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SOME EXTENSION OF BIVARIATE TENSOR-PRODUCT FORMULA

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Abstract. In this article we construct bivariate approximation formulas with scattered data by extensions of bivariate tensor-product Lagrange formula, using spline univariate operators. The graphs of approximation functions are given.

Let $D \subseteq \mathbb{R}^2$ be an arbitrary domain, f a real-valued function defined on D, $Z = \{z_i | \ z_i = (x_i, y_i), \ i = \overline{1, N}\} \subset D$ and $I(f) = \{\lambda_k f | \ k = 1, \ldots, N\}$ a set of informations about f (evaluations of f and of certain of its derivatives at $z_1, \ldots, z_N$).

A general interpolation problem is: for a given function f find a function g that interpolates the data $I(f)$ i.e.

$$\lambda_k f = \lambda_k g, \quad k = \overline{1, N}.$$

Starting from bivariate Lagrange formula for the rectangular grid $\Pi = \{x_0, \ldots, x_m\} \times \{y_0, \ldots, y_n\}$:

$$f(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{u(x)}{(x-x_i)u'(x_i)} \frac{v(y)}{(y-y_j)v'(y_j)} f(x_i, y_j) + (R_{mn}f)(x,y) \qquad (1)$$

where

$$(R_{mn}f)(x,y) = u(x)[x, x_0, \ldots, x_m; f(\cdot, y)] + \sum_{i=0}^{m} \frac{u(x)v(y)}{(x-x_i)u'(x_i)} [y, y_0, \ldots, y_n; f(x_i, \cdot)]$$

with $u(x) = (x - x_0)\ldots(x - x_m)$ and $v(y) = (y - y_0)\ldots(y - y_n)$, there are two generalizations.

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A first generalization of the formula (1) was given by J.F. Steffensen [7]:

$$f(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n_i}\frac{u(x)}{(x-x_i)u'(x_i)}\frac{v_i(y)}{(y-y_j)v_i'(y_j)}f(x_i,y_j)+(R_{m,n_i}f)(x,y) \tag{2}$$

where

$$(R_{m,n_i}f)(x,y)=u(x)[x,x_0,\dots,x_m;f(\cdot,y)]+\sum_{i=0}^{m}\frac{u(x)v_i(y)}{(x-x_i)u_i'(x_i)}[y,y_0,\dots,y_{n_i};f(x_i,\cdot)]$$

with

$$v_i(y)=(y-y_0)\dots(y-y_{n_i}).$$

The interpolation grid here is $\Pi_1=\{(x_i,y_j)|\ i=\overline{0,m},\ j=\overline{0,n_i}\}$.

A second generalization of the Lagrange interpolation formula (1), that is also an extension of the Steffensen formula (2) was given by D.D. Stancu [5]:

$$f(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n_i}\frac{u(x)}{(x-x_i)u'(x_i)}\frac{v_i(y)}{(y-y_{ij})v_i'(y_{ij})}f(x_i,y_{ij})+(R_{m,n_i}f)(x,y) \tag{3}$$

where

$$(R_{m,n_i}f)(x,y)=u(x)[x,x_0,\dots,x_m;f(\cdot,y)]+\sum_{i=0}^{m}\frac{u(x)v_i(y)}{(x-x_i)u'(x_i)}[y,y_{i0},\dots,y_{in_i};f(x,\cdot)]$$

with $v_i(y)=(y-y_{i0})\dots(y-y_{in_i})$ and the interpolation grid

$$\Pi_2=\{(x_i,y_{ij})|\ i=\overline{0,m},\ j=\overline{0,n_i}\}.$$

Remark. 1. *The Steffensen formula (2) does not solve the general interpolation problem,* Π_1 *is only a particular case of the interpolatory set* $\{z_1,\dots,z_N\}$.*. Formula (3) is really a solution of the considered general problem. Indeed, let* $X_k\subset Z$ *be the set of nodes* (x_i,y_i), $i=\overline{1,N}$ *with the same abscises* x_k, *i.e.* $X_k=\{(x_k,y_{kj})|\ j=\overline{0,n_k}\}$ *for all* $k=0,1,\dots,m$. *We have* $X_i\neq X_j$ *for* $i\neq j$ *and* $Z=X_0\cup\dots\cup X_m$.*Thus* $\Pi_2=Z$. *The set* $\{x_0,\dots,x_m\}$ *is obtained by projection of nodes set* Z *on* Ox *axis.*

If L_m^x is the Lagrange's operator for the interpolates nodes $x_0,\dots,x_m$ and $L_{n_i}^y$, $i=\overline{0,m}$ are the Lagrange's operators for the nodes $y_{i0},\dots,y_{in_i}$ respectively, then we have

$$f=L_m^x f+R_m^x f \tag{4}$$

with

$$(L_m^x f)(x,y) = \sum_{i=0}^{m} \frac{u(x)}{(x-x_i)u'(x_i)} f(x_i,y)$$

and

$$f(x_i,\cdot) = (L_{n_i}^y f)(x_i,\cdot) + (R_{n_i}^y f)(x_i,\cdot), \quad i=\overline{0,m} \tag{5}$$

with

$$(L_{n_i}^y f)(x_i,y) = \sum_{j=0}^{n_i} \frac{v_i(y)}{(y-y_{ij})v_i'(y_{ij})} f(x_i,y_{ij}).$$

If the remainder terms are written with the divided differences, from (4) and (5) follows formula (3).

If we make the projection of nodes set Z on Oy axis(see [4],[8]), we consider the Lagrange's operator L_n^y which interpolates the nodes $y_0,\dots,y_n$ and Lagrange's operators $L_{m_i}^x$, $i=\overline{0,n}$ which interpolate the nodes $x_{i0},\dots,x_{im_i}$. In the first level of approximation we use the approximation formula

$$f = L_n^y f + R_n^y f \tag{6}$$

with

$$(L_n^y f)(x,y) = \sum_{i=0}^{n} \frac{v(y)}{(y-y_i)v'(y_i)} f(x,y_i)$$

where $v(y)=(y-y_0)\dots(y-y_n)$. For every $f(x,y_i)$, $i=\overline{0,n}$ we use in a second level of approximation the following formulas

$$f(\cdot,y_i) = (L_{m_i}^x f)(\cdot,y_i) + (R_{m_i}^x f)(\cdot,y_i), \quad i=\overline{0,n} \tag{7}$$

with

$$(L_{m_i}^x f)(x,y_i) = \sum_{j=0}^{m_i} \frac{u_i(x)}{(x-x_{ij})u_i'(x_{ij})} f(x_{ij},y_i).$$

where $u_i(x)=(x-x_{i0})\dots(x-x_{im_i})$. We obtain the following approximation formula

$$f(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{m_i} \frac{u_i(x)}{(x-x_{ij})u_i'(x_{ij})} \frac{v(y)}{(y-y_i)v'(y_i)} f(x_{ij},y_i) + (R_{m_i,n}f)(x,y) \tag{8}$$

where

$$(R_{m_i,n}f)(x,y) = v(y)[y,y_0,\dots,y_n;f(x,\cdot)] + \sum_{i=0}^{n} \frac{u_i(x)v(y)}{(y-y_i)v'(y_i)}[x,x_{i0},\dots,x_{im_i};f(\cdot,y)]$$

Then interpolation grid of formula (8) is

$$\Pi_3 = \{(x_{ij}, y_i) |\ i = \overline{0,n},\ j = \overline{0,m_i}\}.$$

Remark. 2. *Formula (8) is also a solution of the considered general problem. Indeed, let* $Y_k \subset Z$ *be the set of nodes* (x_i, y_i), $i = \overline{1,N}$ *with the same ordinates* y_k, *i.e.* $Y_k = \{(x_{kj}, y_k) |\ j = \overline{0,m_k}\}$ *for all* $k = 0, 1, \dots, n$. *We have* $Y_i \neq Y_j$ *for* $i \neq j$ *and* $Z = Y_0 \cup \cdots \cup Y_n$. *Thus* $\Pi_3 = Z$. *The set* $\{y_0, \dots, y_n\}$ *is obtained by projection of nodes set* Z *on* Oy *axis.*

We denote

$$(P^x f)(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n_i} \frac{u(x)}{(x - x_i)u'(x_i)} \frac{v_i(y)}{(y - y_{ij})v'(y_{ij})} f(x_i, y_{ij}). \tag{9}$$

and

$$(P^y f(x,y) = \sum_{i=0}^{n} \sum_{j=0}^{m_i} \frac{u_i(x)}{(x - x_{ij})u_i'(x_{ij})} \frac{v(y)}{(y - y_i)v'(y_i)} f(x_{ij}, y_i) \tag{10}$$

The interpolation formula generated by the mean of operators P^x and P^y is

$$f(x,y) = (P^M f)(x,y) + (R^M f)(x,y) \tag{11}$$

where

$$(P^M f)(x,y) = \frac{(P^x f)(x,y) + (P^y f)(x,y)}{2} \tag{12}$$

and

$$(R^M f)(x,y) = \frac{(R_{m_i,n} f)(x,y) + (R_{m,n_i} f)(x,y)}{2}$$

The interpolation set of P^M is also Z.

Remark. 3. *Usually the degree* m *of the operator* L_m^x *is more greater than the largest degree of* $L_{n_i}^y$ *i.e.* $m \gg \max\{n_0, \dots, n_m\}$ *and degree* n *of the operator* L_n^y *is more greater than the largest degree of* $L_{m_i}^x$ *i.e.* $n \gg \max\{m_0, \dots, m_n\}$, *which imply a large computational complexity of the polynomials interpolation* $(P^x f)(x,y)$ *and* $(P^y f)(x,y)$. *From this reason and the another ones, in* [1],[2] *instead of Lagrange polynomial operator* L_m^x *is used a spline interpolation operator. In this article, in formula (6), instead of Lagrange polynomial operator* L_n^y *is used a spline interpolation operator.*

Let $S^y_{L,2r-1}$ be the spline interpolation operator of the degree $2r-1$, that interpolates the function f with regard to the variable y at the nodes (x, y_k), $k = \overline{0,n}$ i.e.

$$(S^y_{L,2r-1}f)(x,y) = \sum_{i=0}^{r-1} a_i^x y^i + \sum_{j=0}^{n} b_j^x (y-y_j)_+^{2r-1} \tag{13}$$

for which

$$\begin{cases} (S^y_{L,2r-1}f)(x,y_k) = f(x,y_k), & k=\overline{0,n} \\ (S^y_{L,2r-1}f)^{(0,p)}(x,\alpha) = 0, & p = \overline{r,2r-1},\ \alpha > y_n \end{cases} \tag{14}$$

The spline function of Lagrange type can also be written in the form

$$(S^y_{L,2r-1}f)(x,y) = \sum_{k=0}^{n} s_k(y) f(x,y_k)$$

where s_k are the corresponding cardinal splines i.e., they are of the same form (13), but with the interpolatory conditions

$$s_k(y_j) = \delta_{kj}, \quad k,j = \overline{0,n}.$$

This way, formula (8) becomes

$$f(x,y) = (P^y_S f)(x,y) + (R^y_S f)(x,y) \tag{15}$$

where

$$(P^y_S f)(x,y) = \sum_{i=0}^{n} \sum_{j=0}^{m_i} s_i(y) \frac{u_i(x)}{(x-x_{ij})u_i'(x_{ij})} f(x_{ij}, y_i)$$

and $(R^y_S f)(x,y)$ is the remainder term.

Taking into account that for $f(x,\cdot) \in C^r[y_0, y_n]$

$$(R^y_{L,2r-1}f)(x,y) = \int_{y_0}^{y_n} \varphi_r(y,t) f^{(0,r)}(x,t)dt$$

with

$$\varphi_r(y,t) = \frac{(y-t)_+^{r-1}}{(r-1)!} - \sum_{i=0}^{n} s_i(y) \frac{(y_i-t)_+^{r-1}}{(r-1)!}$$

it follows

Theorem. 4. *If* $f \in C^{0,r}(D)$ *then*

$$(R_S^y f)(x,y) = \int_{y_0}^{y_n} \varphi_r(y,t) f^{(0,r)}(x,t)dt + \tag{16}$$

$$+ \sum_{i=0}^{n} s_i(y) u_i(x)[x, x_{i0}, \dots, x_{im_i}; f(x_i, \cdot)]$$

and if $f \in C^{p+1,r}(D)$ *with* $p = \max\{m_0, \dots, m_n\}$ *we have*

$$(R_S^y f))(x,y) = \int_{y_0}^{y_m} \varphi_r(y,t) f^{(0,r)}(x,t)dt + \tag{17}$$

$$+ \sum_{i=0}^{n} s_i(y) \int_{x_{i0}}^{x_{im_i}} \psi_{m_i}(x,s) f^{(m_i+1,0)}(s, y_i)ds$$

with

$$\psi_{m_i}(x,s) = \frac{(x-s)_+^{m_i}}{m_i!} - \sum_{j=0}^{m_i} \frac{u_i(x)}{(x-x_{ij})u_i'(x_{ij})} \frac{(x_{ij}-s)_+^{m_i}}{m_i!}.$$

If the projection of nodes set is on Ox axis, we have from [1], [2] the interpolation formula

$$f(x,y) = (P_S^x f)(x,y) + (R_S^x f)(x,y) \tag{18}$$

where

$$(P_S^x f)(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n_i} s_i(x) \frac{v_i(y)}{(y-y_{ij})v_i'(y_{ij})} f(x_i, y_{ij})$$

and $(R_S^x f)(x,y)$ is the remainder term.

Theorem. 5. *(*[1], [2]*) If* $f \in C^{r,p+1}(D)$ *with* $p = \max\{n_0, \dots, n_m\}$ *we have*

$$(R_S^x f)(x,y) = \int_{x_0}^{x_m} \varphi_r(x,s) f^{(r,0)}(s,y)ds + \tag{19}$$

$$+ \sum_{i=0}^{m} s_i(x) \int_{y_{i0}}^{y_{in_i}} \psi_{n_i}(y,t) f^{(0,n_i+1)}(x_i, t)$$

with

$$\varphi_r(x,s) = \frac{(x-s)_+^{r-1}}{(r-1)!} - \sum_{i=0}^{m} s_i(x) \frac{(x_i-s)_+^{r-1}}{(r-1)!}$$

$$\psi_{n_i}(y,t) = \frac{(y-t)_+^{n_i}}{n_i!} - \sum_{j=0}^{n_i} \frac{v_i(y)}{(y-y_{ij})v_i'(y_{ij})} \frac{(y_{ij}-t)_+^{n_i}}{n_i!}.$$

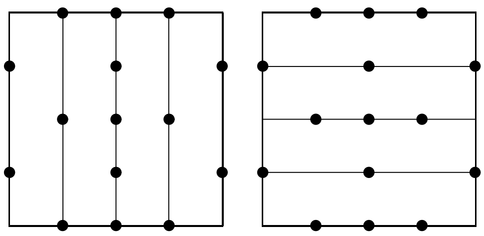


FIGURE 1. The grid I is projected on Ox and respective Oy axis

We obtain the following interpolation formula

$$f(x,y) = (P_S^M f)(x,y) + (R_S^M f)(x,y) \tag{20}$$

where

$$(P_S^M f)(x,y) = \frac{(P_S^x f)(x,y) + (P_S^y f)(x,y)}{2} \tag{21}$$

and

$$(R_S^M f)(x,y) = \frac{(R_S^x f)(x,y) + (R_S^y f)(x,y)}{2}$$

Example 6. . *Let*

$$f : [-1,1] \times [-1,1] \to \mathbb{R}, f(x,y) = \exp(-x^2 - y^2)$$

The sets of nodes are

-grid I

$P_1(-1,-0.5), P_2(-1,0.5), P_3(-0.5,-1), P_4(-0.5,0), P_5(-0.5,1), P_6(0,-1),$ $P_7(0,-0.5), P_8(0,0), P_9(0,0.5), P_{10}(0,1), P_{11}(0.5,-1), P_{12}(0.5,0), P_{13}(0.5,1),$ $P_{14}(1,-0.5), P_{15}(1,0.5)$

-grid II

$P_1(-1,-1), P_2(-1,0), P_3(-1,1), P_4(-0.5,-0.5), P_5(-0.5,0), P_6(-0.5,0.5),$ $P_7(0,-1), P_8(0,-0.5), P_9(0,0), P_{10}(0,0.5), P_{11}(0,1), P_{12}(0.5,-0.5), P_{13}(0.5,0),$ $P_{14}(0.5,0.5), P_{15}(1,-1), P_{16}(1,0), P_{17}(1,1)$

We plot the graph of functions f and the graphs of $P_S^x f$, $P_S^y f$, P_S^M. For a matrix Z we define

$$\|f - Pf\|_{2,Z} = \|\{(f - Pf)(x_i, y_j) | (x_i, y_j) \in Z\}\|_2$$

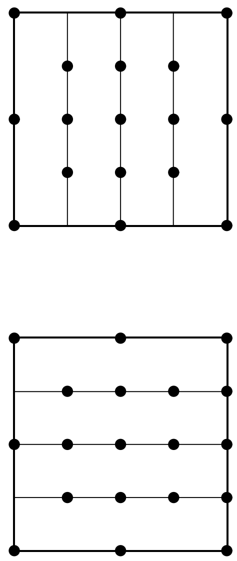
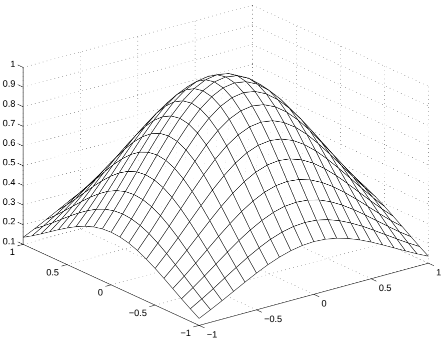


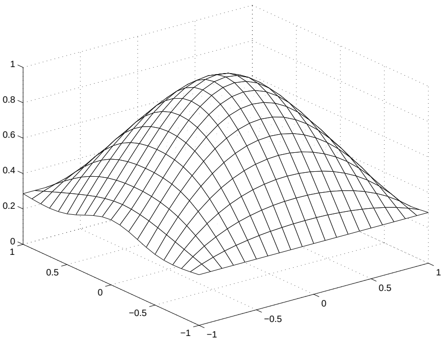
Figure 2. The grid II is projected on Ox and respective Oy axis

If we take $Z = [-1 : 0.1 : 1] \times [-1 : 0.1 : 1]$ *we obtain*

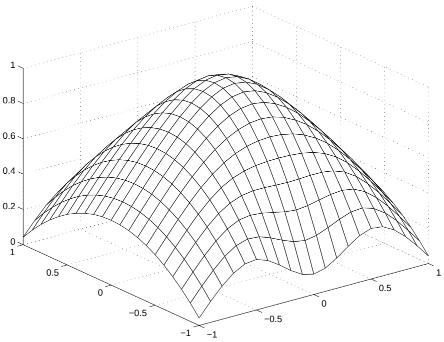
	P	$\|f - Pf\|_{2,Z}$
I	P_S^x	*0.9469*
	P_S^y	*0.8685*
	P_S^M	*0.7326*
II	P_S^x	*1.0244*
	P_S^y	*1.0244*
	P_S^M	*0.5004*



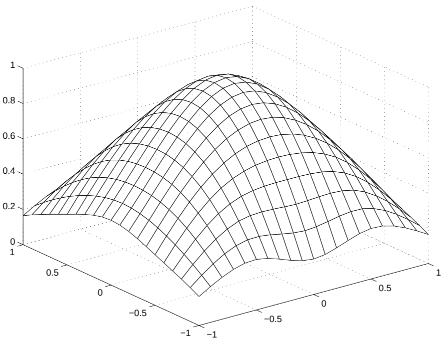
(a) Graph of function f



(b) Graph of $P_S^x f$

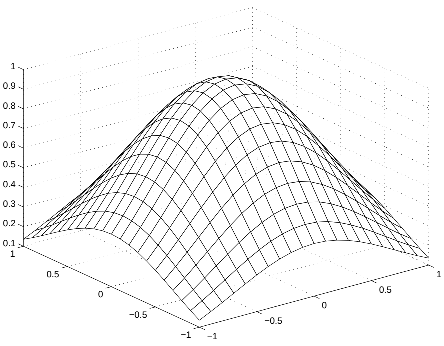


(c) Graph of $P_S^y f$

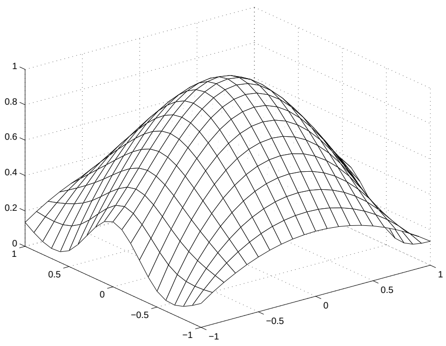


(d) Graph of $P_S^M f$

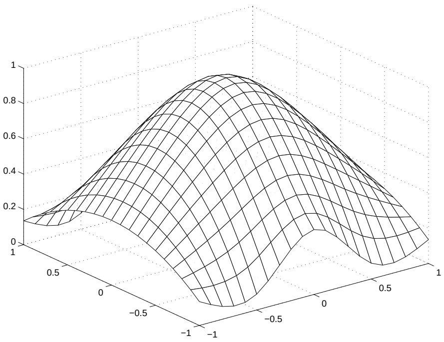
FIGURE 3. The graph of function f and the graphs of $P_S^x f$, $P_S^y f$, $P_S^M f$ for grid I



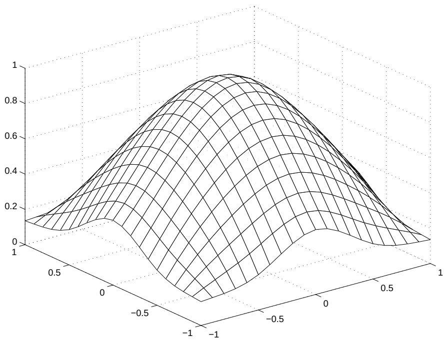
(a) Graph of function f



(b) Graph of $P_S^x f$



(c) Graph of $P_S^y f$



(d) Graph of $P_S^M f$

FIGURE 4. The graph of function f and the graphs of $P_S^x f$, $P_S^y f$, $P_S^M f$ for grid II

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