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BIFURCATIONS OF THE LOGISTIC MAP PERTURBED BY ADDITIVE NOISE

MARCHIS IULIANA

Abstract. In the paper we analyze, mainly numerically, the bifurcations of the logistic map perturbed by different type of additive noise: uniformly and normally distributed random variables. We prove the existence of the stationary density in both cases using some tools from [6], and study the bifurcations. In [4] there are numerical results for the uniform noise case. We extend the simulations for the logistic map perturbed by normally distributed random variables. In this case we get a different bifurcation scenario as in the case of perturbation by uniformly distributed random variables.

1. Basic Notions

Let (X, d) be a metric space and $S : X \to X$ be a discrete dynamical system. Let $x_0 \in X$. Then x_0, $x_1 = S(x_0)$, $x_2 = S(x_1)$,..., $x_n = S(x_{n-1})$,... is the orbit of x_0. In the deterministic case we usually study the orbit of different $x_0 \in X$ to find the dynamics of the system.

Now let ξ_0, ξ_1,...ξ_n,... be independent random variables and we use formula

$$x_{n+1} = S(x_n) + \xi_n, n \in \mathbb{N} \tag{1}$$

to find the orbit of a point from X. In this case the orbit of a point x_0 is different for different realization of the noise. Thus is more adequate to study the change of the initial density function.

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Let $X=\mathbb{R}$, g be the density function of the random variables ξ_0, ξ_1,...ξ_n,... and f_n the density function of x_n. Then we have to find a relation between f_n and f_{n+1}. For this we take an arbitrary bounded, measurable function $h:\mathbb{R}\to\mathbb{R}$ and calculate the expected value of $h(x_{n+1})$ in two different ways. Firstly,

$$\mathbb{E}(h(x_{n+1}))=\int_{\mathbb{R}} h(x)f_{n+1}(x)dx. \tag{2}$$

Secondly,

$$\begin{aligned}\mathbb{E}(h(x_{n+1}))&=\mathbb{E}(h(S(x_n)+\xi_n))\\ &=\int_{\mathbb{R}}\int_{\mathbb{R}} h(S(y)+z)f_n(y)g(z)dydz\\ &=\int_{\mathbb{R}}\int_{\mathbb{R}} h(x)f_n(y)g(x-S(y))dxdy, \end{aligned}\tag{3}$$

using the change of variables $S(y)+z=x$.

From (2) and (3) we get

$$f_{n+1}(x)=\int_{\mathbb{R}} f_n(y)g(x-S(y))dy. \tag{4}$$

We review some notions which we need to study the existence of a stationary density for a random dynamical system. In the followings let $(X,\mathcal{A},\mu)$ be a measure space.

A linear operator $P:L^1\to L^1$ is called **Markov operator** if

(a) $Pf\geq 0$, for all $f\geq 0$, $f\in L^1$;

(b) $\|Pf\|=\|f\|$, for all $f\geq 0$, $f\in L^1$.

A measurable function $K:X\times X\to\mathbb{R}$ is called **stochastic kernel** if

(a) $K(x,y)\geq 0$, for all $x,y\in X$;

(b) $\int_X K(x,y)\mu(dx)=1$, for all $y\in X$.

Let $G\subset\mathbb{R}^d$ be a measurable, unbounded set, $K:G\times G\to\mathbb{R}$ a stochastic kernel. A measurable, nonnegative function $V:G\to\mathbb{R}$, for which

$$\lim_{|x|\to\infty} V(x)=\infty,$$

is called **Liapunov function**.

Returning to formula (4) we observe that

$$Pf(x) = \int_{\mathbb{R}} f(y)g(x - S(y))dy \tag{5}$$

is a Markov operator and

$$K(x, y) = g(x - S(y))$$

is a stochastic kernel. We can write formula (4) in the form $f_{n+1} = Pf_n$, which is equivalent with $f_{n+1} = P^{n+1}f_0$, thus we have to study the sequence $\{P^n\}$.

Let P be a Markov operator. A density function f is a **stationary density** if $Pf = f$.

$\{P^n\}$ is **asymptotically stable** if there exists a unique stationary density f_* such that

$$\lim_{n\to\infty} \|P^n - f_*\| = 0, \ \text{ for every density } f.$$

The proof of the following theorem can be found in [6]. The theorem gives a sufficient condition for the asymptotic stability of $\{P^n\}$.

Theorem 1.1. ([6], Theorem 5.7.1, pg 115) *Let $K : G \times G \to \mathbb{R}$ be a stochastic kernel, P the Markov operator given by (5). If K satisfies*

$$\int_G \inf_{|y|<r} K(x, y)dx > 0, \quad \textit{for all} \ \ r > 0, \tag{6}$$

and there exists a Liapunov function $V : G \to \mathbb{R}$ such that

$$\int_G K(x, y)V(x)dx \leq \alpha V(y) + \beta, \ 0 \leq \alpha < 1, \ \beta \geq 0 \tag{7}$$

for every density f, then $\{P^n\}$ is asymptotically stable.

Consider a dynamical system which depends on a parameter r. A value r_0 of the parameter is a **bifurcation point**, if the system changes its dynamics for this value. There are two approaches in studying bifurcation: the phenomenological ((P)-bifurcation) and the dynamical ((D)-bifurcation) approach.

The (P)-bifurcation approach studies the qualitative changes of stationary densities. In the simulations we study the changes of the shape of the histogram for different values of the parameter. To draw the histogram we start with K initial points X_0^1, X_0^2, ...,X_0^K (K a big natural number) and we calculate the $(N+1)$th point of the orbit of every initial point getting X_N^1, X_N^2,...,X_N^K, $N \in \mathbb{N}$. Then we plot the histogram of the values X_N^1, X_N^2,...,X_N^K: we divide the interval $[0,1]$ into 100 parts and count how many points are in each small interval. We are looking for parameter values r_0, for which the shape of the histogram changes.

The (D)-bifurcation approach focuses on the loss of stability of invariant measures. For this we study the Liapunov exponent which is calculated using the formula

$$\lambda_r(x) = \lim_{n\to\infty} \frac{1}{n} \log \left| \frac{\partial \varphi(n,x)}{\partial x} \right|,$$

where $\varphi(n,x) = x_n$. We are looking for parameter values for which the Liapunov exponent changes its sign.

It is also helpful to draw the bifurcation diagram. For this for every value r of the parameter we start with an arbitrary initial point x_0 and we calculate the points x_1, x_2,...,x_N of the orbit, for N big natural number. Then we calculate x_{N+1},..., x_M, $M > N+1$, and plot the points (r, x_{N+1}), (r, x_{N+2}),...,(r, x_M).

2. The Deterministic Logistic Map

The deterministic case have been intensively studied. In this case the orbit of a point $x_0 \in \mathbb{R}$ can be calculated by the recursive formula

$$x_{n+1} = r x_n (1 - x_n), n \in \mathbb{N}.$$

In the followings we consider $x_0 \in [0,1]$. The bifurcation scenario in this case is well known, see for example [2], [1] or [5]. We summarize this scenario for better comparison of the deterministic and stochastic case. In Figure 1 is plotted the bifurcation diagram and the Liapunov exponent. In simulations we approximate the Liapunov exponent by

$$\lambda_r(x) = \frac{1}{N} \sum_{k=0}^{N-1} \log |r(1 - 2\varphi(k,x)))|,$$

where N is a big natural number. If $0 < r < 1$, there are two fixed points: a stable fixed point 0 and an unstable fixed point $1 - \frac{1}{r}$, so the orbit of each point from [0,1] converges to 0. For $1 < r < 3$ the fixed point 0 becomes unstable and the orbit of

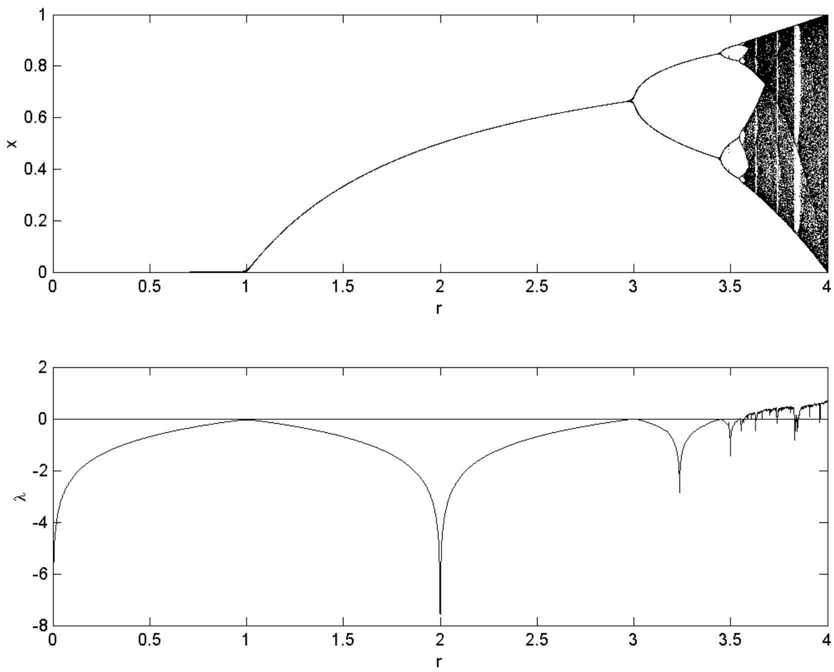

FIGURE 1. *Bifurcation diagram and Liapunov exponent in the deterministic case*

each point converges to $1 - \frac{1}{r}$. Thus $r = 1$ is a bifurcation point. Another bifurcation point is $r = 3$, where the orbit becomes an attractive period-2 orbit. In $r = 3.46$ the period-2 orbit becomes unstable and is replaced by a stable period-4 orbit. We can observe this behavior on the bifurcation diagram. As r increases this period doubling continues. This scenario is illustrated by Figure 2 too, where the shape of the histogram changes from a two-peaked to a four-peaked, then to an eight-peaked form. For $r = 3.57$ the dynamics becomes chaotic. For $r > 3.57$ the chaotic and period doubling behavior alternates. For $r = 3.83$ there is a stable period-3 orbit. If we study the Lyapunov exponent, this becomes zero for $r = 1$, $r = 3$, $r = 3,46$, etc., so these points are bifurcation points with this approach too.

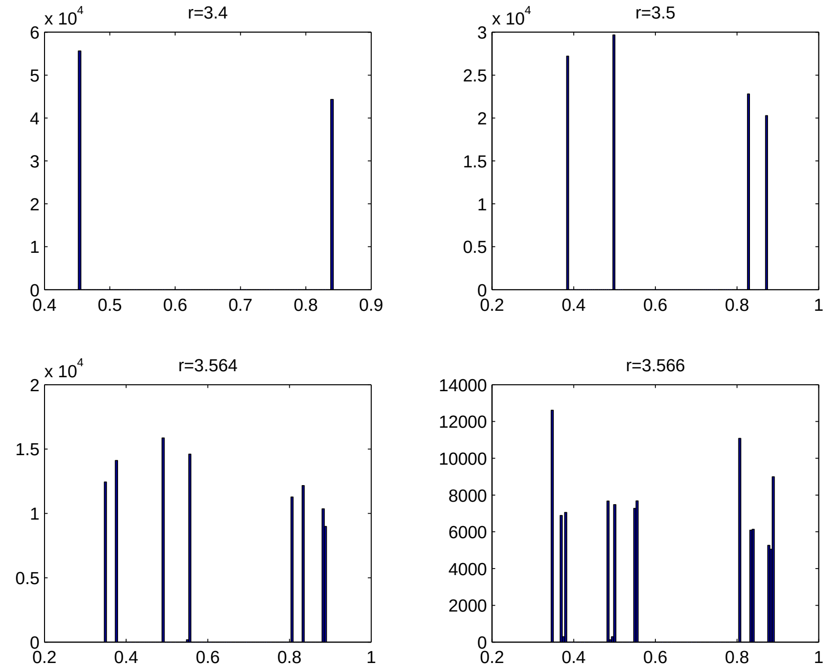

FIGURE 2. *Histogram in the neighborhood of $r = 3.5$ in deterministic case*

3. Perturbation with Uniformly Distributed Random Variables

Consider now the logistic map perturbed by uniformly distributed independent random variables ξ_0, ξ_1,...,ξ_n,... taking values in some interval $[a, b]$. The orbit of a point $x_0 \in [0, 1]$ can be calculated with the formula

$$x_{n+1} = rx_n(1 - x_n) + \xi_n, n \in \mathbb{N}.$$

We study the bifurcation points with two different approaches: the (P)-bifurcation and the (D)-bifurcation approach. In [4] there are some numerical results for this case. We extend them studying the changes of the histogram for different values of r.

Firstly using Theorem 1.1 we prove that for every $r \in (0, 4)$ there exists a stationary density function.

Theorem 3.1. *In case of the logistic map perturbed by uniformly distributed random variables, for every $r \in (0, 4)$ there exists a stationary density function.*

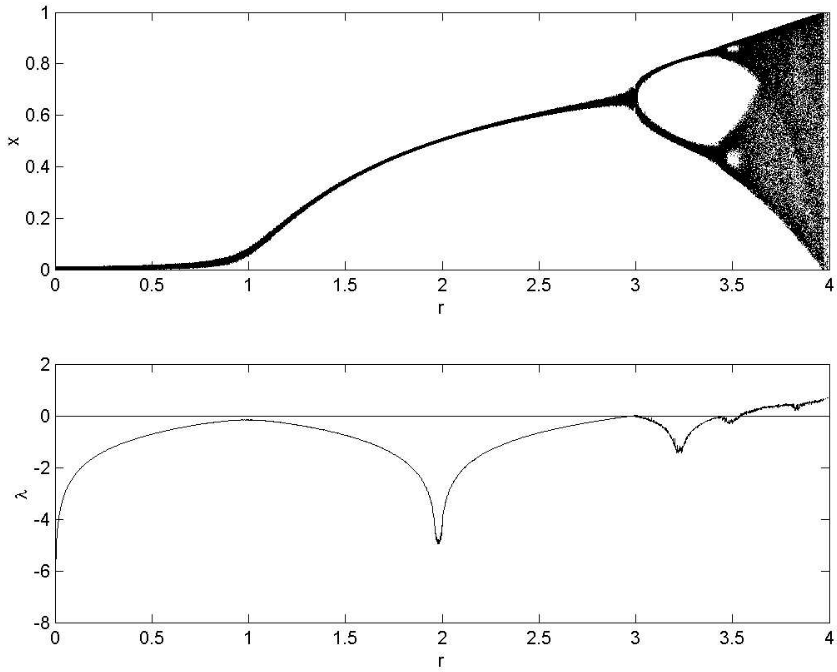

FIGURE 3. *Bifurcation diagram and Liapunov exponent for $a = 0$ and $b = 0.01$*

Proof. Let $V(x) = |x|$ be a Liapunov function. Then

$$\int_{\mathbb{R}} K(x,y)V(x)dx = \int_{\mathbb{R}} |x|g(x-ry)dx = \int_{\mathbb{R}} |s+ry|g(s)ds$$
$$\leq \int_{\mathbb{R}} |s|g(s)ds + |S(y)| \leq \frac{r}{4}V(y) + \frac{a+b}{2} + 1,$$

so $\alpha = \frac{r}{4}$ and $\beta = \frac{a+b}{2} + 1$, and $\alpha < 1$, if $r < 4$. So by Theorem 1.1 there exists a stationary density. □

In Figure 3 we plotted the bifurcation diagram for $a = 0$ and $b = 0.01$. The Liapunov exponent is not 0 in $r = 1$, so this point is not a (D)-bifurcation point. Studying the histogram in neighborhood of $r = 1$ leads to the conclusion, that this is not a (P)-bifurcation point, too (see Figure 4). So the dynamics of the system in $r = 1$ is different as in the deterministic case, where this point was a bifurcation point. In

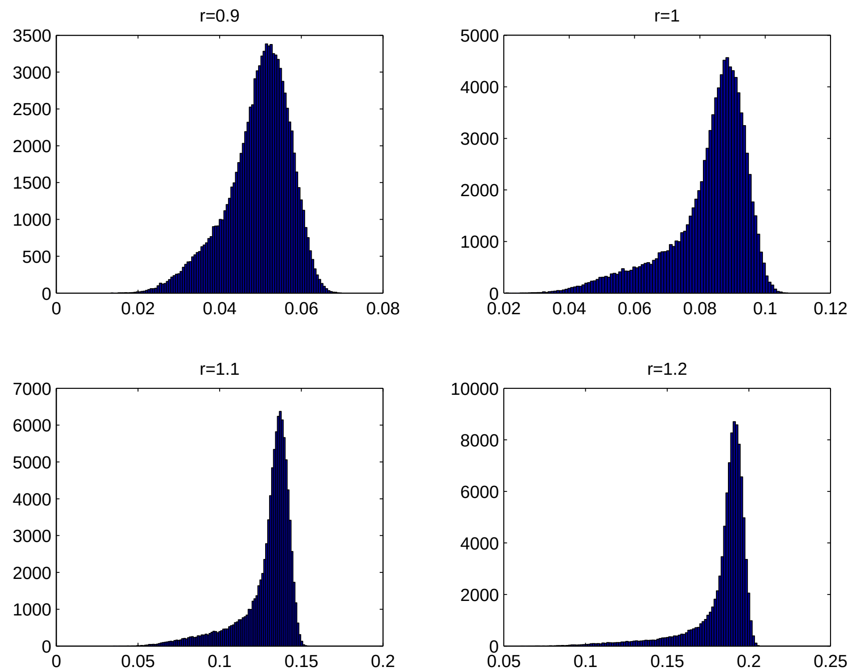

FIGURE 4. *Histogram in the neighborhood of* $r = 1$ *for* $a = 0$, $b = 0.01$

$r = 3$ the Liapunov exponent is also not 0, but this point is a (P)-bifurcation point, as the histogram changes its shape from a one-peaked to a two-peaked form (see Figure 5). Another (P)-bifurcation occurs between $r = 3.4$ and $r = 3.5$, where we observe a transition from a two-peaked histogram to a four-peaked histogram (see Figure 6). But the Liapunov exponent remains negative in this case, too.

Even if the Liapunov exponent for the deterministic case and for the small noise case is close to each other, the behavior of single trajectories can be very different, as the Liapunov exponent measures only the exponential of convergence (divergence) of two neighboring trajectories.

It is interesting that for $b \geq 0.05$ the period doubling behavior disappears (see Figure 7). Studying the histogram for values between 3.4 and 3.6 we observe that the shape doesn't become four-peaked as in the case of $b = 0.01$ (see Figure 8).

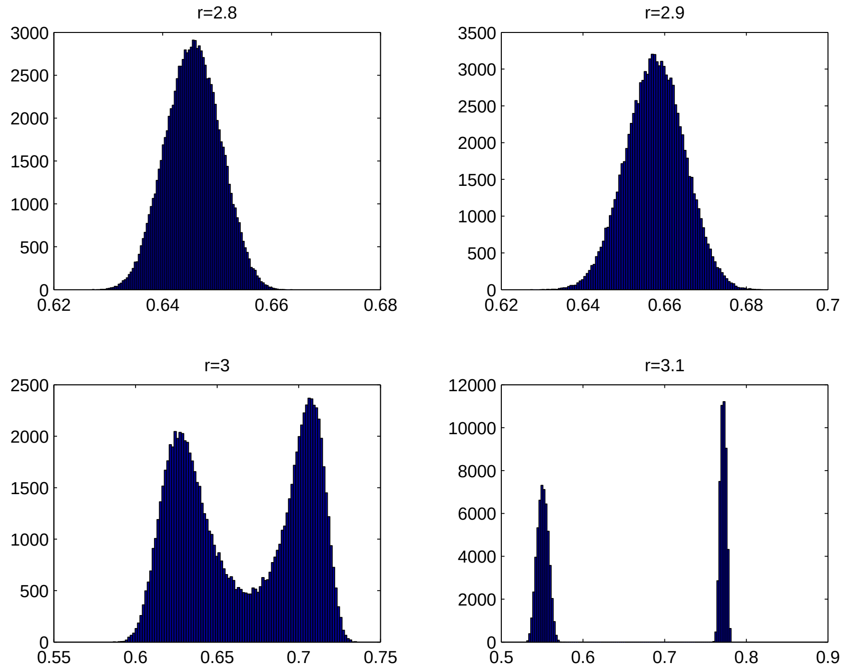

FIGURE 5. *Histogram in the neighborhood of* $r = 3$ *for* $a = 0$, $b = 0.01$

4. **Perturbation with Normally Distributed Random Variables**

Now consider ξ_0, ξ_1,...,ξ_n,... to be normally distributed independent random variables with mean m and variance σ^2.

Using Theorem 1.1 we prove that for every $r \in (0, 4)$ there exists a stationary density function.

Theorem 4.1. *In case of the logistic map perturbed by normally distributed random variables, for every* $r \in (0, 4)$ *there exists a stationary density function.*

Proof. Let $V(x) = |x|$ be a Liapunov function. Then

$$\int_{\mathbb{R}} K(x,y)V(x)dx = \int_{\mathbb{R}} |x|g(x-ry)dx = \int_{\mathbb{R}} |s+ry|g(s)ds$$
$$\leq \int_{\mathbb{R}} |s|g(s)ds + |S(y)| \leq \frac{r}{4}V(y) + m + 1,$$

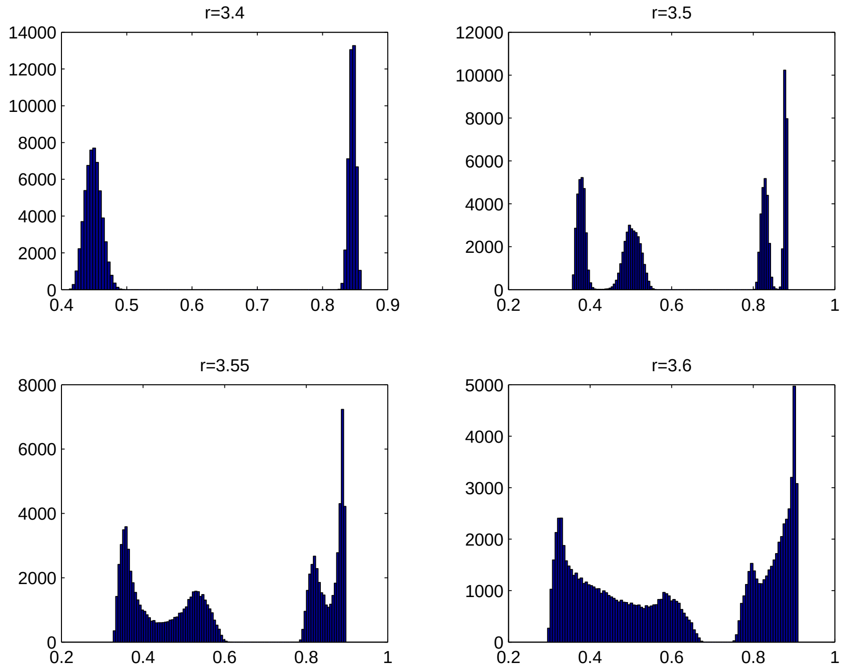

FIGURE 6. *Histogram in the neighborhood of* $r = 3.5$ *for* $a = 0$, $b = 0.01$

so $\alpha = \dfrac{r}{4}$ and $\beta = m + 1$ in Theorem 1.1. We have to have $\alpha < 1$, so $r < 4$. □

In Figure 9 we see the bifurcation diagram and Liapunov exponent for $m = 0$ and $\sigma = 0.0001$. Comparing with Figure 1 we see that for small noise the bifurcation scenario is similar with the scenario in deterministic case. Here $r = 1$ is a bifurcation point as in the deterministic case (see Figure 10).

If the noise is bigger ($\sigma = 0.001$) the phenomena in $r = 1$ is interesting. Observe in Figure 11 that in neighborhood of $r = 1$ seems to be a chaotic region. The Lyapunov exponent becomes positive in $r = 1$.

In case of $\sigma = 0.01$ this region becomes larger (see Figure 12). The histogram in neighborhood of $r = 1$ (Figure 13) also tells this, see the histograms for $r = 0.9$, $r = 1$ and $r = 1.1$, where the values are spread to a large interval. Note that for $r = 1$ the scale of the $0x$ axis is multiplied by 10^{307}! $r = 3$ is a (P)-bifurcation

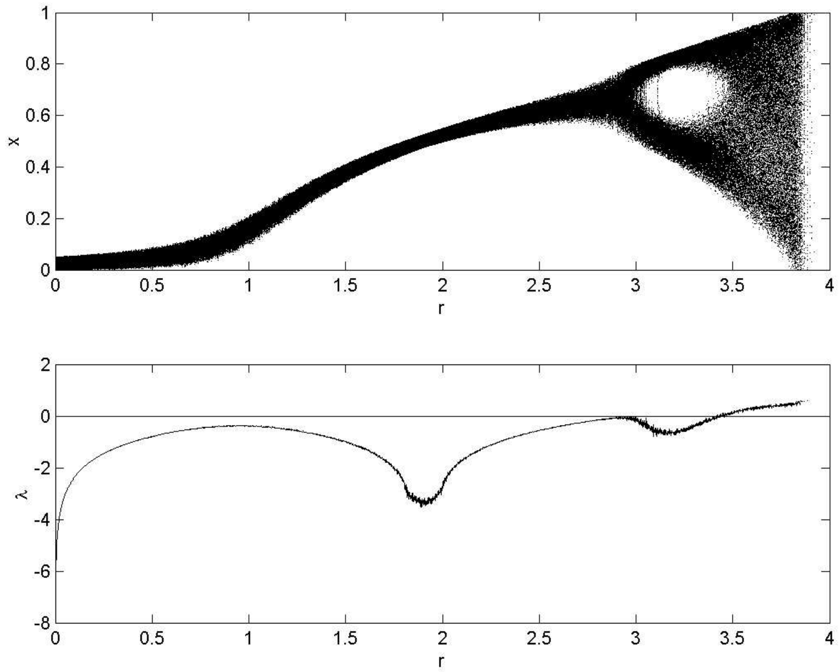

FIGURE 7. *Bifurcation diagram and Liapunov exponent for* $a = 0$ *and* $b = 0.05$

point, because the histogram changes its shape from one-peaked to two-peaked form, but this is not a (D)-bifurcation, because the Liapunov exponent stays negative. It is interesting that between 3.5 and 3.6 the Liapunov exponent changes its sign several times (see the zoomed in part of Figure 12 in Figure 15), so these point are (D)-bifurcation points, but the histogram doesn't changes its shape (Figure 16), so they are not (P)-bifurcation points.

We don't observe the period doubling behavior in this case (Figure 17) similarly with the case of the perturbation with uniformly distributed random variables on the interval $[a, b]$ for $b \geq 0.05$. So if the noise becomes bigger the period doubling behavior disappears. It is also interesting that for $r < 1.2$ the points of the orbit can have negative values too (as the random variables added can be negative), but for $r > 1.2$ the points are positive.

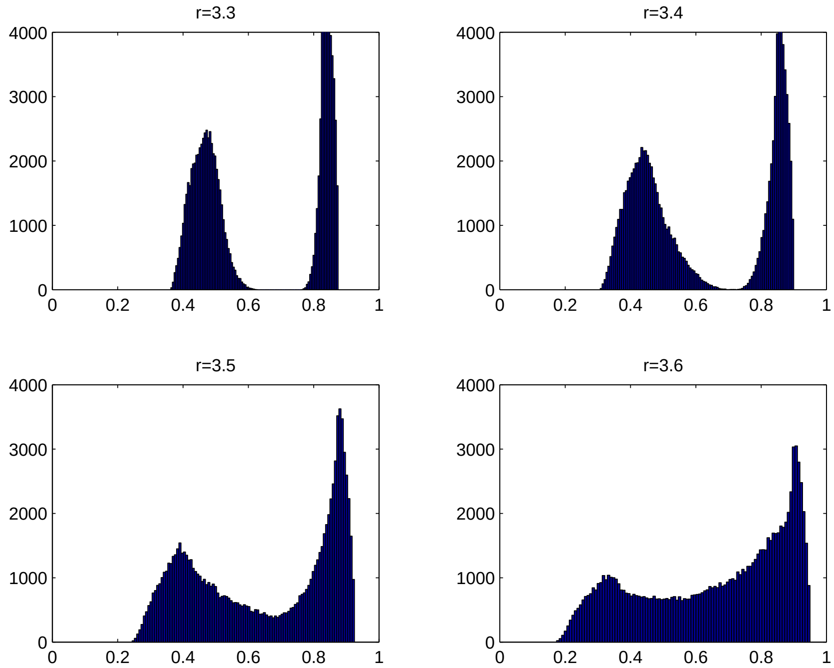

FIGURE 8. *Histogram in the neighborhood of* $r = 3.5$ *for* $a = 0$ *and* $b = 0.05$

Now change the mean of the normally distributed random variables. If the mean becomes positive the chaotic region around $r = 1$ disappears (see Figure 18 for $m = 0.01$ and $\sigma = 0.01$). If the mean becomes negative the length of the interval of the values of r for which we get a chaotic behavior increases as the mean decreases. In Figure 19 we observe, that for $m = -0.01$ the chaotic region is larger than in the case of $m = 0$.

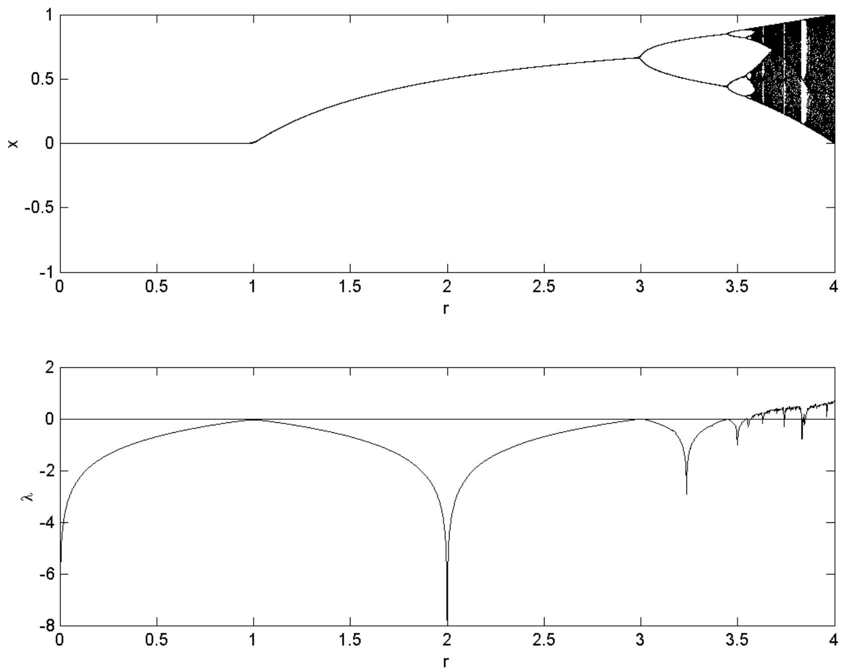

FIGURE 9. *Bifurcation diagram and Liapunov exponent for* $m = 0$ *and* $\sigma = 0.0001$

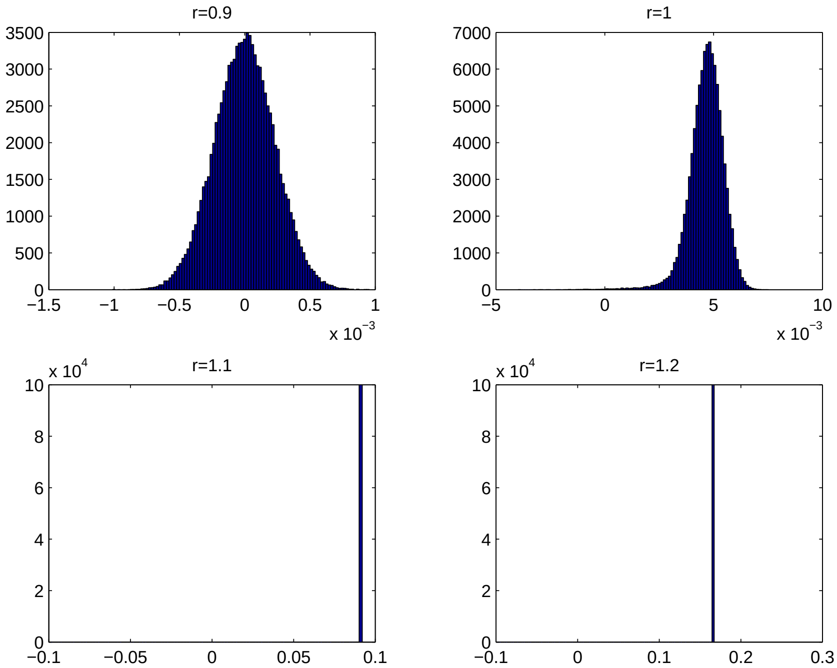

Figure 10. *Histogram in the neighborhood of* $r = 1$ *for* $m = 0$ *and* $\sigma = 0.0001$

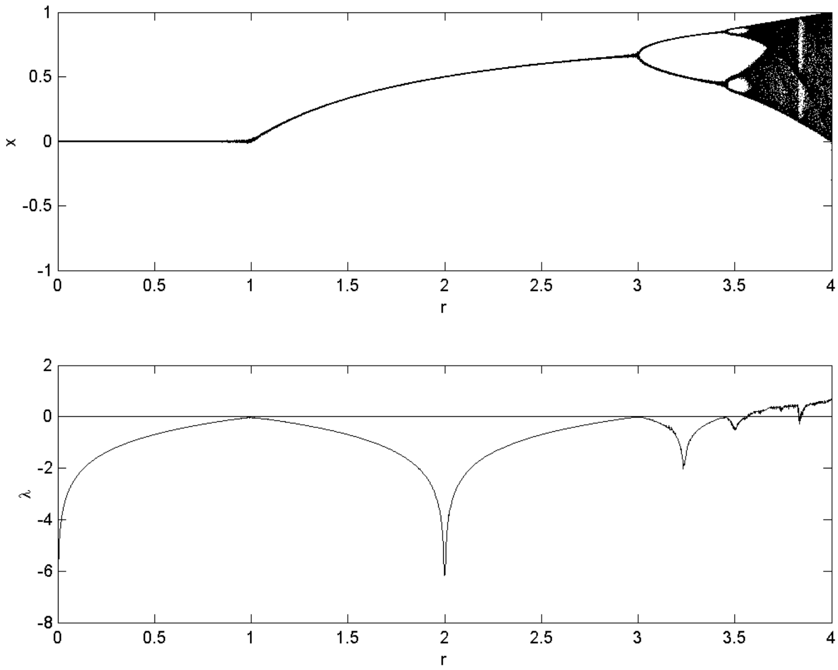

Figure 11. *Bifurcation diagram and Liapunov exponent for* $m = 0$ *and* $\sigma = 0.001$

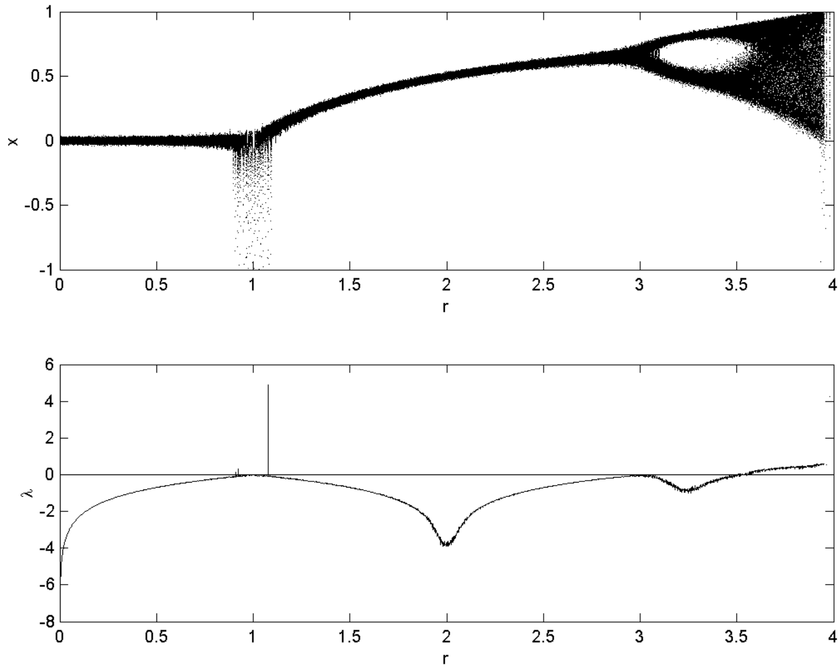

FIGURE 12. *Bifurcation diagram and Liapunov exponent for* $m = 0$ *and* $\sigma = 0.01$

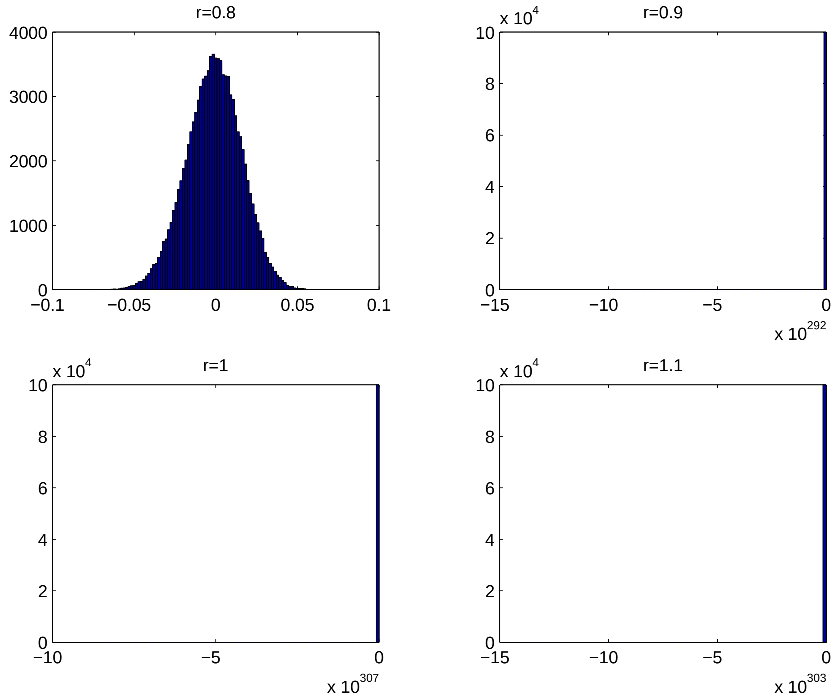

FIGURE 13. *Histogram in the neighborhood of* $r = 1$ *for* $m = 0$ *and* $\sigma = 0.01$

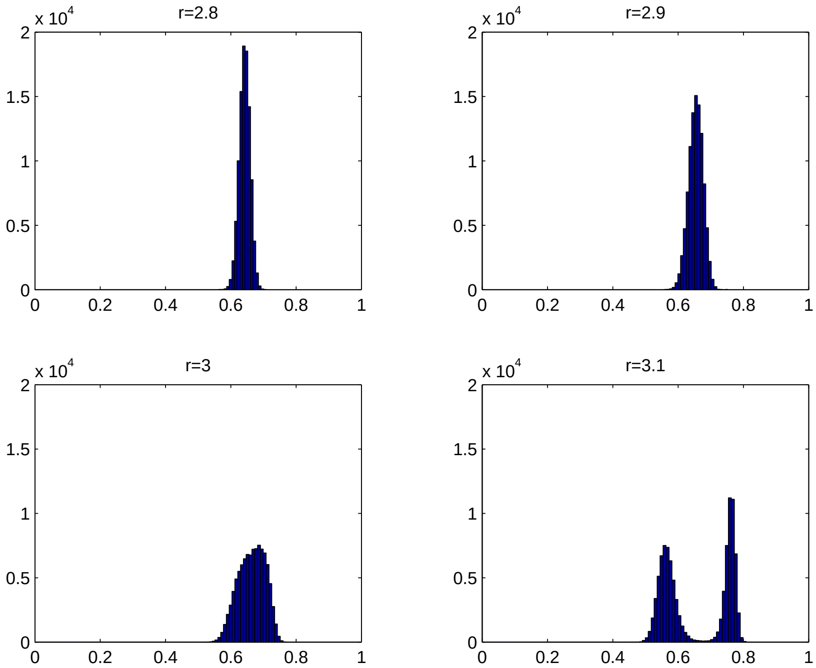

FIGURE 14. *Histogram in the neighborhood of* $r = 3$ *for* $m = 0$ *and* $\sigma = 0.01$

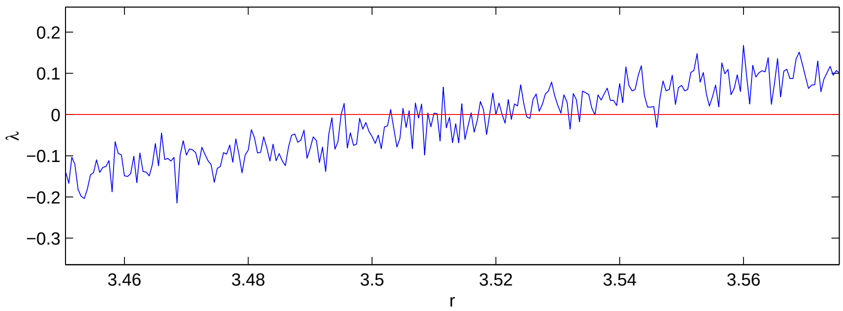

FIGURE 15. *Liapunov exponent for* $m = 0$ *and* $\sigma = 0.01$

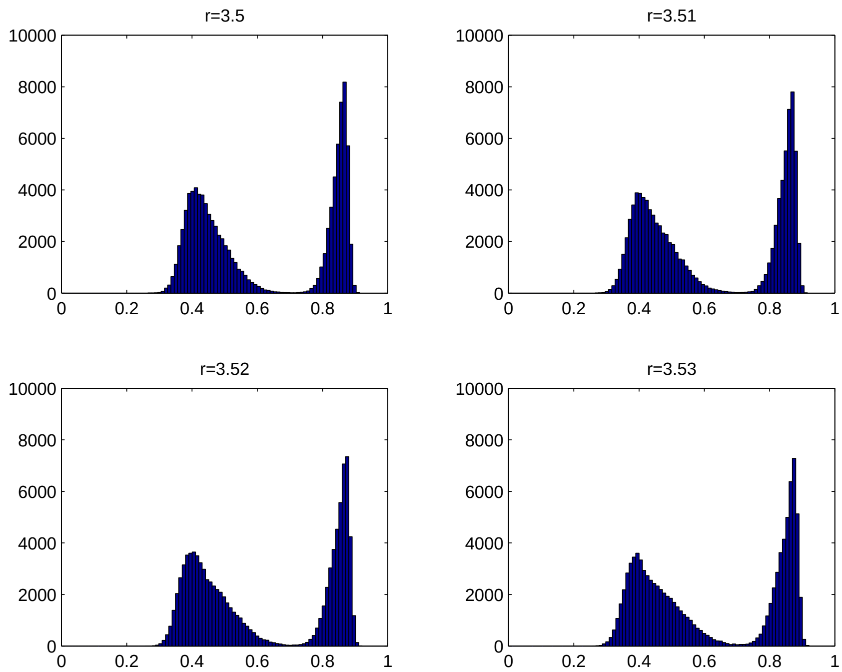

FIGURE 16. *Histogram in the neighborhood of* $r = 3.5$ *for* $m = 0$ *and* $\sigma = 0.01$

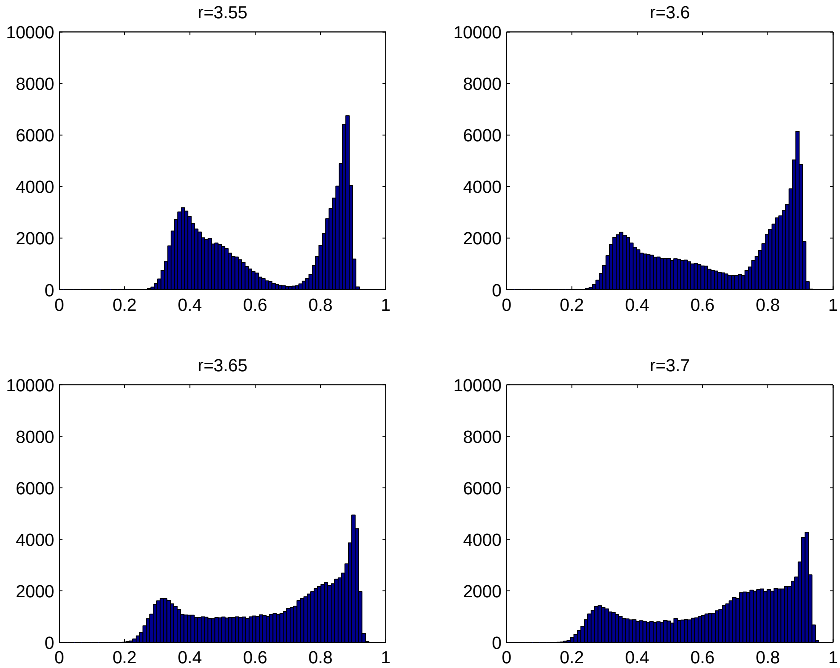

FIGURE 17. *Histogram in the neighborhood of* $r = 3.55$ *for* $m = 0$ *and* $\sigma = 0.01$

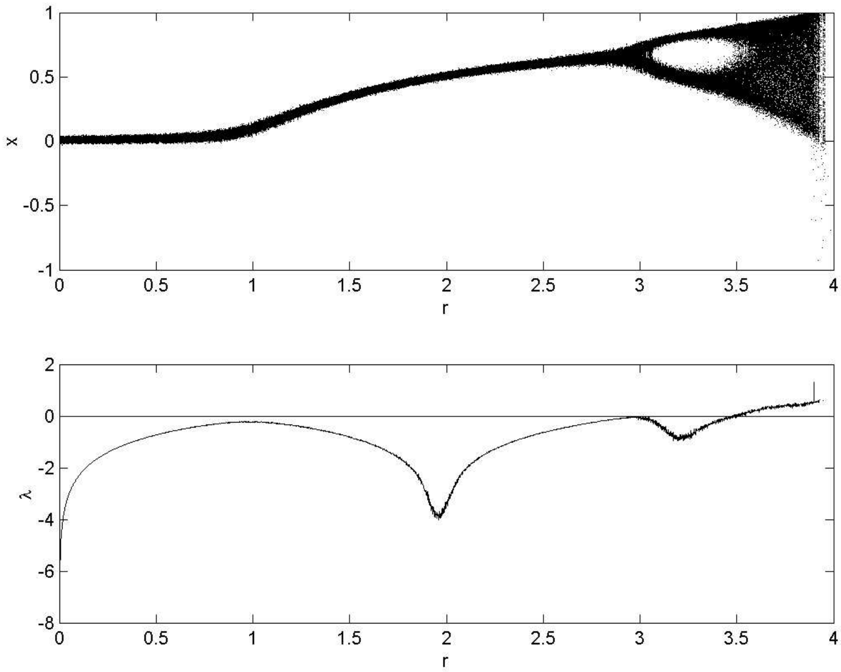

FIGURE 18. *Bifurcation diagram and Liapunov exponent for* $m = 0.01$ *and* $\sigma = 0.01$

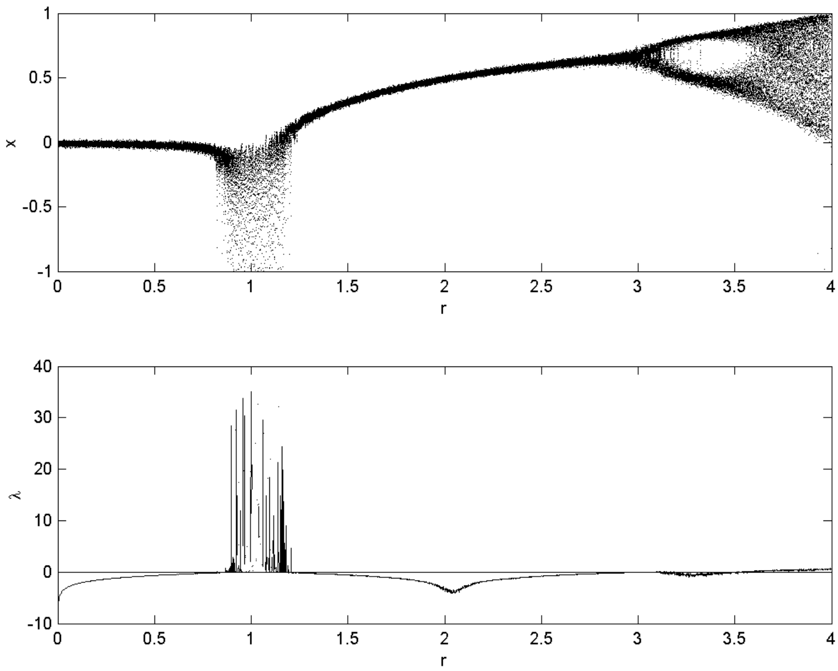

FIGURE 19. *Bifurcation diagram and Liapunov exponent for* $m = -0.01$ *and* $\sigma = 0.01$

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Babeş-Bolyai University, Kogălniceanu 1, Cluj-Napoca, Romania

E-mail address: imarchis@math.ubbcluj.ro