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ON EDGE-CONNECTIVITY OF INSERTED GRAPHS

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Abstract. The aim of this paper is to estimate the edge-connectivity of the inserted graph with the help of the degree of vertices of the inserted graph and the edge-connectivity of the original graph.

1. **Introduction**

Throughout the paper we consider ordinary graphs (finite, undirected, with no loops or multiple edges) and G denotes a graph with vertex set V_G and edge set E_G. Each member of $V_G \cup E_G$ will be called an element of G. A graph G is called trivial graph if it has a vertex set with single vertex and a null edge set. If e be an edge of a graph G with end vertices x and y, then we denote the edge e, by $e = xy$.

We introduce the notions of box graph $B(G)$ and inserted graph $I(G)$ of a non-trivial graph G in [3]. It is an elementary basic fact that the inserted graph $I(G)$ of a non-trivial connected graph G in connected. The edge-connectivity $\lambda(G)$ of a graph G is the least number of edges whose removal disconnects G; and a set of $\lambda(G)$ edges satisfying this condition is called a minimal separating edge set of G. Clearly, G is m-edge-connected if and only if $\lambda(G) \geq m$.

In §2, we recall some definitions and results which will be used in §3 and also give an example of edge-connectivity of a graph G and its inserted graph $I(G)$.

In [1], we investigate the relations between the connectivity and edge-connectivity of a graph and its inserted graph. In §3 of this paper we obtain more

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results about edge-connectivity and give an alternative proof of some corollaries stated in [1].

2. **Preliminaries**

Definition 2.1. [3] *A graph can be constructed by inserting a new vertex on each edge of G, the resulting graph is called Box graph of G, denoted by $B(G)$.*

Definition 2.2. [3] *Let I_G be the set of all inserted vertices in $B(G)$. A graph $I(G)$ with vertex set I_G is called the inserted graph in which any two vertices are adjacent if they are joined by a path of length two in $B(G)$.*

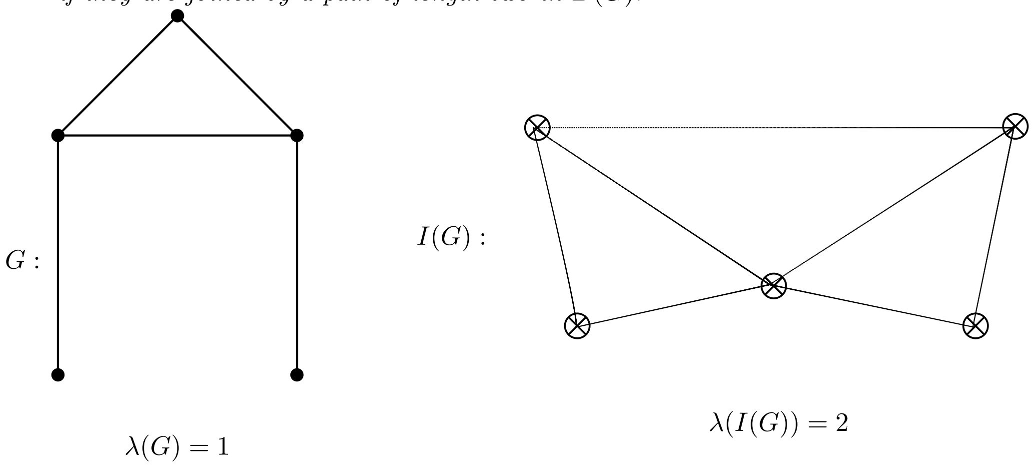

Figure 1 : The edge - connectivity of a graph and its inserted graph

These concepts are illustrated for a graph G and its inserted graph $I(G)$ in the Fig.1. Here $\bigotimes$ marked vertices are the newly inserted vertices.

Now we recall the following theorems:

Theorem 2.3. [4] *A graph G is m-edge-connected if and only if for every non-empty proper subset A of the vertex set V_G of the graph G, the number of edges joining A and $V_G - A$ is at least m.*

The next observation is due to Whitney [5].

Theorem 2.4. *For any graph G, $\lambda(G) \leq min\ deg\ G$.*

The order of a graph is the cardinality of its vertex set. If $G^{'}$ is a subgraph of G and $V_{G'}$, V_G are the vertex sets of $G^{'}$ and G respectively, then the degree of

$G^{'}$ in G is the number of all edges of G joining vertices in $V_{G'}$ with the vertices in $V_G - V_{G'}$.

3. Edge-connectivity of $I(G)$

To begin with let us prove the following lemma.

Lemma 3.1. *If*

$$\lambda(I(G)) < \lambda(G)[\frac{\lambda(G)+1}{2}],$$

then there exists a connected subgraph of G of order 2 and degree $\lambda(I(G)$ in G.

Proof: Let Y denote any nonempty proper subset of the edge set E_G of G. Thus Y induces a nonempty proper subset $\overline{Y}$ of the vertex set $V_{I(G)}$. For each vertex u in G, denote the number of edges of Y incident with u by $\delta(u)$ and the number of edges of $E_G - Y$ incident with u by $\delta^{'}(u)$; and set $W = \{u; \delta(u) > 0, \delta^{'}(u) > 0\}$. Suppose that each connected subgraph of G with two vertices has degree at least $\lambda(I(G)) + 1$ in G. We shall show that

$$\sum_{u \in W} \delta(u)\delta^{'}(u) \geq \lambda(I(G)) + 1.$$

First, suppose that no two vertices of W are adjacent. Now from the Theorem 2.4, $deg\ \ u \geq \lambda(G)$ for every vertex $u \in W$. Thus one of the numbers $\delta(u)$ and $\delta^{'}(u)$ must be $[\frac{\lambda(G)+1}{2}]$. Consequently,

$$\sum_{u \in W} \delta(u)\delta^{'}(u) \geq [\frac{\lambda(G)+1}{2}] \sum_{u \in W} \delta_u(u),$$

where δ_u means δ or $\delta^{'}$. From the $\lambda(G)$-edge-connectivity of G it follows that

$$\sum_{u \in W} \delta_u(u) \geq \lambda(G),$$

and hence

$$\sum_{u \in W} \delta(u)\delta^{'}(u) \geq \lambda(G)[\frac{\lambda(G)+1}{2}] > \lambda(I(G)).$$

Suppose now that two adjacent vertices, say v and w, belonging to W. We assume that the degree of the subgraph generated by v and w is at least $\lambda(I(G)) + 1$

in G, i.e.

$$\delta(v) + \delta^{'}(v) + \delta(w) + \delta^{'}(w) \geq \lambda(I(G)) + 3.$$

Since for any natural numbers N_1 and N_2, $N_1 N_2 \geq N_1 + N_2 - 1$, we may write

$$\sum_{u \in W} \delta(u)\delta^{'}(u) \geq \delta(v)\delta^{'}(v) + \delta(w)\delta^{'}(w) \geq \delta(v) + \delta^{'}(v) - 1 + \delta(w) + \delta^{'}(w) - 1 \geq \lambda(I(G)) + 1.$$

By application of Theorem 2.3, the inequality

$$\sum_{u \in W} \delta(u)\delta^{'}(u) \geq \lambda(I(G)) + 1$$

proved above for a set W derived from an arbitrary proper subset $\overline{Y}$ of $V_{I(G)}$ shows that $I(G)$ is $(\lambda(I(G)) + 1)$-edge-connected, which is by definition impossible.

Therefore there exist a connected subgraph $G^{'}$ of G of order 2 and of degree at most $\lambda(I(G))$; if this degree becomes smaller than $\lambda(I(G))$, then the corresponding vertex of $I(G)$ have degree smaller than $\lambda(I(G))$, contradicting the Theorem 2.4. Hence $G^{'}$ has precisely the degree $\lambda(I(G))$ in G.

We now show that Corollaries 3.5 and 3.6 of [1] follows from the above Lemma.

Corollary 3.2. [1] *$\lambda(I(G)) \geq 2\lambda(G) - 2$.*

Proof: We prove the corollary by the method of contradiction.
Suppose that $\lambda(I(G)) < 2\lambda(G) - 2$. Since

$$2\lambda(G) - 2 \leq \lambda(G)[\frac{\lambda(G)) + 1}{2}],$$

Lemma 3.1 implies the existence of a connected subgraph $G^{'}$ of G with two vertices of degree $\lambda(I(G))$ in G; since this degree is smaller than $2\lambda(G) - 2$, the degree of one of the vertices of $G^{'}$ is at most $\lambda(G) - 1$, contradicting Theorem 2.4.

Corollary 3.3. [1] *If $\lambda(G) \neq 2$, then $\lambda(I(G)) = 2\lambda(G) - 2$ if and only if there exist two adjacent vertices in G with degree $\lambda(G)$.*

Proof: For $\lambda(G) \neq 2$,

$$2\lambda(G) - 2 < \lambda(G)[\frac{\lambda(G)) + 1}{2}].$$

Hence by using Lemma 3.1, it follows that if

$$\lambda(I(G)) = 2\lambda(G) - 2,$$

then there exist two adjacent vertices v, w in G so that

$$deg \quad v + deg \quad w = \lambda(I(G)) + 2.$$

Since both v and w have degree $\geq \lambda(G)$ and

$$deg \quad v + deg \quad w = 2\lambda(G),$$

it follows immediately by Theorem 2.4 that

$$deg \quad v = deg \quad w = \lambda(G).$$

Conversely, if v and w are adjacent vertices of G and

$$deg \quad v = deg \quad w = \lambda(G),$$

then the vertex in $I(G)$ corresponding to the edge joining v and w has degree $2\lambda(G)-2$. Hence by Theorem 2.4,

$$\lambda(I(G)) \leq 2\lambda(G) - 2.$$

Now by Corollary 3.2, it follows that

$$\lambda(I(G)) = 2\lambda(G) - 2.$$

Corollary 3.4. *If* $\lambda(G) \geq 3$, *then* $\lambda(I(G)) = 2\lambda(G)-1$ *only if there exist two adjacent vertices in* G*, one of degree* $\lambda(G)$ *and the other of degree* $\lambda(G) + 1$ *.*

Proof: Proof is similar to that of Corollary 3.3

This procedure can be continued finitely as the graph is finite. Now we prove the following significant theorem.

Theorem 3.5. *If*

$$min \quad deg \quad I(G) \leq \lambda(G)[\frac{\lambda(G)+1}{2}],$$

then $\lambda(I(G)) = min \quad deg \quad I(G)$*. Also if*

$$min \quad deg \quad I(G) \geq \lambda(G)[\frac{\lambda(G)+1}{2}],$$

then

$$\lambda(G)[\frac{\lambda(G)+1}{2}] \leq \lambda(I(G)) \leq min \;\; deg \;\; I(G).$$

Proof: Theorem 2.4 implies $\lambda(I(G)) \leq min \;\; deg \;\; I(G)$. Now for the case

$$min \;\; deg \;\; I(G) \leq \lambda(G)[\frac{\lambda(G)+1}{2}]$$

suppose that $\lambda(I(G)) < min \;\; deg \;\; I(G)$. Then Lemma 3.1 asserts that there exists a connected subgraph of order 2 and degree $\lambda(I(G))$ in G; this means that there is a vertex in $I(G)$ of degree $\lambda(I(G))$, violating the assumed inequality. Consequently,

$$\lambda(I(G)) = min \;\; deg \;\; I(G).$$

For the case

$$min \;\; deg \;\; I(G) \geq \lambda(G)[\frac{\lambda(G)+1}{2}],$$

it remains to be shown that

$$\lambda(G)[\frac{\lambda(G)+1}{2}] \leq \lambda(I(G)).$$

Suppose on the contrary that

$$\lambda(G)[\frac{\lambda(G)+1}{2}] > \lambda(I(G)).$$

Then by Lemma 3.1 some vertex in $I(G)$ has degree $\lambda(I(G))$. Hence

$$min \;\; deg \;\; I(G) \leq \lambda(I(G)).$$

Thus it follows that

$$\lambda(G)[\frac{\lambda(G)+1}{2}] \leq \lambda(I(G)),$$

contradicting the inequality assumed above.

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