive constant D_3 satisfying

$$\frac{\partial V_0}{\partial t} \leq D_3[|a'(t)| + b'_+(t) + |c'(t)| + |d'(t)|][F_4(x) + y^2 + z^2 + w^2] \leq D_4\gamma_0 V_0,$$

by using the inequality (2.12), where $D_4 = \frac{D_3}{D_1}$.Therefore one can find a positive constant D_5 such that

$$\dot{V_0} \leq -2D_5(y^2 + z^2 + w^2) + (\Delta_2 y + z + \Delta_1 w)p(t,x,y,z,w) + D_4\gamma_0 V_0.$$

Let $D_6 = \max(\Delta_2, 1, \Delta_1)$, then

$$\begin{aligned}\dot{V_0} \quad & \leq -2D_5(y^2 + z^2 + w^2) + \sqrt{3}D_6(y^2 + z^2 + w^2)^{1/2}\,|p(t,x,y,z,w)| + D_4\gamma_0 V_0 \\ & \leq -2D_5(y^2 + z^2 + w^2) + \sqrt{3}D_6(y^2 + z^2 + w^2)^{1/2}\,\{p_1(t) \\ & + p_2(t)[F_4(x) + y^2 + z^2 + w^2]^{\delta/2} + \Delta(y^2 + z^2 + w^2)^{1/2}\big\} + D_4\gamma_0 V_0.\end{aligned}$$

Let Δ be fixed, in what follows, to satisfy $\Delta = \frac{D_5}{\sqrt{3}D_6}$ with this limitation on Δ we have

$$\begin{aligned}\dot{V_0} \leq \quad & -D_5(y^2 + z^2 + w^2) + \sqrt{3}D_6(y^2 + z^2 + w^2)^{1/2}\,\{p_1(t) \\ & + p_2(t)[F_4(x) + y^2 + z^2 + w^2]^{\delta/2}\big\} + D_4\gamma_0 V_0.\end{aligned} \tag{2.24}$$

Note that

$$[F_4(x) + y^2 + z^2 + w^2]^{\delta/2} \leq 1 + [F_4(x) + y^2 + z^2 + w^2]^{1/2}. \tag{2.25}$$

From (2.24) and (2.25) we find

$$\dot{V}_0 \leq \ -D_5(y^2+z^2+w^2)+\sqrt{3}D_6(y^2+z^2+w^2)^{1/2}[p_1(t)+p_2(t)]$$

$$+\sqrt{3}D_6 p_2(t)[F_4(x)+y^2+z^2+w^2]+D_4\gamma_0 V_0$$

3. **Completion of the Proof**

We define

$$V(t,x,y,z,w)=\exp\left(-\int_0^t \gamma(\tau)d\tau\right)V_0(t,x,y,z,w), \tag{3.1}$$

where

$$\gamma(t)=D_4\gamma_0+\frac{2\sqrt{3}D_6}{D_1}[p_1(t)+p_2(t)]. \tag{3.2}$$

Then it is easy to see that there exist two functions $U_1(r), U_2(r)$ satisfying

$$U_1(\|\overline{x}\|)\leq V(t,x,y,z,w)\leq U_2(\|\overline{x}\|), \tag{3.3}$$

for all $\overline{x}\in R^4$ and $t\in R^+$ where $U_1(r)$ is a continuous increasing positive definite function, $U_1(r)\to\infty$ as $r\to\infty$ and $U_2(r)$ is a continuous increasing function.

From (3.1), (2.15), (3.2) and (2.12) we have

$$\dot{V}=\exp\left(-\int_0^t \gamma(\tau)d\tau\right)\left[\dot{V}_0-\gamma(t)V_0\right]$$

$$\leq \exp\left(-\int_0^t \gamma(\tau)d\tau\right)\Big\{-D_5(y^2+z^2+w^2)+\sqrt{3}D_6(y^2+z^2+w^2)^{1/2}[p_1(t)+p_2(t)]$$

$$-\sqrt{3}D_6[p_1(t)+p_2(t)][F_4(x)+y^2+z^2+w^2+2k]\Big\}$$

$$\leq \exp\left(-\int_0^t \gamma(\tau)d\tau\right)\{-D_5(y^2+z^2+w^2)$$

$$-\sqrt{3}D_6[p_1(t)+p_2(t)]\left[\left(\sqrt{y^2+z^2+w^2}-\frac{1}{2}\right)^2-\frac{1}{4}+2k]\right]\Bigg\}.$$

Setting $k\geq\frac{1}{8}$, we can find a positive constant D_7 such that

$$\dot{V}\leq -D_7(y^2+z^2+w^2)=-U(\|\overline{x}\|). \tag{3.4}$$

From inequalities (3.3) and (3.4) it follows that all the solutions $(x(t), y(t), z(t), w(t))$ of (2.1) are uniformly bounded [12; Theorem 10.2].

Auxiliary Lemma

We consider a system of differential equations

$$\dot{\overline{x}} = F(t, \overline{x}) + G(t, \overline{x}), \tag{3.5}$$

where $F(t, \overline{x})$ and $G(t, \overline{x})$ are continuous vector functions on $R^+ \times Q$ (Q is an open set in R^n). We assume

$$\|G(t, \overline{x})\| \leq G_1(t, \overline{x}) + G_2(\overline{x}),$$

where $G_1(t, \overline{x})$ is non-negative continuous scalar function on $R^+ \times Q$ and $\int_0^t G_1(\tau, \overline{x}) d\tau$ is bounded for all t whenever $\overline{x}$ belongs to any compact subset of Q and $G_2(\overline{x})$ is a non-negative continuous scalar function on Q.

The following lemma is a simple extension of the well-known result obtained by Yoshizawa [12; Theorem 14.2].

Lemma 3. *Suppose that there exists a non-negative continuously differentiable scalar function $V(t, \overline{x})$ on $R^+ \times Q$ such that $\dot{V}_{(3.5)}\ (t, \overline{x}) \leq -U(\|\overline{x}\|)$, where $U(\|\overline{x}\|)$ is positive definite with respect to a closed set Ω of Q. Moreover, suppose that $F(t, \overline{x})$ of system (3.5) is bounded for all t when $\overline{x}$ belongs to an arbitrary compact set in Q and that $F(t, \overline{x})$ satisfies the following two conditions with respect to Ω*

(1) $F(t, \overline{x})$ tends to a function $H(\overline{x})$ for $\overline{x} \in \Omega$ as $t \to \infty$, and on any compact set in Ω this convergence is uniform;

(2) Corresponding to each $\varepsilon > 0$ and each $\overline{y} \in \Omega$, there exist a δ, $\delta = \delta(\varepsilon, \overline{y})$ and a $T = T(\varepsilon, \overline{y})$ such that if $t \geq T$ and $\|\overline{x} - \overline{y}\|\ \delta$, we have $\|F(t, \overline{x}) - F(\varepsilon, \overline{y})\| < \varepsilon$. And suppose that

(3) $G_2(\overline{x})$ is positive definite with respect to a closed set Ω of Q.

Then every bounded solution of (3.5) approaches the largest semi-imvariant set of the system $\dot{\overline{x}} = H(\overline{x})$ contained in Ω as $t \to \infty$.

Proof. (See [7]) From (2.1) we set F and G in (3.5) as follows

$$F(t,\overline{x}) = \begin{bmatrix} y \\ z \\ w \\ -a(t)f_1(x,y,z,w)w - b(t)f_2(x,y,z) - c(t)f_3(x,y) - d(t)f_4(x) \end{bmatrix},$$

$$G(t,\overline{x}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ p(t,x,y,z,w) \end{bmatrix}.$$

Thus from (xi) we find

$$\|G(t,\overline{x})\| \le p_1(t) + p_2(t)[F_4(x) + y^2 + z^2 + w^2]^{\delta/2} + \Delta(y^2 + z^2 + w^2)^{1/2}.$$

Let

$$G_1(t,\overline{x}) = p_1(t) + p_2(t)[F_4(x) + y^2 + z^2 + w^2]^{\delta/2} \text{ and } G_2(\overline{x}) = \Delta(y^2 + z^2 + w^2)^{1/2}.$$

Then $F(t,\overline{x})$ and $G(t,\overline{x})$ clearly satisfy the conditions of Lemma 3.

Now $U(\|\overline{x}\|)$ in (3.4) is positive definite with respect to the closed set $\Omega = \{(x,y,z,w) \mid x \in R^+, y = 0, z = 0, w = 0\}$, it follows that, in Ω,

$$F(t,\overline{x}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -d(t)f_4(x) \end{bmatrix}.$$

From (i) and (x), we have $d(t) \to d_\infty$ as $t \to \infty$ where $0 \le d_0 < d_\infty \le D$. If we set

$$H(\overline{x}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -d_\infty f_4(x) \end{bmatrix}, \tag{3.6}$$

then the conditions on $H(\overline{x})$ of Lemma 3 are satisfied. Moreover $G_2(\overline{x})$ is positive definite with respect to a closed set Ω.

Since all of the solutions of (2.1) are bounded, it follows from Lemma 3 that every solution of (2.1) approaches the largest semi-imvariant set of the system $\dot{\overline{x}} = H(\overline{x})$ contained in Ω as $t \to \infty$. From (3.6), $\dot{\overline{x}} = H(\overline{x})$ is the system

$$\dot{x} = 0, \dot{y} = 0, \dot{z} = 0, \dot{w} = -d_\infty f_4(x),$$

which has the solutions $x = k_1, y = k_2, z = k_3, w = k_4 - d_\infty f_4(k_1)(t - t_0)$. To remain in Ω; $k_2 = k_3 = 0$ and $k_4 - d_\infty f_4(k_1)(t - t_0) = 0$ for all $t \geq t_0$ which implies $k_1 = k_4 = 0$.

Therefore the only solution of $\dot{\overline{x}} = H(\overline{x})$ remaining in Ω is $\overline{x} = \overline{0}$, that is, the largest semi-invariant set of $\dot{\overline{x}} = H(\overline{x})$ contained in Ω is the point $(0, 0, 0, 0)$. Then it follows that

$$x(t) \to 0, y(t) \to 0, z(t) \to 0, w(t) \to 0 \quad \text{as} \quad t \to \infty,$$

which are equivalent to (1.4).

This completes the proof of the theorem.

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