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ON SOME INTEGRAL EQUATIONS WITH DEVIATING ARGUMENT

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Abstract. The purpose of this paper is to study the following functional equation with modified argument:

$$x(t) = g(t, hx(t), x(t), x(0)) + \int_{-\theta t}^{\theta t} K(t, s, x(s))ds,$$

where $\theta \in (0,1), t \in [-T, T], T > 0$.

1. **Introduction**

Let (X, d) be a metric space and $A : X \to X$ an operator.We shall use the following notations:
$F_A := \{x \in X \mid Ax = x\}$ the fixed points set of A.
$I(A) := \{Y \in P(X) \mid A(Y) \subset Y\}$ the family of the nonempty invariant subsets of A.
$A^{n+1} = A \circ A^n, A^0 = 1_X, A^1 = A, n \in N$.

Definition 1.1. [4] *An operator A is weakly Picard operator(WPO)if the sequence*

$$(A^n x)_{n \in N}$$

converges, for all $x \in X$ and the limit(which depend on x) is a fixed point of A.

Definition 1.2. [4],[1] *If the operator A is WPO and $F_A = \{x^*\}$then by definition A is Picard operator.*

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Definition 1.3. [4] *If A is WPO, then we consider the operator*

$$A^{\infty} : X \to X, A^{\infty}(x) = \lim_{n\to\infty} A^n x.$$

We remark that $A^{\infty}(X) = F_A$.

Definition 1.4. *[1] Let be A an WPO and* $c > 0$*.The operator A is c-WPO if* $d(x, A^{\infty}x) \leq d(x, Ax)$.

We have the following characterization of the WPOs

Theorem 1.1. [4]*Let* (X, d) *be a metric space and* $A : X \to X$ *an operator. The operator A is WPO (c-WPO) if and only if there exists a partition of X,*

$$X = \bigcup_{\lambda\in\Lambda} X_{\lambda}$$

such that
(a)$X_{\lambda} \in I(A)$
(b)$A \mid: X_{\lambda} \to X_{\lambda}$*is a Picard(c-Picard) operator,for all* $\lambda \in \Lambda$.

For the class of c-WPOs we have the following data dependence result.

Theorem 1.2. [4] *Let* (X, d) *be a metric space and* $A_i : X \to X, i = 1, 2$ *an operator.We suppose that :*
(i)the operator A_i *is* $c_i - WPO$*i=1,2.*
(ii)there exists $\eta > o$ *such that*

$$d(A_1x, A_2x) \leq \eta, (\forall)x \in X.$$

Then

$$H(F_{A_1}, F_{A_2}) \leq \eta max\{c_1, c_2\}.$$

Here stands for Hausdorff-Pompeiu functional

We have

Lemma 1.1. [4],[1] *Let*$(X, d, \leq)$ *be an ordered metric space and* $A : X \rightarrow X$ *an operator such that:*

a)A is monotone increasing.

b)A is WPO.

Then the operator A^{∞} *is monotone increasing.*

2. **Main results**

Data dependence for functional-integral equations was study in [2],[3],[4],[1].

Let$(X, \|\cdot\|)$ a Banach space and the space $C([-T,T],X)$ endowed with the Bieleski norm $\|\cdot\|_\tau$ defined by

$$\|x\|_\tau = \max_{t\in[-T,T]} \|x(t)\|e^{-\tau(t+T)}.$$

In[1] Viorica Muresan was study the following functional integral equation:

$$x(t) = g(t, h(x)(t), x(t), x(0)) + \int_0^t K(t, s, x(\theta s))ds, t \in [0, b], \theta \in [0, 1]$$

by the weakly Picard operators technique.

We consider the following functional-integral equations with modified argument:

$$x(t) = g(t, hx(t), x(t), x(0)) + \int_{-\theta t}^{\theta t} K(t, s, x(s))ds, \tag{1}$$

where:

i)$t \in [-T,T],, T > 0$.

ii)$h : C([-T,T],X) \longrightarrow C([-T,T],X), g \in C([-T,T] \times X^3, X), K \in C([-T,T] \times [-T,T] \times X^2, X)$.

We suppose that the following conditions are satisfied:

(c_1) there exists $l > 0$ such that

$$\|hx(t) - hy(t)\| \leq l\|x(t) - y(t)\|,$$

for all $x, y \in C([-T,T],X), t \in [-T,T]$.

(c_2) There exists $l_1 > 0, l_2 > 0$ such that

$$\|g(t,u_1,v_1,w) - g(t,u_2,v_2,w)\| \leq l_1\|u_1 - u_2\| + l_2 \|v_1 - v_2\| .$$

for all $t \in [-T,T], u_i, v_i, w \in X, i = 1,2$.

(c_3) There exists $l_3 > 0$ such that

$$\|K(t,s,u) - K(t,s,u_1)\| \leq l_3\|u - u_1\|,$$

for all $t, s \in [-T,T], u, u_1 \in X$.

$(c_4) l_1 l + l_2 < 1$.

$(c_5) g(0,h(x)(0),x(0),x(0)) = x(0)$ for any $x \in C([-T,T],X)$.

Let $A : C([-T,T],X) \longrightarrow C([-T,T],X)$ be defined by

$$Ax(t) = g(t,hx(t),x(t),x(0)) + \int_{-\theta t}^{\theta t} K(t,s,x(s))ds \tag{2}$$

Let $\lambda \in X$ and $X_\lambda = \{x \in C([-T,T],X) \mid x(0) = \lambda\}$. Then $C([-T,T],X) = \bigcup_{\lambda \in X} X_\lambda$ is a partition of $C([-T,T],X)$.From c_5 we have that $X_\lambda \in I(A)$.

For studding of data dependence we consider the following equations

$$x(t) = g_1(t,hx(t),x(t),x(0)) + \int_{-\theta t}^{\theta t} K_1(t,s,x(s))ds \tag{3}$$

$$x(t) = g_2(t,hx(t),x(t),x(0)) + \int_{-\theta t}^{\theta t} K_2(t,s,x(s))ds \tag{4}$$

Theorem 2.1. *We consider the equation (1) under following conditions:*

(i)The conditions $c_1 - c_5$ are satisfied.

(ii)The operators $h(\cdot), g(t,\cdot,\cdot,\cdot), K(t,s,\cdot,\cdot)$ are monotone increasing.

(iii)There exists $\eta_1, \eta_2 > 0$ such that

$$\|g_1(t,u,v,w) - g_2(t,u,v,w)\| < \eta_1,$$

$$\|K_1(t,s,u) - K_2(t,s,)\| \leq \eta_2$$

for all $t \in [-T, T], u, v, w \in X$. *Then:*

(a)For all x,y solutions of (1) with $x(0) \leq y(0)$ *we have* $x(t) \leq y(t)$, *for all* $t \in [-T, T]$.

(b) $H(S_1, S_2) \leq \dfrac{\eta_1 + 2\eta_2 T}{(1 - l_1 l - l_2 - \frac{l_3}{\tau})}$, *where* S_1, S_2 *is the solutions set of(3),(4).*

Proof We denote with A_λ the restriction of the operator A at X_λ.First we show that $A_\lambda$is a contraction map onX_λ.From$c_1 - c_5$ we have that

$$\|A_\lambda x(t) - A_\lambda y(t)\| \leq (l_1 l + l_2)\|x(t) - y(t)\| + \int_{-\theta t}^{\theta t} \|K(t, s, x(s)) - K(t, s, y(s))\| \, dsleq$$

$$\leq (l_1 l + l_2)\|x - y\|_\tau e^{\tau(t+T)} + l_3 \|x - y\|_\tau \int_{-\theta t}^{\theta t} e^{\tau(t+T)} ds.$$

So A is c-WPO with

$$c = \frac{1}{1 - l_1 l - l_2 - \dfrac{l_3}{\tau}}.$$

Using the theorem 1.2 we obtain (b).

For proof of (a) let be x,y solutions for(1) with $x(0) \leq y(0)$.Then $x \in X_{x(0)}, y \in X_{y(0)}$.We define

$$\widetilde{x}(t) = x(0), t \in [0, b]$$

$$\widetilde{y}(t) = y(0), t \in [0, b]$$

We have

$$\widetilde{x}(0) \in X_{x(0)}, \widetilde{y}(0) \in X_{y(0)}, \widetilde{x}(0) \leq \widetilde{y}(0).$$

From lemma 1.1 we obtain that the operator A^∞ is increasing.It follows that

$$A^\infty(\widetilde{x}(0)) \leq A^\infty(\widetilde{x}(0))$$

i.e $x \leq y$

Next we define φ -contraction notion and use this for estimate distance between two weakly Picard operators.

Let $\varphi : R_+ \longrightarrow R_+$.

Definition 2.1. [5] *φ is a strict comparison function if φ satisfies the following:*

i) φ is continuous.

ii)φ is monotone increasing.

iii) $\varphi^n(t) \longrightarrow 0$, for all $t > 0$.

iv) t-$\varphi(t) \longrightarrow \infty$,for $t \longrightarrow \infty$.

Let (X, d) be a metric space and $f : X \longrightarrow X$ an operator.

Definition 2.2. [5] *The operator f is called a strict φ-contraction if:*

(i) φ is a strict comparison function.

(ii)$d(f(x), f(y)) \leq \varphi(d(x, y))$, for all $x, y \in X$.

Theorem 2.2. [5] *Let (X, d) be a complete metric space,$\varphi : R_+ \longrightarrow R_+$ a strict comparison and $f, g : X \longrightarrow X$ two orbitally continuous operators.We suppose that:*

(i) $d(f(x), f^2(x)) \leq \varphi(d(x, f(x)))$ for any $x \in X$ and $d(g(x), g^2(x)) \leq \varphi(d(x, g(x)))$ for any $x \in X$.

(ii)there exists $\eta > 0$ such that $d(f(x), g(x)) \leq \eta$, for any $x \in X$

Then:

(a) f,g are weakly Picard operators.

(b)$H(F_f, F_g) \leq \tau_\eta$where $\tau_\eta = sup\{t \mid t - \varphi(t) \leq \eta\}$.

Theorem 2.3. *We suppose that condition (c_5) is verified and the following conditions are satisfied:*

(H_1) there exists φ a strict comparison function such that

$$(i)\|hx(t) - hy(t)\| \leq \|x(t) - y(t)\|,$$

for all $x, y \in C([-T, T], X), t \in [-T, T]$.

$$(ii)g(t, u_1, v_1, w) - g(t, u_2, v_2, w)\| \leq a\varphi(\|u_1 - u_2\|) + b\varphi(\|v_1 - v_2\|).$$

for all $t \in [-T, T], u_i, v_i, w \in X, i = 1, 2$

$$(iii)\,\|K(t, s, u) - K(t, s, u_1)\| \leq l(t, s)\varphi(\|u - u_1\|),$$

for all $t, s \in [-T, T], u, u_1, \in X$*,where* $l(t, \cdot) \in L^1[-T, T]$.

(H_2) *There exists* $\eta_1, \eta_2 > 0$ *such that*

$$\|g_1(t, u, v, w) - g_2(t, u, v, w)\| \leq \eta_1,$$

$$\|K_1(t, s, u) - K_2(t, s,)\| \leq \eta_2$$

for all $t \in [-T, T], u, v, w \in X$.

(H_3)

$$a + b + \max_{t\in[-T,T]} \int_{-T}^{T} l(t, s)ds \leq 1$$

Then:

(i)the equation (1) has at least solution.

(ii)$H(S_1, S_2) \leq \tau_\eta$ *where* $\eta = \eta_1 + 2T\eta_2, S_1, S_2$ *is the solutions set of(3),(4).*

ProofLet be$A_1, A_2 : C([-T, T], X) \longrightarrow C([-T, T], X)$,

$$A_1x(t) = g_1(t, hx(t), x(t), x(0)) + \int_{-\theta t}^{\theta t} K_1(t, s, x(s))ds$$

$$A_2x(t) = g_2(t, hx(t), x(t), x(0)) + \int_{-\theta t}^{\theta t} K_2(t, s, x(s))ds.$$

From

$$\left\|A_ix(t) - A_i^2x(t)\right\| \leq \|g_i(t, hx(t), x(t), x(0)) - g_i(t, hA_ix(t), A_ix(t), A_ix(0))\| +$$

$$+ \int_{-\theta t}^{\theta t} \|K_i(t, s, x(s)) - K_i(t, s, A_ix(s))\| \, ds$$

$$\leq a\varphi(\|hx(t) - hA_ix(t)\|) + b\varphi(\|x(t) - A_ix(t)\|)+$$

$$+ \int_{-\theta t}^{\theta t} l(t, s)\varphi(\|x(s) - A_ix(s))\| \, ds \leq a\varphi(\|x(t) - A_ix(t)\|) + b\varphi(\|x(t) - A_ix(t)\|)+$$

$$\int_{-\theta t}^{\theta t} l(t, s)\varphi(\|x(s) - A_ix(s))\| \, ds \leq (a + b + \max_{t\in[-T,T]} \int_{-T}^{T} l(t, s)ds)\varphi(\|x - A_ix\|_C) \leq$$

$$\leq \varphi(\|x - A_i x\|_C)$$

we have that

$$\|A_i x - A_i^2 x\|_C \leq \varphi(\|x - A_i x\|_C), i = \overline{1,2}.$$

Here $\|\cdot\|_C$is the Chebyshev norm on $C([-T,T],X)$.

We note that $\|A_1 x - A_2 x\|_C \leq \eta_1 + 2T\eta_2$. From this, using the theorem 2.2 we have the conclusions.

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