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UNBOUNDED SOLUTIONS OF EQUATION $\dot{y}(t) = \beta(t)[y(t-\delta) - a(t)y(t-\tau)]$

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Abstract. This contribution is devoted to asymptotic behavior (for $t \to \infty$) of solutions of first-order differential equation with two delays

$$\dot{y}(t) = \beta(t)[y(t-\delta) - a(t)y(t-\tau)].$$

Representation of solutions in an exponential form is discussed and inequalities for such solutions are given. As a consequence, existence of unbounded solutions is proved. An overview of known results and illustrative examples are considered, too.

1. Introduction

1.1. **The aim of the contribution.** In this contribution we deal with asymptotic behavior of solutions to a linear homogeneous differential equation with two delayed terms containing two discrete delays

$$\dot{y}(t) = \beta(t)[y(t-\delta) - a(t)y(t-\tau)] \tag{1}$$

for $t \to \infty$. In (1) $\delta, \tau \in \mathbb{R}^+$, $\mathbb{R}^+ := (0, +\infty)$, $\tau > \delta$, $\beta : I_{-1} \to \mathbb{R}^+$ is a continuous function, $I_{-1} := [t_0 - \tau, \infty)$, $t_0 \in \mathbb{R}$ and $a : I \to [0,1]$, where $I := [t_0, \infty)$, is a continuous function. The symbol " $\dot{}$ " denotes (at least) the *right-hand* derivative. Similarly, if necessary, the value of a function at a point of I_{-1} is understood (at least) as value of the corresponding limit *from the right*. We show that increasing solutions of (1) have representation

$$\exp\left[\int_{t_0-\tau}^{t} \tilde{\varepsilon}(s)\beta(s)\,\mathrm{d}s\right] \tag{2}$$

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with a function $\tilde{\varepsilon} : I_{-1} \to (0,1)$. Such representation we call *exponential*. Representation (2) is then specified and a criterion connecting it with an integral inequality is formulated. Since the equation considered is linear, the corresponding statements formulated for increasing solutions are (under obvious modification) valid for decreasing solutions etc. Let us note that close investigation of asymptotic behaviour of a solution of delayed functional differential equations is performed e.g. in papers [1]–[24]. The studied Eq. (1) (with $a \equiv 1$) occurs e.g. in the number theory [23].

The contribution is organized as follows: In Section 2 a basic auxiliary inequality is studied and the relationship of its solutions with solutions of Eq. (1) is established. Exponential representation of monotone solutions is discussed in Section 3. Section 4 contains main results of the paper concerning inequalities for solutions of Eq. (1) and existence of unbounded solutions. An overview of known results and illustrative examples are contained in Section 5. The paper ends with an open problem formulated in Section 6.

1.2. **Some definitions.** Let us shortly recall basic definitions. Let $\mathcal{C} := C([-\tau, 0], \mathbb{R})$ be Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}$ equipped with the supremum norm.

A function $y(t)$ is said to be a *solution of Eq.* (1) *on* $[\nu - \tau, \nu + A)$ with $\nu \in I$ and $A > 0$, if $y \in C([\nu-\tau, \nu+A), \mathbb{R}) \cap C^1([\nu, \nu+A), \mathbb{R})$, and $y(t)$ satisfies the Eq. (1) for $t \in [\nu, \nu + A)$.

For given $\nu \in I$, $\varphi \in \mathcal{C}$, we say that $y(\nu, \varphi)$ is a *solution* of Eq. (1) *through* (ν, φ) (or that $y(\nu, \varphi)$ *corresponds to the initial point* ν), if there is an $A > 0$ such that $y(\nu, \varphi)$ is a solution of Eq. (1) on $[\nu - \tau, \nu + A)$ and $y(\nu, \varphi)(\nu + \theta) = \varphi(\theta)$ for $\theta \in [-\tau, 0]$.

Due to linearity of equation (1), the solution $y(\nu, \varphi)$ is unique and is defined on $[\nu - \tau, \infty)$, i.e. in previous definitions we can put $A := \infty$.

2. **An auxiliary inequality**

Auxiliary inequality

$$\dot{\omega}(t) \le \beta(t)[\omega(t - \delta) - a(t)\omega(t - \tau)] \tag{3}$$

plays a main role in analysis of equation (1). A function $\omega(t)$ is said to be a *solution of* (3) *on* $[\nu - \tau, \nu + A)$ with $\nu \in I$ and $A > 0$, if $\omega \in C([\nu - \tau, \nu + A), \mathbb{R}) \cap C^1([\nu, \nu + A), \mathbb{R})$, and $\omega(t)$ satisfies the inequality (3) for $t \in [\nu, \nu + A)$.

2.1. **Inequalities between solutions of inequality (3) and equation (1).** Below we discuss some properties of solutions of inequalities of the type (3) and inequalities between solutions of (1) and inequality (3).

Theorem 1. *Suppose that* $\omega(t)$ *is a solution of inequality* (3) *on* I_{-1}. *Then there exists a solution* $y(t)$ *of* (1) *on* I_{-1} *such that an inequality*

$$y(t) \geq \omega(t) \tag{4}$$

holds on I_{-1}. *In particular, a solution* $y(t_0, \phi)$ *of Eq.* (1) *with* $\phi \in \mathcal{C}$ *defined by relation*

$$\phi(\theta) := \omega(t_0 + \theta), \quad \theta \in [-\tau, 0], \tag{5}$$

is a such solution.

Proof. Let $\omega(t)$ be a solution of inequality (3) on I_{-1}. Let us show that the solution $y(t) := y(t_0, \phi)(t)$ of (1) satisfies inequality (4) i.e.

$$y(t_0, \phi)(t) \geq \omega(t) \tag{6}$$

on I_{-1}. Due to definition of $y(t)$ we have $y(t) \equiv \omega(t)$, $t \in [t_0 - \tau, t_0]$ and (4) holds on initial interval $[t_0 - \tau, t_0]$. Define on I_{-1} a continuous function

$$W(t) := y(t) - \omega(t).$$

Function W is continuously differentiable on I. Then (taking into account inequality (3)) the estimation

$$\dot{W}(t) = \dot{y}(t) - \dot{\omega}(t) \geq Z(t)$$

with

$$Z(t) := \beta(t)[y(t-\delta) - a(t)y(t-\tau)] - \beta(t)[\omega(t-\delta) - a(t)\omega(t-\tau)] = \beta(t)[W(t-\delta) - a(t)W(t-\tau)]$$

is valid on I . Let $t \in (t_0, t_0 + \delta]$. In view of (5) $W(t - \delta) \equiv W(t - \tau) \equiv 0$, $Z(t) \equiv 0$ and $\dot{W}(t) \geq 0$, i.e. (4) holds on $(t_0, t_0 + \delta]$. Let $t \in (t_0 + \delta, t_0 + \tau]$. In this case $W(t - \tau) \equiv 0$ and

$$Z(t) \equiv \beta(t)[y(t - \delta) - \omega(t - \delta)] = \beta(t)W(t - \delta) \geq 0.$$

Consequently, $\dot{W}(t) \geq 0$, i.e. (4) holds on $(t_0 + \delta, t_0 + \tau]$, too. Let us show that inequality $\dot{W}(t) \geq 0$ holds on the whole interval I. For it *suppose the contrary*, i.e. suppose existence of a point $t_1 > t_0 + \tau$ such that

$$\begin{aligned} \dot{W}(t) &\geq 0, \quad t \in [t_0, t_1), \\ \dot{W}(t_1) &= 0, \\ \dot{W}(t) &< 0, \quad t \in (t_1, t_1 + \varepsilon), \end{aligned} \tag{7}$$

where $\varepsilon < \delta$ is a small positive number. Due to continuity of $W(t)$ on I_{-1}, our construction and suppositions, such point t_1 exists. Let $t_2 \in (t_1, t_1 + \varepsilon)$. Taking into account that $W(t)$ is nondecreasing on $[t_0, t_1]$ we conclude $W(t_2 - \delta) \geq W(t_2 - \tau) \geq 0$. Then

$$\dot{W}(t_2) = \dot{y}(t_2) - \dot{\omega}(t_2) \geq Z(t_2) = \beta(t_2)[W(t_2 - \delta) - a(t_2)W(t_2 - \tau)] \geq \\ \beta(t_2)(1 - a(t_2))W(t_2 - \tau) \geq 0.$$

The resulting inequality $\dot{W}(t_2) \geq 0$ contradicts (7). $\square$

Remark 1. *Let us note that an affirmation, opposite in a sense with the statement of Theorem 1 is obvious. Namely, if a solution $y(t)$ of (1) on I_{-1} is given, then there exists a solution $\omega(t)$ of inequality (3) on I_{-1} such that inequality*

$$\omega(t) \geq y(t) \tag{8}$$

holds on I_{-1}, since it can be put $\omega(t) \equiv y(t)$.

2.2. **A comparison lemma.** Let us consider an inequality of the type (3)

$$\dot{\omega}^*(t) \le \beta_1(t)[\omega^*(t-\delta) - a_1(t)\omega^*(t-\tau)] \tag{9}$$

where $\beta_1 : I_{-1} \to \mathbb{R}^+$ and $a_1 : I \to [0,1]$ are continuous functions satisfying inequalities $\beta_1(t) \le \beta(t)$, $a_1(t) \ge a(t)$ on I_{-1}. The following comparison lemma will be used below.

Lemma 1. *Let the inequality* (9) *have a nondecreasing positive solution on I_{-1}. Then this solution is a solution of the inequality* (3) *on I_{-1}, too.*

Proof. Let ω^* be a nondecreasing solution of inequality (9) on I_{-1}. Then

$$\begin{aligned}\dot{\omega}^* \le \beta_1(t)[\omega^*(t-\delta) - a_1(t)\omega^*(t-\tau)] &\le \beta(t)[\omega^*(t-\delta) - a_1(t)\omega^*(t-\tau)]\\ &\le \beta(t)[\omega^*(t-\delta) - a(t)\omega^*(t-\tau)].\end{aligned}$$

Consequently, the function $\omega := \omega^*$ solves the inequality (3), too. □

2.3. **A solution of the inequality (3).** It is easy to get a solution of inequality (3) in an exponential form.

Lemma 2. *Suppose that there exists a function $\varepsilon : I_{-1} \to \mathbb{R}$, continuous on $I_{-1} \setminus \{t_0\}$ with at most first order discontinuity at the point $t = t_0$ and satisfying on I the inequality*

$$\exp\left[-\int_{t-\delta}^{t} \varepsilon(s)\beta(s)\,\mathrm{d}s\right] \ge \varepsilon(t) + a(t)\exp\left[-\int_{t-\tau}^{t} \varepsilon(s)\beta(s)\,\mathrm{d}s\right]. \tag{10}$$

Then on I_{-1}, there exists a solution $\omega(t) = \omega_{\mathrm{e}}(t)$ of inequality (3) *having the form*

$$\omega_{\mathrm{e}}(t) := \exp\left[\int_{t_0-\tau}^{t} \varepsilon(s)\beta(s)\,\mathrm{d}s\right]. \tag{11}$$

Proof. Inequality (10) follows immediately from inequality (3) if a possible solution $\omega(t)$ is taken in the form (11). □

3. **Properties of solutions of equation (1)**

In this part we prove auxiliary results concerning solutions of equation (1).

Lemma 3. *Let $\varphi \in \mathcal{C}$ is increasing and positive on $[-\tau, 0]$. Then the corresponding solution $y(t^*, \varphi)(t)$ of* (1) *with $t^* \in I$ is increasing in $[t^* - \tau, \infty)$, too.*

Proof. Immediately, from the form of (1), we get $\operatorname{sign} \dot{y}(t^*, \varphi)(t^*) = +1$ in the case when the function φ increases on $[-\tau, 0]$. The case $\dot{y}(t^*, \varphi)(t^{**}) = 0$ for a $t^{**} \in (t^*, \infty)$ and simultaneously $\operatorname{sign} \dot{y}(t^*, \varphi)(t) \neq 0$ on interval $t \in (t^*, t^{**})$ is impossible because, as it follows from (1) and from the properties of function φ, the inequality $y(t^{**} - \delta) > y(t^{**} - \tau)$ holds and, consequently,

$$y(t^{**} - \delta) - a(t^{**})y(t^{**} - \tau) \neq 0.$$

I.e. $\dot{y}(t^*, \varphi)(t^{**}) \neq 0$. $\square$

3.1. **Exponential representation of solutions of equation (1).**

Theorem 2. *Every continuously increasing on I_{-1} and continuously differentiable on $I_{-1} \setminus \{t_0\}$ solution $y(t)$ of (1) with $y(t_0 - \tau) = 1$ is on I_{-1} representable in exponential form:*

$$y(t) = \exp\left[\int_{t_0-\tau}^{t} \tilde{\varepsilon}(s)\beta(s)\,\mathrm{d}s\right] \tag{12}$$

where $\tilde{\varepsilon} : I_{-1} \to \mathbb{R}_+ := [0, \infty)$ is a continuous function on $I_{-1} \setminus \{t_0\}$ with at most first order discontinuity at t_0 and $0 < \tilde{\varepsilon}(t) < 1$ on I.

Proof Let $\varphi \in \mathcal{C}$, $\varphi(t_0 - \tau) = 1$ be increasing and continuously differentiable initial function generating solution $y(t) = y(t_0, \varphi)(t)$. By Lemma 3 is $y(t)$ increasing in I_{-1}. Define

$$\tilde{\varepsilon}(t) := \begin{cases} \dfrac{\varphi'(t)}{\beta(t)\varphi(t)} & \text{on} \quad [t_0 - \tau, t_0), \\[2ex] \dfrac{\dot{y}(t)}{\beta(t)y(t)} & \text{on} \quad I. \end{cases}$$

Then on I_{-1} representation (12) holds. Really, for $t \in [t_0 - \tau, t_0)$ we have

$$\exp\left[\int_{t_0-\tau}^{t} \tilde{\varepsilon}(s)\beta(s)\,\mathrm{d}s\right] = \exp\left[\ln \frac{\varphi(t)}{\varphi(t_0 - \tau)}\right] = \varphi(t)$$

and for $t \in I$

$$\exp\left[\int_{t_0-\tau}^{t} \tilde{\varepsilon}(s)\beta(s)\,\mathrm{d}s\right] = \exp\left[\int_{t_0-\tau}^{t_0} \tilde{\varepsilon}(s)\beta(s)\,\mathrm{d}s + \int_{t_0}^{t} \tilde{\varepsilon}(s)\beta(s)\,\mathrm{d}s\right] =$$

$$\exp\left[\ln \frac{\varphi(t_0)}{\varphi(t_0 - \tau)} + \ln \frac{y(t)}{y(t_0)}\right] = y(t).$$

Function $\tilde{\varepsilon}$ is on $[t_0-\tau, t_0)$ nonnegative, since obviously $\varphi > 0$, $\varphi' \geq 0$ and $\beta > 0$. Positivity of $\tilde{\varepsilon}$ on I is obvious, too since

$$\tilde{\varepsilon}(t) = \frac{\dot{y}(t)}{\beta(t)y(t)} = \frac{y(t-\delta) - a(t)y(t-\tau)}{y(t)} > \frac{(1-a(t))y(t-\tau)}{y(t)} \geq 0,$$

i.e. $\tilde{\varepsilon}(t) > 0$. Moreover, on I,

$$\tilde{\varepsilon}(t) = \frac{\dot{y}(t)}{\beta(t)y(t)} = \frac{y(t-\delta) - a(t)y(t-\tau)}{y(t)} \leq \frac{y(t-\delta)}{y(t)} < \frac{y(t)}{y(t)} = 1,$$

i.e. $\tilde{\varepsilon}(t) < 1$. $\square$

Below is given a modification of previous result.

Corollary 1. *There exists continuously increasing on I_{-1} and continuously differentiable on $I_{-1} \setminus \{t_0\}$ solution $y(t)$ of (1) with $y(t_0-\tau) = 1$, representable in exponential form (12), where*

$$\tilde{\varepsilon} : I_{-1} \to (0,1)$$

is a continuous function on $I_{-1} \setminus \{t_0\}$ with at most first order discontinuity at t_0.

The proof remains exactly the same if the initial function $\varphi \in \mathcal{C}$ is defined as

$$\varphi(\theta) := \exp\left[\int_{t_0-\tau}^{t_0+\theta} \varepsilon^*(s)\beta(s)\,\mathrm{d}s\right], \quad \theta \in [-\tau, 0],$$

where $\varepsilon^* : [t_0-\tau, t_0] \to (0,1)$ is a continuous function. Then we can define corresponding function $\tilde{\varepsilon}$ e.g. in the following way:

$$\tilde{\varepsilon}(t) := \begin{cases} \varepsilon^*(t) & \text{on} \quad [t_0-\tau, t_0), \\[2ex] \dfrac{y(t-\delta) - a(t)y(t-\tau)}{y(t)} & \text{on} \quad [t_0, \infty). \end{cases}$$

Remark 2. *From the statement of Theorem 2 it follows that every continuously increasing on I_{-1} and continuously differentiable on $I_{-1} \setminus \{t_0\}$ solution $y(t)$ of (1) with $y(t_0-\tau) = 1$ satisfies on I the inequality*

$$y(t) < \exp\left[\int_{t_0-\tau}^{t} \beta(s)\,\mathrm{d}s\right]. \tag{13}$$

Moreover (as it follows from Corollary 1) there exists continuously increasing on I_{-1} and continuously differentiable on $I_{-1} \setminus \{t_0\}$ solution $y(t)$ of (1) with $y(t_0-\tau) = 1$, such that inequality (13) holds on $I_{-1} \setminus \{t_0-\tau\}$.

4. **Main results**

The purpose of this part is to give an equivalence between existence of a certain type of exponential behavior of solutions of (1) and existence of a solution of inequality (3). The following result can be useful in the case when we need a concrete inequality for indicated solution $y = y(t)$ of (1).

4.1. **Two equivalent statements.**

Theorem 3. *Let $q : I_{-1} \to (0,1)$ be a given function such that the integral $\int_{t_0-\tau}^{t} q(s)\beta(s)\,\mathrm{d}s$ exists for any $t \in I_{-1}$. Then the following two statements are equivalent:*

- **a)** *There exists a continuously increasing on I_{-1} and continuously differentiable on $I_{-1} \setminus \{t_0\}$ solution $y = y(t)$ of (1) representable in the form*

$$y(t) = \exp\left[\int_{t_0-\tau}^{t} \tilde{\varepsilon}(s)\beta(s)\,\mathrm{d}s\right] \tag{14}$$

  on I_{-1}, where $\tilde{\varepsilon} : I_{-1} \to (0,1)$ is a continuous function on $I_{-1} \setminus \{t_0\}$ with at most first order discontinuity at the point $t = t_0$, such that

$$y(t) \geq \exp\left[\int_{t_0-\tau}^{t} q(s)\beta(s)\,\mathrm{d}s\right] \tag{15}$$

  on I_{-1}.
- **b)** *There exists a function $\varepsilon : I_{-1} \to (0,1)$ continuous on $I_{-1} \setminus \{t_0\}$ with at most first order discontinuity at the point $t = t_0$ such that*

$$\int_{t_0-\tau}^{t} \varepsilon(s)\beta(s)\mathrm{d}s \geq \int_{t_0-\tau}^{t} q(s)\beta(s)\mathrm{d}s \tag{16}$$

  on I_{-1}, and satisfying the integral inequality (10) on I.

Proof

Part b) $\Longrightarrow$ a). In this case there exists (by Lemma 2) a solution $\omega(t) \equiv \omega_{\mathrm{e}}(t)$ of inequality (10) given by formula (14). Define

$$\varphi(\theta) := \omega_{\mathrm{e}}(t_0 + \theta), \;\; \theta \in [-\tau, 0].$$

Since $\varphi \in \mathcal{C}$ is increasing and positive on $[-\tau, 0]$, then (by Lemma 3) solution $y(t) = y(t_0, \varphi)(t)$ is increasing in I_{-1} and, by Theorem 1, satisfies on I_{-1} inequality (4), i.e.

$$y(t) \geq \exp\left[\int_{t_0-\tau}^{t} \varepsilon(s)\beta(s)\,\mathrm{d}s\right], \quad t \in I_{-1}.$$

Now is the inequality (15) a straightforward consequence of inequality (16). The part $b) \implies a)$ is proved.

Part a) $\implies$ b). Let $y(t)$ be a solution of (1) on I_{-1}, having form (14), with properties indicated in the part a). Then on $I_{-1} \setminus \{t_0\}$:

$$\dot{y}(t) = \tilde{\varepsilon}(t)\beta(t) \cdot \exp\left[\int_{t_0-\tau}^{t} \tilde{\varepsilon}(s)\beta(s)\,\mathrm{d}s\right].$$

Let us put $y(t)$ into (1). Then on I:

$$\tilde{\varepsilon}(t) = \exp\left[-\int_{t-\delta}^{t} \tilde{\varepsilon}(s)\beta(s)\,\mathrm{d}s\right] - a(t)\exp\left[-\int_{t-\tau}^{t} \tilde{\varepsilon}(s)\beta(s)\,\mathrm{d}s\right].$$

Define function $\varepsilon : I_{-1} \setminus \{t_0\} \to (0,1)$ as $\varepsilon := \tilde{\varepsilon}$, and rewrite the last equality. For $t \in I$ we get

$$\exp\left[-\int_{t-\delta}^{t} \varepsilon(s)\beta(s)\,\mathrm{d}s\right] = \varepsilon(t) + a(t)\exp\left[-\int_{t-\tau}^{t} \varepsilon(s)\beta(s)\,\mathrm{d}s\right],$$

i.e. the integral inequality (10) holds on I. Moreover, due to (15) we have

$$y(t) = \exp\left[\int_{t_0-\tau}^{t} \varepsilon(s)\beta(s)\,\mathrm{d}s\right] \geq \exp\left[\int_{t_0-\tau}^{t} q(s)\beta(s)\,\mathrm{d}s\right],$$

i.e. the inequality (16) holds, too. This ends the proof. $\square$

Remark 3. *Note that Theorem 3 remains valid if, instead of the supposition $q : I_{-1} \to (0,1)$, a more general supposition $q : I_{-1} \to \mathbb{R}$ is used. But for some specifications of the function q the equivalence between statements a) and b) can lose sense since the existence of solution $y = y(t)$ satisfying inequality* (15) *can follows directly from the statements of Theorem 2 or Corollary 1. E.g. the choice $q(t) := 0$ gives no new information as well as the choice $q(t) := \tilde{\varepsilon}(t)$. Theorem 3 generalizes and improves Theorem 2 from* [14]*, where the equation* (1) *with $a(t) \equiv 1$ was investigated. The authors are grateful to R. Hakl for corresponding remark during discussions on*

International Conference on Nonlinear Operators, Differential Equations and Applications in Cluj-Napoca, Romania, August 2004, indicating a gap in formulation of this result.

Remark 4. *Let us underline that Theorem 3 together with Remark 2 give for solution $y(t)$ on I_{-1} estimation*

$$\exp\left[\int_{t_0-\tau}^{t} q(s)\beta(s)\,\mathrm{d}s\right] \le y(t) \le \exp\left[\int_{t_0-\tau}^{t} \beta(s)\,\mathrm{d}s\right].$$

4.2. **Sufficient conditions for divergence.** Conditions guarantee existence of unbounded solution can be derived easily from previous results. Let us formulate some of them. From Theorem 1 we get

Theorem 4. *Suppose that $\omega(t)$ is a solution of inequality* (3) *on I_{-1} such that*

$$\limsup_{t\to\infty} \omega(t) = +\infty.$$

Then there exists unbounded solution $y(t)$ of (1) *on I_{-1}.*

From Lemma 2, Theorem 1 and Theorem 3 (putting $q(t) := \varepsilon(t)$) we get

Theorem 5. *Suppose there exists a function $\varepsilon : I_{-1} \to \mathbb{R}$, continuous on $I_{-1} \setminus \{t_0\}$ with at most first order discontinuity at the point $t = t_0$ satisfying $\int^{\infty} \varepsilon(s)\beta(s)\mathrm{d}s = \infty$, and on I the inequality* (10). *Then there exists unbounded solution $y(t)$ of* (1) *on I_{-1} satisfying inequality*

$$y(t) \ge \exp\left[\int_{t_0-\tau}^{t} \varepsilon(s)\beta(s)\,\mathrm{d}s\right] \tag{17}$$

on I_{-1}. If, moreover ε is on $[t_0-\tau, t_0]$ positive then there exists increasing unbounded solution $y(t)$ of (1) *on I_{-1}, satisfying inequality* (17).

5. Summary of known results and examples

5.1. **Known results relative to equation (1).** Let us recall some known partial results concerning equation (1). In paper [12] conditions for convergence of all solutions of equation (1) with $a(t) \equiv 1$ and $\delta = 1$, i.e. the equation

$$\dot{y}(t) = \beta(t)[y(t) - y(t-\tau)]. \tag{18}$$

are given. We reproduce one result as the first statement of following theorem. The second part concerns of equation (1) with $a(t) \equiv 1$, i.e. the equation

$$\dot{y}(t) = \beta(t)[y(t-\delta) - y(t-\tau)]. \tag{19}$$

and follows from results given in [3, 6].

Theorem 6. *Let for all $t \in I_{-1}$ and a constant $p > 1$:*

$$\beta(t) \le \frac{1}{\tau} - \frac{p}{2t} \ . \tag{20}$$

Then each solution of (18) *corresponding to the initial point t_0 converges.*
Let for all $t \in I_{-1}$ exists a constant ρ such that

$$\beta(t) \le \rho < \frac{1}{\tau - \delta} \ . \tag{21}$$

Then each solution of (19) *corresponding to the initial point t_0 converges.*

In the paper [14] is proved following result concerning existence of unbounded increasing solutions of (19).

Theorem 7. *Let for all $t \in I_{-1}$ with sufficiently large t_0 and for a constant $p \in (0,1)$:*

$$\beta(t) \ge \frac{1}{\tau - \delta} - \frac{p}{2t} \ . \tag{22}$$

Then there exists an increasing and unbounded solution of (19) *as $t \to \infty$.*

5.2. **Examples.** In this part we give two examples to demonstrate the influence of the coefficient a to appearance of unbounded solutions.

Example 1. *The first remark is obvious - the presence of coefficient a in Eq.* (1) *enlarges, in the case $a(t) \not\equiv 1$, the range for coefficient β. Consider the following result to illustrate this phenomenon.*

Theorem 8. *Let for all $t \in I_{-1}$ inequalities*

$$\beta(t) \ge \frac{1}{\tau - \delta} + \frac{p}{t} \ , \quad 0 \le a(t) \le 1 - \frac{b}{t^2} \tag{23}$$

with constants $p \in \mathbb{R}$, $b \in \mathbb{R}^+$ hold. Then there exists increasing and unbounded solution $y(t)$ of (1) *as $t \to \infty$.*

Proof. *Let us verify that the integral inequality* (10) *have (for sufficiently large values* t*) a solution of the form* $\varepsilon(t) := \alpha/t$ *with* $\alpha \in \mathbb{R}^+$*. Put in* (10)

$$\beta(t) := \frac{1}{\tau-\delta} + \frac{p}{t}\ , \quad a(t) := 1 - \frac{b}{t^2}\ , \quad \varepsilon(t) := \frac{\alpha}{t}\ .$$

Then the left-hand side $\mathcal{L}(t)$ *of* (10) *equals*

$$\mathcal{L}(t) \equiv \exp\left[-\int_{t-\delta}^{t} \varepsilon(s)\beta(s)\,\mathrm{d}s\right] = \exp\left[-\int_{t-\delta}^{t} \frac{\alpha}{s}\left[\frac{1}{\tau-\delta} + \frac{p}{s}\right]\mathrm{d}s\right] = \left(\frac{t-\delta}{t}\right)^{\dfrac{\alpha}{\tau-\delta}} \cdot \exp\left[\frac{-\alpha\delta p}{t(t-\delta)}\right].$$

Now we asymptotically decompose $\mathcal{L}(t)$ *for* $t \to \infty$ *with sufficient accuracy for further application. We get:*

$$\mathcal{L}(t) = \left[1 - \frac{\alpha\delta}{(\tau-\delta)t} + \frac{\alpha\delta^2}{2(\tau-\delta)} \cdot \left(\frac{\alpha}{\tau-\delta} - 1\right)\frac{1}{t^2} + O\left(\frac{1}{t^3}\right)\right] \times \left[1 - \frac{\alpha\delta p}{t^2} + O\left(\frac{1}{t^3}\right)\right]$$

$$= 1 - \frac{\alpha\delta}{(\tau-\delta)t} + \left[\frac{\alpha\delta^2}{2(\tau-\delta)} \cdot \left(\frac{\alpha}{\tau-\delta} - 1\right) - \alpha\delta p\right]\frac{1}{t^2} + O\left(\frac{1}{t^3}\right)$$

where O *is the Landau order symbol. Decomposition of the right-hand side* $\mathcal{R}(t)$ *of* (10) *leads to*

$$\mathcal{R}(t) \equiv \varepsilon(t) + a(t)\exp\left[-\int_{t-\tau}^{t} \varepsilon(s)\beta(s)\,\mathrm{d}s\right]$$

$$= \frac{\alpha}{t} + \left(1 - \frac{b}{t^2}\right) \cdot \exp\left[-\int_{t-\tau}^{t} \frac{\alpha}{s}\left[\frac{1}{\tau-\delta} + \frac{p}{s}\right]\mathrm{d}s\right]$$

$$= \frac{\alpha}{t} + \left(1 - \frac{b}{t^2}\right) \cdot \left(\frac{t-\tau}{t}\right)^{\dfrac{\alpha}{\tau-\delta}} \cdot \exp\left[\frac{-\alpha\tau p}{t(t-\tau)}\right]$$

$$= \frac{\alpha}{t} + \left(1 - \frac{b}{t^2}\right) \cdot \left[1 - \frac{\alpha\tau}{(\tau-\delta)t} + \frac{\alpha\tau^2}{2(\tau-\delta)} \cdot \left(\frac{\alpha}{\tau-\delta} - 1\right)\frac{1}{t^2} + O\left(\frac{1}{t^3}\right)\right] \times \left[1 - \frac{\alpha\tau p}{t^2} + O\left(\frac{1}{t^3}\right)\right]$$

$$= 1 + \frac{\alpha}{t} - \frac{\alpha\tau}{\tau-\delta} \cdot \frac{1}{t} + \left[\frac{\alpha\tau^2}{2(\tau-\delta)} \cdot \left(\frac{\alpha}{\tau-\delta} - 1\right) - \alpha\tau p - b\right]\frac{1}{t^2} + O\left(\frac{1}{t^3}\right).$$

Comparing $\mathcal{L}(t)$ and $\mathcal{R}(t)$, we see that for $\mathcal{L}(t) \geq \mathcal{R}(t)$ it is necessary to compare coefficients of the terms t^{-2} because coefficients of the terms t^0 and t^{-1} are equal. It means we need the inequality

$$\frac{\alpha\delta^2}{2(\tau-\delta)} \cdot \left(\frac{\alpha}{\tau-\delta} - 1\right) - \alpha\delta p > \frac{\alpha\tau^2}{2(\tau-\delta)} \cdot \left(\frac{\alpha}{\tau-\delta} - 1\right) - \alpha\tau p - b.$$

We see that for sufficiently small positive α this inequality holds since taking limit for $\alpha \to 0^+$, the limiting inequality $0 > -b$ is valid due to positivity of b. Consequently, a function

$$\omega_e(t) := \exp\left[\int_{t_0-\tau}^{t} \varepsilon(s)\beta(s)\mathrm{d}s\right] = \exp\left[\int_{t_0-\tau}^{t} \frac{\alpha}{s}\left(\frac{1}{\tau-\delta} + \frac{p}{s}\right)\mathrm{d}s\right]$$

is (under supposition that t_0 is sufficiently large) a positive solution of the integral inequality (3) *and, moreover, it is easy to verify that $\omega_e(\infty) = +\infty$. Let us show that this solution solves every inequality of the type* (10) *(perhaps starting with a different value t_0) if the above* fixed functions *β and a (defined at beginning of the proof) are changed by any functions β and a specifying in formulation of theorem by inequalities* (23)*. This statement is a straightforward consequence of Lemma 1 if in its formulation*

$$\beta_1(t) := \frac{1}{\tau-\delta} + \frac{p}{t}\ , \quad a_1(t) := 1 - \frac{b}{t^2}\ .$$

Finally, by Theorem 4 with $\omega := \omega_{\mathrm{e}}$, there exists increasing and unbounded solution $y(t)$ of (1) *as $t \to \infty$.* $\square$

Remark 5. *The discussed above influence of the coefficient a can be now treated as follows. Slight perturbation of the coefficient $a(t) := 1$, in situation when Theorem 8 holds, leads to substantial enlargement or the range of the coefficient β (compare inequalities* (22) *and* (23)*) such that the property of existence of increasing unbounded solutions remains preserved.*

Example 2. *Let us show that unbounded increasing solution of* (1) *as $t \to \infty$ can exists even in the case when the inequality* (20) *holds. This can be caused due to smallness of a.*

Theorem 9. *Put $\beta(t) := 1/\sqrt{t}$ on I_{-1} with $t_0 > \tau$. Let there exists a constant $q \in (0,1)$ such that $a : I \to [0,q]$. Then there exists increasing and unbounded solution $y(t)$ of* (1) *as $t \to \infty$.*

Proof. *Let us verify that the integral inequality* (10) *have a solution given by formula $\varepsilon(t) := 1/\sqrt{t}$. We proceed similarly as in the proof of Theorem 8. The left-hand side $\mathcal{L}(t)$ of* (10) *equals*

$$\mathcal{L}(t) \equiv \exp\left[-\int_{t-\delta}^{t} \varepsilon(s)\beta(s)\,\mathrm{d}s\right] = \exp\left[-\int_{t-\delta}^{t} \frac{1}{s}\,\mathrm{d}s\right] = \exp\left[-\ln\frac{t}{t-\delta}\right] = 1-\frac{\delta}{t}$$

Computation of the right-hand side $\mathcal{R}(t)$ of (3) *leads to*

$$\mathcal{R}(t) \equiv \varepsilon(t) + a(t)\exp\left[-\int_{t-\tau}^{t} \varepsilon(s)\beta(s)\,\mathrm{d}s\right] = \frac{1}{\sqrt{t}} + a(t)\exp\left[-\int_{t-\tau}^{t} \frac{1}{s}\,\mathrm{d}s\right] =$$

$$\frac{1}{\sqrt{t}} + a(t)\exp\left[-\ln\frac{t}{t-\tau}\right] = \frac{1}{\sqrt{t}} + a(t)\left(1-\frac{\tau}{t}\right) < \frac{1}{\sqrt{t}} + q\left(1-\frac{\tau}{t}\right).$$

Inequality $\mathcal{L}(t) \geq \mathcal{R}(t)$ will be valid if

$$1-\frac{\delta}{t} > \frac{1}{\sqrt{t}} + q\left(1-\frac{\tau}{t}\right).$$

This inequality obviously holds for sufficiently large t since, by supposition, $q < 1$. So, function

$$\omega_e(t) := \exp\left[\int_{t_0-\tau}^{t} \varepsilon(s)\beta(s)\,\mathrm{d}s\right] = \exp\left[\int_{t_0-\tau}^{t} \frac{1}{s}\,\mathrm{d}s\right] = \frac{t}{t_0-\tau}$$

is (under supposition that t_0 is sufficiently large) a solution of the integral inequality (3) *and $\omega_e(\infty) = +\infty$. By Theorem 5, there exists increasing and unbounded solution $y(t)$ of* (1) *as $t \to \infty$ satisfying inequality $y(t) \geq t/(t_0-\tau)$.* □

6. Open problem

Problem 1. *Comparing inequalities* (22), (23) *the following open question arises. Can be the affirmation of Theorem 8 improved in the following sense? Exists a function b^* satisfying on I_{-1} inequalities*

$$1-\frac{b}{t^2} < b^*(t) < 1$$

such that formulated statement remains valid if for the function a inequalities

$$0 \le a(t) \le b^*(t)$$

on I_{-1} hold?

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