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DATA DEPENDENCE FOR SOME INTEGRAL EQUATIONS VIA WEAKLY PICARD OPERATORS

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Abstract. In this paper we study data dependence for the following integral equation:

$$u(x) = h(x, u(0)) + \int_0^{x_1} \cdots \int_0^{x_m} K(x, s, u(\theta_1 s_1, \cdots, \theta_m s_m))ds,$$

$$x \in \prod_{i=1}^{m} [0, b_i], \theta_i \in (0, 1), (\forall)i = \overline{1, m}$$

by using c-WPOs.

1. **Introduction**

Let (X, d) be a metric space and $A : X \to X$ an operator. We shall use the following notations:

$F_A := \{x \in X \mid A(x) = x\}$ the fixed points set of A.

$I(A) := \{Y \in P(X) \mid A(Y) \subset Y\}$ the family of the nonempty invariant subsets of A.

$A^{n+1} = A \circ A^n, A^0 = 1_X, A^1 = A, n \in N$.

Definition 1.1. [1] *An operator A is weakly Picard operator (WPO) if the sequence*

$$(A^n(x))_{n \in N}$$

converges, for all $x \in X$ and the limit (which depend on x) is a fixed point of A.

Definition 1.2. [1] *If the operator A is WPO and $F_A = \{x^*\}$ then by definition A is Picard operator.*

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Definition 1.3. [1] *If A is WPO, then we consider the operator*

$$A^\infty : X \to X, A^\infty(x) = \lim_{n\to\infty} A^n(x).$$

We remark that $A^\infty(X) = F_A$.

Definition 1.4. [1] *Let be A an WPO and $c > 0$. The operator A is c-WPO if*

$$d(x, A^\infty(x)) \leq c \cdot d(x, A(x)).$$

We have the following characterization of the WPOs:

Theorem 1.1. [1]*Let (X, d) be a metric space and $A : X \to X$ an operator. The operator A is WPO (c-WPO) if and only if there exists a partition of X,*

$$X = \bigcup_{\lambda\in\Lambda} X_\lambda$$

such that

(a) $X_\lambda \in I(A)$
(b) $A \mid X_\lambda : X_\lambda \to X_\lambda$ *is a Picard (c-Picard) operator, for all* $\lambda \in \Lambda$.

For the class of c-WPOs we have the following data dependence result:

Theorem 1.2. [1] *Let (X, d) be a metric space and $A_i : X \to X, i = \overline{1,2}$ operators. We suppose that:*

(i) *the operator A_i is $c_i - WPO$, $i = \overline{1,2}$.*
(ii) *there exists $\eta > 0$ such that*

$$d(A_1(x), A_2(x)) \leq \eta, (\forall)x \in X.$$

Then

$$H(F_{A_1}, F_{A_2}) \leq \eta \; max\{c_1, c_2\}.$$

Here stands for Hausdorff-Pompeiu functional.

We have:

Lemma 1.1. [1], [3] *Let $(X, d, \leq)$ be an ordered metric space and $A : X \to X$ an operator such that:*

a) A is monotone increasing.

b) A is WPO.

Then the operator A^∞ is monotone increasing.

Lemma 1.2. [1], [3] *Let $(X, d, \leq)$ be an ordered metric space and $A, B, C : X \to X$ such that :*

(i) $A \leq B \leq C$.

(ii) the operators A,B,C are WPOs.

(iii) the operator B is monotone increasing.

Then

$$x \leq y \leq z \Longrightarrow A^\infty(x) \leq B^\infty(y) \leq C^\infty(z).$$

2. **Main results**

Data dependence for functional integral equations was studied [1], [2], [3]. In what follows we consider the integral equation

$$u(x) = h(x, u(0)) + \int_0^{x_1} \cdots \int_0^{x_m} K(x, s, u(\theta_1 s, \cdots, \theta_m s))ds, \tag{1}$$

where

$$x \in \prod_{i=1}^{m} [0, b_i], \theta_i \in (0, 1), (\forall) i = \overline{1, m}.$$

We denote $D = \prod_{i=1}^{m} [0, b_i]$.

Theorem 2.1. *We suppose that:*

(i) $h \in C(D \times R)$ and $K \in C(D \times D \times R)$.

(ii) $h(0, \alpha) = \alpha, (\forall)\alpha \in R$.

(iii) there exists $L_K > 0$ such that

$$|K(x, s, u_1) - K(x, s, u_2)| \leq L_K |u_1 - u_2|,$$

for all $x, s \in D$ *and* $u_1, u_2 \in R$.

In these conditions the equation (1) has in C(D) an infinity of solutions.

Moreover if

(iv) $h(x, \cdot)$ *and* $K(x, s, \cdot)$ *are monotone increasing for all* $x, s \in D$

then if u *and* v *are solutions of the equation (1) such that* $u(0) \leq v(0)$ *we have* $u \leq v$.

Proof. Consider the operator

$$A : (C(D), \|\cdot\|_B) \to (C(D), \|\cdot\|_B),$$

$$A(u)(x) := h(x, u(0)) + \int_0^{x_1} \cdots \int_0^{x_m} K(x, s, u(\theta_1 s, \cdots, \theta_m s)) ds.$$

Here $\|u\|_B = \max_{x \in D} |u(x)| e^{-\sum_{i=1}^{m} x_i}$.

Let $\lambda \in R$ and $X_\lambda = \{u \in C(D) \mid u(0) = \lambda\}$. Then

$$C(D) = \bigcup_{\lambda \in R} X_\lambda.$$

is a partition of $C(D)$ and $X_\lambda \in I(A)$, for all $\lambda \in R$.

For all $u, v \in X_\lambda$, we have have

$$|A(u)(x) - A(v)(x)| \leq \frac{L_K}{\tau^m \theta_1 \cdots \theta_m} e^{\tau \sum_{i=1}^{m} x_i} \|u - v\|_B .$$

So the restriction of the operator A on X_λ is a c-Picard operator with $c = (1 - \frac{L_K}{\tau^m \theta_1 \cdots \theta_m})^{-1}$, for a suitable choices of τ such that $\frac{L_K}{\tau^m \theta_1 \cdots \theta_m} < 1$.

If $u \in R$ then we denote by $\widetilde{u}$ the constant operator

$$\widetilde{u} : C(D) \to C(D)$$

defined by

$$\widetilde{u}(t) = u.$$

If $u, v \in C(D)$ are the solutions of (1) with $u(0) \leq v(0)$ then $\widetilde{u(0)} \in X_{u(0)}, \widetilde{v(0)} \in X_{v(0)}$.

By lema 1.1 we have that

$$\widetilde{u(0)} \leq \widetilde{v(0)} \Longrightarrow A^{\infty}(\widetilde{u(0)}) \leq A^{\infty}(\widetilde{v(0)}).$$

But

$$u = A^{\infty}(\widetilde{u(0)}), v = A^{\infty}(\widetilde{v(0)}).$$

So, $u \leq v$.

Theorem 2.2. *Let $h_i \in C(D \times R)$ and $K_i \in C(D \times D \times R)$, $i = \overline{1,3}$ satisfy the conditions (i), (ii), (iii) from the Theorem 2.1. We suppose that*

(a) $h_2(x, \cdot)$ and $K_2(x, s, \cdot)$ are monotone increasing, for all $x, s \in D$.

(b) $h_1 \leq h_2 \leq h_3$ and $K_1 \leq K_2 \leq K_3$.

Let u_i be a solution of the equation (1) corresponding to h_i and K_i.

Then

$$u_1(0) \leq u_2(0) \leq u_3(0) \quad \textit{imply} \quad u_1 \leq u_2 \leq u_3.$$

Proof. The proof follows from Lemma 1.2.

For studding of data dependence we consider the following equations:

$$u(x) = h_1(x, u(0)) + \int_0^{x_1} \cdots \int_0^{x_m} K_1(x, s, u(\theta_1 s_1, \cdots, \theta_m s_m))ds \tag{2}$$

$$u(x) = h_2(x, u(0)) + \int_0^{x_1} \cdots \int_0^{x_m} K_2(x, s, u(\theta_1 s_1, \cdots, \theta_m s_m))ds \tag{3}$$

Theorem 2.3. *We consider (2), (3) under the following conditions:*

(i) $h_i \in C(D \times R)$ and $K_i \in C(D \times D \times R)$, $i = \overline{1,2}$.

(ii) $h_i(0, \alpha) = \alpha, (\forall)\alpha \in R$, $i = \overline{1,2}$.

(iii) there exists $L_{K_i} > 0$, $i = \overline{1,2}$ such that

$$|K_i(x, s, u_1) - K_i(x, s, u_2)| \leq L_{K_i}|u_1 - u_2|, \quad i = \overline{1,2}$$

for all $x, s \in D$ and $u_1, u_2 \in R$.

(iv) $(\exists)\eta_1, \eta_2 > 0$ such that

$$|h_1(x,u) - h_2(x,u)| \le \eta_1,$$

$$|K_1(x,s,u) - K_2(x,s,u)| \le \eta_2,$$

$(\forall) x, s \in D, u \in R.$

If $S_1,\ S_2$ are the solutions sets of the equations (2), (3), then we have:

$$H(S_1,S_2) \le (\eta_1 + \eta_2 \prod_{i=1}^{m} b_i) \max_{i=\overline{1,2}} \left\{ \frac{1}{1 - \frac{L_{K_i}}{\tau^m \theta_1 \cdots \theta_m}} \right\},$$

for $\tau > \max_{i=\overline{1,2}} \left\{ \sqrt[m]{\frac{L_{K_i}}{\theta_1 \cdots \theta_m}} \right\}$.

Proof. We consider the following operators:

$$A_i : (C(D), \|\cdot\|_B) \to (C(D), \|\cdot\|_B)),$$

$$A_i u(x) := h_i(x, u(0)) + \int_0^{x_1} \cdots \int_0^{x_m} K_i(x, s, u(\theta_1 s, \cdots, \theta_m s)) ds,\ i = \overline{1,2}$$

From:

$$|A_1(u)(x) - A_2(u)(x)| \le |h_1(x, u(0)) - h_2(x, u(0))| +$$

$$\int_0^{x_1} \cdots \int_0^{x_m} \|K_1(x, s, u(\theta_1 s \cdot \theta_m s)) - K_2(x, s, u(\theta_1 s, \cdots \theta_m s))\| ds \le$$

$$\le \eta_1 + \eta_2 \prod_{i=1}^{m} b_i.$$

we have that $\|A(u) - A(v)\|_B \le \eta_1 + \eta_2 \prod_{i=1}^{m} b_i$

Like in the proof of Theorem 1.2 we obtain that the operators $A_i, i = \overline{1,2}$ are c_i-WPOs with $c_i = \left(1 - \frac{L_{K_i}}{\tau^m \theta_1 \cdots \theta_m}\right)^{-1}$, $\tau > \max_{i=\overline{1,2}} \left\{ \sqrt[m]{\frac{L_{K_i}}{\theta_1 \cdots \theta_m}} \right\}$.

From this and by Theorem 1.2. we have conclusion.

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