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THE EXPONENTIAL MAP ON THE SECOND ORDER TANGENT BUNDLE

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Abstract. On the 2-tangent (or 2-jet) bundle T^2M of a Riemannian manifold endowed with geometrical objects as in [1] and [2], the first variation of the energy $E = \int\limits_0^1 \langle \dot{c}, \dot{c} \rangle$ is computed and the conditions such that its extremal curves should be invariant to the group of homotheties are determined. In these conditions, by using homotheties, we define the exponential map on T^2M.

1. **Introduction**

The geometry of the second order tangent bundle T^2M (called as well "2-osculator bundle" and denoted by Osc^2M), constructed by R. Miron and Gh. Atanasiu, ([12]-[17]) represents the geometry of the jet-space J_0^2M, endowed with characteristic geometrical objects as: 2-tangent structure, nonlinear connections and N-linear connections. This construction allows the prolongation to T^2M of Riemannian and Finslerian structures of M. Within this geometrical framework, V. Balan and P.Stavrinos ([3], [4], [18]), defined geodesics of T^2M as stationary curves of the distance Lagrangian $L(c) = \sqrt{\langle \dot{c}, \dot{c} \rangle}$ and deduced their equations. In these papers, the authors use linear connections D with the property that the 2-tangent structure J is absolutely parallel with respect to D.

A notion which plays a major role in our considerations is that of homogeneity of a function given on T^2M (respectively, of a vector field on T^2M), defined and studied by M. de Leon and E. Vasquez, [5], R. Miron, [7], Gh. Atanasiu, [2].

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In this paper, we define geodesics as extremal curves of the energy Lagrangian $E = \langle \dot{c}, \dot{c} \rangle$ (not of the distance Lagrangian, as in [3], [4], [18]), deduce their equations (Theorem 5), study the conditions that an exponential map could be defined on T^2M (Theorem 7) and construct this application. It is worth mentioning the following facts:

1. for Lagrangians defined on T^2M, the integral of action $I(c)$ essentially depends on the parametrization of the curve c; this is why the classical technique of defining the exponential map (which relies on re-parametrizations) is here replaced by a technique which uses the group of homotheties;

2. throughout the paper, by N- linear connection we shall mean (as in Gh. Atanasiu, [1]) a linear connection which preserves by parallelism the distributions generated by a nonlinear connection N, but is not necessarily compatible with J.

2. **The 2-tangent bundle** T^2M

Let M be a real differentiable manifold of dimension n and class $\mathcal{C}^\infty$; the coordinates of a point $x \in M$ in a local chart (U, ϕ) will be denoted by $\phi(x) = (x^i)$, $i = 1, ..., n$. Let (Osc^2M, π^2, M) be its *2-osculator bundle* ([12]-[17]), which will be called in the following, *2-tangent bundle* and denoted by (T^2M, π^2, M), ([1], [2]. A point of T^2M will have in a local chart the coordinates $(x^i, y^{(1)i}, y^{(2)i})$.

Let N be a nonlinear connection on T^2M, given by its coefficients $(\underset{(1)}{N}{}^i_j, \underset{(2)}{N}{}^i_j)$, [1], [7], [8]. Then, the adapted basis to N is

$$\mathcal{B} = \{\delta_i := \frac{\delta}{\delta x^i} = \frac{\delta}{\delta y^{(0)i}}, \delta_{1i} := \frac{\delta}{\delta y^{(1)i}}, \delta_{2i} := \frac{\delta}{\delta y^{(2)i}}\},$$

where

$$\begin{cases} \dfrac{\delta}{\delta x^i} = \dfrac{\partial}{\partial x^i} - \underset{(1)}{N}{}^j_i \dfrac{\partial}{\partial y^{(1)j}} - \underset{(2)}{N}{}^j_i \dfrac{\partial}{\partial y^{(2)j}} \\ \dfrac{\delta}{\delta y^{(1)i}} = \dfrac{\partial}{\partial y^{(1)i}} - \underset{(1)}{N}{}^j_i \dfrac{\partial}{\partial y^{(2)j}} \\ \dfrac{\delta}{\delta y^{(2)i}} = \dfrac{\partial}{\partial y^{(2)i}}. \end{cases} \tag{1}$$

The dual basis of $\mathcal{B}$ is $\mathcal{B}^* = \{dx^i, \delta y^{(1)i}, \delta y^{(2)i}\}$, given by

$$\delta y^{(0)i} = dx^i,\ \delta y^{(1)i} = dy^{(1)i} + \underset{(1)}{M}{}^i_j dx^j,\ \delta y^{(2)i} = dy^{(2)i} + \underset{(1)}{M}{}^i_j dy^{(1)j} + \underset{(2)}{M}{}^i_j dx^j. \tag{2}$$

The bases above correspond to the direct-sum decomposition

$$\begin{aligned} T_u\left(T^2M\right) &= N_u \oplus N_{1u} \oplus V_{2u}, \\ T_u^*\left(T^2M\right) &= N_u^* \oplus N_{1u}^* \oplus V_{2u}^*, \forall u \in T^2M. \end{aligned} \tag{3}$$

Then, a vector field $X \in \mathcal{X}\left(T^2M\right)$ is represented in the local adapted basis as

$$X = X^{(0)i}\delta_i + X^{(1)i}\delta_{1i} + X^{(2)i}\delta_{2i}, \tag{4}$$

with the three right terms (called *d-vector fields*) belonging to the distributions N, N_1 and V_2 respectively.

A 1-form $\omega \in \mathcal{X}^*\left(T^2M\right)$ will be decomposed as

$$\omega = \omega_i^{(0)}dx^i + \omega_i^{(1)}\delta y^{(1)i} + \omega_i^{(2)}\delta y^{(2)i}. \tag{5}$$

Similarly, a tensor field $T \in \mathcal{T}_s^r\left(T^2M\right)$ can be split with respect to (3) into components , named *d-tensor fields.*

The $\mathcal{F}\left(T^2M\right)$-linear mapping $J : \mathcal{X}\left(T^2M\right) \to \mathcal{X}\left(T^2M\right)$ given by

$$J\left(\delta_i\right) = \delta_{1i}, J\left(\delta_{1i}\right) = \delta_{2i}, J\left(\delta_{2i}\right) = 0 \tag{6}$$

is called the *2-tangent structure on* T^2M,[7], [8].

Let

$$H = \left\{h_t \mid h_t : \mathbb{R} \to \mathbb{R},\ t \in \mathbb{R}_+^*\right\}$$

be the *group of homotheties*, ([1], [5], [7]), of the real numbers set. Then, H acts on T^2M as a one-parameter group of transformations, as follows:

$$\begin{aligned} \left(h_t, u\right) \mapsto h_t\left(u\right) : H \times T^2M &\to T^2M, \text{ where} \\ h_t\left(x, y^{(1)}, y^{(2)}\right) &= \left(x, ty^{(1)}, t^2y^{(2)}\right). \end{aligned} \tag{7}$$

A function $f : T^2M \to \mathbb{R}$, which is differentiable on $\widetilde{T^2M}$ and continuous on the null-section $0 : M \to T^2M$ is called *homogeneous of degree* r ($r \in \mathbb{Z}$) (or, shortly, r-homogeneous) on the fibres of T^2M, if

$$f \circ h_t = t^r f,\ \ \forall t \in \mathbb{R}_+^*, \tag{8}$$

A vector field $X \in \mathcal{X}\left(T^2M\right)$, $X = X^{(0)i}\dfrac{\partial}{\partial x^\iota} + X^{(1)i}\dfrac{\partial}{\partial y^{(1)\iota}} + X^{(2)i}\dfrac{\partial}{\partial y^{(2)\iota}}$, is *r-homogeneous* , [1], if and only if $X^{(0)i}$ are $(r-1)$-homogeneous, $X^{(1)i}$ are r-homogeneous and $X^{(2)i}$ are $(r+1)$-homogeneous functions.

3. N- linear connections

An *N-linear connection D*, [1], is a linear connection on T^2M, which preserves by parallelism the distributions N, N_1 and V_2. An N- linear connection, in the sense of the definition above, is not necessarily compatible to the 2-tangent structure J (an N-linear connection which is also compatible to J is called, [1], a *JN-linear connection*).

An N- linear connection is locally given by its coefficients

$$D\Gamma\left(N\right) = \left(\underset{(00)}{L}{}^{i}{}_{jk}, \underset{(10)}{L}{}^{i}{}_{jk}, \underset{(20)}{L}{}^{i}{}_{jk}, \underset{(01)}{C}{}^{i}{}_{jk}, \underset{(11)}{C}{}^{i}{}_{jk}, \underset{(21)}{C}{}^{i}{}_{jk}, \underset{(02)}{C}{}^{i}{}_{jk}, \underset{(12)}{C}{}^{i}{}_{jk}, \underset{(22)}{C}{}^{i}{}_{jk}\right), \tag{9}$$

where

$$\begin{cases} D_{\delta_k}\delta_j = \underset{(00)}{L}{}^{i}{}_{jk}\delta_i,\ D_{\delta_k}\delta_{1j} = \underset{(10)}{L}{}^{i}{}_{jk}\delta_{1i},\ D_{\delta_k}\delta_{2j} = \underset{(20)}{L}{}^{i}{}_{jk}\delta_{2i} \\ D_{\delta_{1k}}\delta_j = \underset{(01)}{C}{}^{i}{}_{jk}\delta_i,\ D_{\delta_{1k}}\delta_{1j} = \underset{(11)}{C}{}^{i}{}_{jk}\delta_{1i},\ D_{\delta_{1k}}\delta_{2j} = \underset{(21)}{C}{}^{i}{}_{jk}\delta_{2i} \\ D_{\delta_{2k}}\delta_j = \underset{(02)}{C}{}^{i}{}_{jk}\delta_i,\ D_{\delta_{2k}}\delta_{1j} = \underset{(12)}{C}{}^{i}{}_{jk}\delta_{1i},\ D_{\delta_{2k}}\delta_{2j} = \underset{(22)}{C}{}^{i}{}_{jk}\delta_{2i} \end{cases}. \tag{10}$$

In the particular case when D is J-compatible, we have

$$\begin{aligned} \underset{(00)}{L}{}^{i}{}_{jk} &= \underset{(10)}{L}{}^{i}{}_{jk} = \underset{(20)}{L}{}^{i}{}_{jk} =: L^{i}{}_{jk}, \\ \underset{(01)}{C}{}^{i}{}_{jk} &= \underset{(11)}{C}{}^{i}{}_{jk} = \underset{(21)}{C}{}^{i}{}_{jk} = \underset{(1)}{C}{}^{i}{}_{jk}, \\ \underset{(02)}{C}{}^{i}{}_{jk} &= \underset{(12)}{C}{}^{i}{}_{jk} = \underset{(22)}{C}{}^{i}{}_{jk} = \underset{(2)}{C}{}^{i}{}_{jk}. \end{aligned}$$

The torsion tensor of an N- linear connection D, $T\left(X,Y\right) = D_XY - D_YX - [X,Y]$, is well determined by the following components, which are d-tensors of $(1,2)$-type ([1], [7], [8]:

$$v_\gamma T(\delta_{\beta k}, \delta_{\alpha j}) =: \underset{(\alpha\beta)}{\overset{(\gamma)}{T}}{}^{i}{}_{jk}\delta_{\gamma i},\ \ \alpha,\beta,\gamma = 1,2;$$

the detailed expressions of $\underset{(\alpha\beta)}{\overset{(\gamma)}{T}}{}^{i}{}_{jk}$ can be found in [1].

The curvature of the N- linear connection D, namely, $R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z$, is completely determined by its components (which are d-tensors):

$$R(\delta_{\gamma l}, \delta_{\beta k})\delta_{\alpha j} = \underset{(\alpha\beta\gamma)}{R}{}_{j\,kl}^{\,i}\delta_{\alpha i},\ \alpha,\beta,\gamma = 0,1,2.$$

4. **Metric structures and geodesics on T^2M**

A *Riemannian metric* on T^2M is a tensor field G of type $(0,2)$, which is non-degenerate at each point $p \in T^2M$ and is positively defined on T^2M.

If G is a Riemannian metric on T^2M, we denote

$$\langle X, Y\rangle := G(X,Y),\ \forall X,Y \in \mathcal{X}(T^2M). \tag{11}$$

In this paper, we shall only consider metrics in the form

$$G = \underset{(0)}{g}{}_{ij}dx^i \otimes dx^j + \underset{(1)}{g}{}_{ij}\delta y^{(1)i} \otimes \delta y^{(1)j} + \underset{(2)}{g}{}_{ij}\delta y^{(2)i} \otimes \delta y^{(2)j}, \tag{12}$$

this is, so that the distributions N, N_1 and V_2 generated by the nonlinear connection N are orthogonal in pairs with respect to G.

An N- linear connection D is called *metrical* if $D_X G = 0$, $\forall X \in \mathcal{X}(T^2M)$. This means

$$X\langle Y,Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle,\ \forall X,Y,Z \in \mathcal{X}\left(T^2M\right). \tag{13}$$

In the following, we shall consider throughout the paper T^2M endowed with:

- a nonlinear connection N;
- a Riemannian metric G;
- a metrical N- linear connection D with coefficients 9.

Let $c : [0,1] \to T^2M$, $c(t) = \left(x^i(t), y^{(1)i}(t), y^{(2)i}(t)\right)$ be a piecewise smooth curve and $0 = t_0 < t_1 < ... < t_k = 1$ a division of $[0,1]$ so that $c_{|[t_{i-1},t_i]}$ be of class C^∞ on each interval $[t_{i-1}, t_i]$. Let us denote $c(0) = p$, $c(1) = q$.

A *variation* of c (with fixed endpoints) is a mapping $\alpha : (-\varepsilon, \varepsilon) \times [0,1] \to T^2M$, (where $\varepsilon > 0$), with the properties

1. $\alpha(0,t) = c(t)$, $\forall t \in [0,1]$;
2. α is continuous on each $(-\varepsilon,\varepsilon) \times [t_{i-1}, t_i]$, $\forall i = 1,...,k$.

Let $\overline{\alpha}$ the mapping defined on $(-\varepsilon, \varepsilon)$ by

$$\overline{\alpha}(u)(t) = \alpha(u, t).$$

If α is a variation with fixed endpoints of c, then the vector field $W \in \mathcal{X}\left(T^2M\right)$ along c, given by

$$W(t) = \frac{\partial \alpha}{\partial u}(0, t) \tag{14}$$

is called the *deviation vector field* , [3], [4], [18], attached to α. We obviously have

$$W(0) = W(1) = 0.$$

Let us denote, as in [3], [4], [18], $V = \dot{c}$. Then, V locally writes

$$\dot{c} = V = V^{(\alpha)i}\delta_{\alpha i},$$

with

$$V^{(0)i} = \frac{dx^i}{dt}, \; V^{(1)i} = \frac{\delta y^{(1)i}}{dt}, \; V^{(2)i} = \frac{\delta y^{(2)i}}{dt}.$$

Let also

$$A := \frac{DV}{dt} = A^{(0)i}\delta_i + A^{(1)i}\delta_{1i} + A^{(2)i}\delta_{2i}, \tag{15}$$

be the covariant acceleration, where, for $X \in \mathcal{X}\left(T^2M\right)$, we denoted

$$\frac{DX}{dt} := D_{\dot{c}}X,$$

and

$$\Delta_t X = X(t_+) - X(t_-), \; t \in [0,1], \; X \in \mathcal{X}\left(T^2M\right), \tag{16}$$

the *jump* of $X \in \mathcal{X}\left(T^2M\right)$ in t.

The *energy* of the curve c is

$$E(c) = \int_0^1 \underset{(0)}{g}_{ij} V^{(0)i}V^{(0)j} + \underset{(1)}{g}_{ij} V^{(1)i}V^{(1)j} + \underset{(2)}{g}_{ij} V^{(2)i}V^{(2)j} dt, \tag{17}$$

this is, $E(c) = \int_0^1 \langle V, V\rangle \, dt$.

Definition 1. *We call a* geodesic *of* T^2M, *a critical path* $c : [0,1] \to T^2M$ *of the energy* E, *which is* C^∞*-smooth on the whole* $[0,1]$.

By a direct computation, taking into account the metricity of the N- linear connection D, we obtain

Theorem 2. ***(The first variation of the energy):*** *If* $c : [0,1] \to T^2M$ *is a piecewise smooth path and* $\alpha : (-\varepsilon, \varepsilon) \times [0,1] \to T^2M$ *is a variation with fixed endpoints of* c, *then*

$$\frac{1}{2}\frac{dE\left(\overline{\alpha}\left(u\right)\right)}{du}\rfloor_{u=0} = -\sum_{i=0}^{k-1}\left\langle W, \Delta_{t_i}V\right\rangle + \int_0^1 \left\langle T\left(W,V\right),V\right\rangle - \left\langle W,A\right\rangle dt. \tag{18}$$

Remark 3. *If the curve* c *is* C^{∞}*-smooth on the whole* $[0,1]$, *then*

$$\frac{1}{2}\frac{dE\left(\overline{\alpha}\left(u\right)\right)}{du}\rfloor_{u=0} = \int_0^1 \left\langle T\left(W,V\right),V\right\rangle - \left\langle W,A\right\rangle dt.$$

In order to deduce the equations of the geodesics of T^2M, in (18), we write the term $\left\langle T\left(W,V\right),V\right\rangle$ in the form $\langle F, W\rangle$; in local coordinates, one obtains

Theorem 4. 1. $F = \sum\limits_{\alpha=0}^{2} F^{(\alpha)i}\delta_{\alpha i}$ *given by*

$$F^{(\alpha)i} = \underset{(\alpha)}{g}{}^{il}\,\underset{(\gamma)}{g}{}_{kh}\,\overset{(\gamma)}{\underset{(\beta\alpha)}{T}}{}^{k}_{jl}V^{(\beta)j}V^{(\gamma)h}, \quad \alpha = 0,1,2 \tag{19}$$

is a vector field, globally defined along c.

2. *There holds the equality*

$$\left\langle T\left(W,V\right),V\right\rangle = \left\langle W,F\right\rangle. \tag{20}$$

3. *The vector field* F *does not depend on the variation* α *of* c.

Taking into account the previous theorem, we get

$$\frac{1}{2}\frac{dE\left(\overline{\alpha}\left(u\right)\right)}{du}\rfloor_{u=0} = -\sum_{i=0}^{k-1}\left\langle W, \Delta_{t_i}V\right\rangle - \int_0^1 \left\langle W, F - A\right\rangle dt.$$

We have proved this way

Theorem 5. *The* C^{∞}*-smooth curve* $c : [0,1] \to T^2M$, $t \mapsto \left(x^i\left(t\right), y^{(1)i}\left(t\right), y^{(2)i}\left(t\right)\right)$ *is a geodesic of* T^2M *if and only if*

$$\frac{D}{dt}\frac{dc}{dt} = F, \tag{21}$$

or, in local coordinates,

$$\begin{aligned} \frac{DV^{(0)i}}{dt} &= F^{(0)i}, \\ \frac{DV^{(1)i}}{dt} &= F^{(1)i}, \\ \frac{DV^{(2)i}}{dt} &= F^{(2)i}. \end{aligned} \tag{22}$$

It will be useful to write the last equalities in the following form

$$\begin{aligned} \frac{dV^{(0)i}}{dt} + \underset{(00)}{L}{}^{i}_{jk} V^{(0)k} V^{(0)j} + \underset{(01)}{C}{}^{i}_{jk} V^{(1)k} V^{(0)j} + \underset{(02)}{C}{}^{i}_{jk} V^{(2)k} V^{(0)j} &= F^{(0)i}, \\ \frac{dV^{(1)i}}{dt} + \underset{(10)}{L}{}^{i}_{jk} V^{(0)k} V^{(1)j} + \underset{(11)}{C}{}^{i}_{jk} V^{(1)k} V^{(1)j} + \underset{(12)}{C}{}^{i}_{jk} V^{(2)k} V^{(1)j} &= F^{(1)i}, \\ \frac{dV^{(2)i}}{dt} + \underset{(00)}{L}{}^{i}_{jk} V^{(0)k} V^{(2)j} + \underset{(01)}{C}{}^{i}_{jk} V^{(1)k} V^{(2)j} + \underset{(02)}{C}{}^{i}_{jk} V^{(2)k} V^{(2)j} &= F^{(2)i}. \end{aligned} \tag{23}$$

5. Invariance to homotheties of the equations of geodesics

We consider the homotheties h_λ in (7).

Definition 6. *Let* $c : [0,1] \to T^2M$, $t \mapsto c(t) = \left(x(t), y^{(1)}(t), y^{(2)}(t)\right)$ *be an arbitrary curve and* $\lambda > 0$ *a real number. We call the* homothetic *of* c *the curve*

$$\bar{c} : \left[0, \frac{1}{\lambda}\right] \to T^2M, \ \bar{c}\left(\frac{1}{\lambda}t\right) := h_\lambda(c(t)), \tag{24}$$

and we denote

$$\bar{c} = h_\lambda(c).$$

Let us remark that $h_\lambda(c) \neq h_\lambda \circ c$.

$\bar{c} = h_\lambda(c)$ locally writes

$$\begin{aligned} \overline{x}^i\left(\frac{1}{\lambda}t\right) &= x^i(t), \\ \overline{y}^{(1)i}\left(\frac{1}{\lambda}t\right) &= \lambda y^{(1)i}(t), \\ \overline{y}^{(2)i}\left(\frac{1}{\lambda}t\right) &= \lambda^2 y^{(2)i}(t). \end{aligned} \tag{25}$$

If we suppose that

$$\underset{(1)}{N}{}^i_j \text{ are 1-homogeneous, } \underset{(2)}{N}{}^i_j \text{ are 2-homogeneous,} \tag{26}$$

(which implies that $\underset{(1)}{M}{}^i_j$ are 1-homogeneous and, $\underset{(2)}{M}{}^i_j$ are 2-homogeneous), then, δ_i are 1-homogeneous, δ_{1i} are 0-homogeneous and δ_{2i} are –1-homogeneous; consequently, the tangent vectors of $\overline{c}$ are given by

$$\overline{V}^{(\alpha)i}\left(\frac{t}{\lambda}\right) = \lambda^{\alpha+1}V^{(\alpha)i}\left(t\right),\ \alpha = 0, 1, 2. \tag{27}$$

or $\overline{V}\left(\frac{t}{\lambda}\right) = \lambda h^*_\lambda V\left(t\right).$

If we claim that, for any geodesic c of T^2M, the homothetic $\overline{c}$ should be a geodesic, too, we obtain:

Theorem 7. *Let $\underset{(1)}{N}{}^i_j$ be 1-homogeneous , $\underset{(2)}{N}{}^i_j$ be 2-homogeneous. If:*

1. $\underset{(00)}{L}{}^i_{jk}, \underset{(10)}{L}{}^i_{jk}, \underset{(20)}{L}{}^i_{jk}$ *are homogeneous of degree* 0;
2. $\underset{(01)}{C}{}^i_{jk}, \underset{(11)}{C}{}^i_{jk}, \underset{(21)}{C}{}^i_{jk}$ *- homogeneous of degree* -1;
3. $\underset{(02)}{C}{}^i_{jk}, \underset{(12)}{C}{}^i_{jk}, \underset{(22)}{C}{}^i_{jk}$ *- homogeneous of degree* -2;
4. $\underset{(\alpha)}{g}{}_{ij}$ *- homogeneous of degree* $-\alpha$, $\alpha = 0, 1, 2$,

then the equations of the geodesics of T^2M are invariant to the homotheties (24).

Proof. 1., 2. and 3. can be obtained by a direct computation.

In order to prove 4., we must take into account that:

- $V^{(\alpha)i}$ are $(\alpha+1)$-homogeneous;
- in the expression of $F^{(\alpha)i}$, the term $\underset{(\beta\alpha)}{\overset{(\gamma)}{T}}{}^k_{jl}V^{(\beta)j}V^{(\gamma)h}$ is homogeneous of degree $\gamma - \beta - \alpha + \beta + 1 + \gamma + 1 = 2\gamma - \alpha + 2$;
- if $\underset{(\gamma)}{g}{}_{kh}$ are homogeneous of degree $-\gamma$, then $\underset{(\gamma)}{g}{}^{kh}$ are homogeneous of degree $+\gamma$.

$\square$

6. **The exponential map of T^2M**

It is known, [7], that for regular Lagrangians defined on T^2M, the integral of action $I(c)$ essentially depends on the parametrization of the curve c (the Zermelo conditions); consequently, the equations of geodesics (22) are generally not invariant to re-parametrizations of the form $t \mapsto \dfrac{t}{\lambda}$, $\lambda > 0$. This is why, instead of the classical technique of defining the exponential map (which relies on such re-parametrizations), we shall use the homotheties $c \mapsto \overline{c}$ as defined above.

Let us remark, for the beginning, that the equations of geodesics (22) constitute a system of $6n$ ODE system with the unknown (real) functions x^i, $y^{(1)i}$, $y^{(2)i}$, $V^{(0)i}$, $V^{(1)i}$, $V^{(2)i}$. This allows us to state an existence and uniqueness result.

For $p \in T^2M$, let us denote in the following, $p := (x^i, y^{(1)i}, y^{(2)i})$ its coordinates in a local chart and, for $X \in \mathcal{X}(T^2M)$, $X := (X^{(0)i}, X^{(1)i}, X^{(2)i})$.

Let $p_1 := (x_1^i, y_1^{(1)i}, y_1^{(2)i}) \in T^2M$ and $V_1 := (V_1^{(0)i}, V_1^{(1)i}, V_1^{(2)i}) \in T_{p_1}\left(T^2M\right)$ be arbitrary. There holds

Theorem 8. *There exists a neighbourhood W of $(p_1, V_1) \in \mathbb{R}^{6n}$ and a real number $\varepsilon > 0$ so that, for any $(p_0, V_0) \in W$, the system (22) has a unique solution*

$$t \mapsto (p\,(t)\,, V\,(t))$$

defined for $t \in (-\varepsilon, \varepsilon)$ and which satisfies the initial conditions

$$p\,(0) = p_0, \;\; V\,(0) = V_0. \tag{28}$$

Furthermore, the solution depends smoothly on the initial conditions (28).

In the conditions of Theorem 7, if c is a geodesic of T^2M, then $\overline{c} = h_\lambda\,(c)$ is also a geodesic. We are now able to state

Theorem 9. *In the conditions of Theorem 7, for any $p_0 \in T^2M$ there is an $\varepsilon > 0$ so that, for any tangent vector $V \in T_{p_0}\left(T^2M\right)$, with $\|V\| < \varepsilon$, there exists the geodesic*

$$c : (-2, 2) \to T^2M, \, t \mapsto \left(x^i\,(t)\,, y^{(1)i}\,(t)\,, y^{(2)i}\,(t)\right)$$

with the initial conditions

$$c\,(0) = p_0, \; \frac{dc}{dt}\,(0) = V.$$

Definition 10. *The point* $c(1) := \left(x^i(1), y^{(1)i}(1), y^{(2)i}(1)\right)$ *is called the* exponential of $V \in T_{p_0}\left(T^2M\right)$ *in* p_0 *and will be denoted by*

$$c(1) = \exp_{p_0}(V). \tag{29}$$

Let us prove Theorem 9:

Let $p_0 \in T^2M$, $\varepsilon > 0$ and $V \in T_{p_0}\left(T^2M\right)$ with $\|V\| < \varepsilon$ be arbitrary. Then, according to Theorem 8, for any $\overline{p}_0 \in T^2M$ and for any $\overline{V} \in T_{\overline{p}_0}\left(T^2M\right)$, there uniquely exists the geodesic $c_{\overline{V}} : (-2\varepsilon_2, 2\varepsilon_2) \to T^2M$ with

$$c_{\overline{V}}(0) = \overline{p}_0, \ \frac{dc_{\overline{V}}}{dt}(0) = \overline{V}. \tag{30}$$

We set

$$\begin{aligned} \overline{p}_0 \quad &: \quad = h_{\frac{1}{\varepsilon_2}}(p_0) \\ \overline{V} \quad &: \quad = \frac{1}{\varepsilon_2} h^*_{\frac{1}{\varepsilon_2}, p_0}(V) \in T_{\overline{p}_0}\left(T^2M\right) \\ \varepsilon \quad &< \quad \varepsilon_1\varepsilon_2. \end{aligned} \tag{31}$$

($\overline{V}$ is the tangent vector field of $h_{\frac{1}{\varepsilon_2}}(c)$).

Because $\|V\| < \varepsilon$ and according to (31), we have

$$\left\|\overline{V}\right\| = \frac{1}{\varepsilon_2}\|V\| < \varepsilon_1;$$

consequently, there uniquely exists the geodesic $c_{\overline{V}}$ with the initial conditions (30). Furthermore, if $|t| < 2$, then $|\varepsilon_2 t| < 2\varepsilon_2$, which allows us to define

$$c(t) := h_{\varepsilon_2}\left(c_{\overline{V}}(t)\right) : (-2, 2) \to T^2M,$$

then c is obviously a geodesic and is uniquely defined by the above equality. Furthermore,

$$c(0) = h_{\varepsilon_2}\left(c_{\overline{V}}(0)\right) = h_{\varepsilon_2}\left(\overline{p}_0\right) = \left(h_{\varepsilon_2} \circ h_{\frac{1}{\varepsilon_2}}\right)(p_0) = p_0.$$

Let $\overline{Z}$ be the tangent vector field of $c_{\overline{V}}$; then, $\overline{Z}(0) = \overline{V} = \frac{1}{\varepsilon_2} h^*_{\frac{1}{\varepsilon_2}, p_0}(V)$; taking into account that $h^*_{\varepsilon_2} = \left(h^*_{\frac{1}{\varepsilon_2}}\right)^{-1}$, we get

$$\frac{dc}{dt}(0) = \varepsilon_2 h^*_{\varepsilon_2, \overline{p}_0}\left(\overline{Z}\right) = \varepsilon_2 \frac{1}{\varepsilon_2} h^*_{\varepsilon_2, \overline{p}_0}\left(h^*_{\frac{1}{\varepsilon_2}, p_0}(V)\right) = V,$$

which completes the proof.

It is worth mentioning that:

1. The exponential map in $p \in T^2M$ is generally defined only for small values of $\|V\|$. If it exists, the value $\exp_p(V)$ is unique.
2. If c is a geodesic of T^2M with $p_0 = c(0)$, $V = \dot{c}(0)$, then

$$c(t) = \exp_p(tV). \tag{32}$$

7. Example

Let (M, g) be a Riemannian manifold, (T^2M, π^2, M), its second order tangent bundle and $\widetilde{T^2M} = T^2M \backslash \{0\}$, i.e., T^2M without its null section. We consider the following geometric objects on $\widetilde{T^2M}$:

- the *canonical nonlinear connection* N, [8], given by its dual coefficients $\underset{(1)}{M}{}^i_j = \gamma^i_{jk} y^{(1)k}$, $\underset{(2)}{M}{}^i_j = \dfrac{1}{2}\left\{\mathbb{C}\left(\gamma^i_{jk} y^{(1)k}\right) + \underset{(1)}{M}{}^i_k \underset{(1)}{M}{}^k_j\right\}$,

where $\gamma^i_{jk} = \gamma^i_{jk}(x)$ are the Christoffel symbols of g and $\mathbb{C} = y^{(1)i}\dfrac{\partial}{\partial x^i} + 2y^{(2)i}\dfrac{\partial}{\partial y^{(1)i}}$;

- the *homogeneous N-lift* of the metric g, defined by prof. Gh. Atanasiu, [2],

$$\overset{o}{G} = g_{ij} dx^i \otimes dx^j + \frac{a^2}{\left\|y^{(1)}\right\|^2} g_{ij}\, \delta y^{(1)i} \otimes \delta y^{(1)j} + \frac{a^4}{\left\|y^{(1)}\right\|^4} g_{ij} \delta y^{(2)i} \otimes \delta y^{(2)j},$$

  where $\left\|y^{(1)}\right\| = \sqrt{g_{ij} y^{(1)i} y^{(1)j}}$;

- the *canonical N-linear connection*, $D\overset{o}{\Gamma}(N)$, [2], given by the coefficients

$$\underset{(00)}{L}{}^i_{jk} = \underset{(10)}{L}{}^i_{jk} = \underset{(20)}{L}{}^i_{jk} = \gamma^i_{jk}(x), \quad \underset{(01)}{C}{}^i_{jk} = 0,$$

$$\underset{(11)}{C}{}^i_{jk} = -\frac{1}{\left\|y^{(1)}\right\|^2}\left(\delta^i_j y^{(1)}_k + \delta^i_k y^{(1)}_l - g_{jk} y^{(1)i}\right), \quad \underset{(21)}{C}{}^i_{jk} = 2\,\underset{(11)}{C}{}^i_{jk},$$

$$\underset{(02)}{C}{}^i_{jk} = \underset{(12)}{C}{}^i_{jk} = \underset{(22)}{C}{}^i_{jk} = 0.$$

By a direct calculus, one proves that the conditions of Theorem 7 are accomplished; consequently, if $\widetilde{T^2M}$ is endowed with these structures, the exponential map can be defined on $\widetilde{T^2M}$.

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