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# CONTINUATION METHODS FOR INTEGRAL EQUATIONS IN LOCALLY CONVEX SPACES

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**Abstract**. The continuation method is used to investigate the existence of solutions to integral equations in locally convex spaces.

## 1. **Introduction**

In this article we study the problem of the existence of solutions for the Fredholm integral equation

$$x(t)=\int_0^1 K(t,s,x(s))ds, \qquad t\in[0,1]. \tag{1.1}$$

and the Volterra integral equation

$$x(t)=\int_0^t K(t,s,x(s))ds, \qquad t\in[0,1] \tag{1.2}$$

where the functions $x, K$ have values in a locally convex space.

In paper [2] the above equations are studied using fixed point theorems for self-maps. Our approach is based on the continuation method.

The results presented in this paper extend and complement those in [2]-[5].

We finish this section by stating the main result from [1] which will be used in the next section.

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For a map $H : D \times [0,1] \to X$, where $D \subset X$, we will use the following notations:

$$\begin{aligned}
&\Sigma = \{(x,\lambda) \in D \times [0,1] : H(x,\lambda) = x\},\\
&S = \{x \in D : H(x,\lambda) = x \text{ for some } \lambda \in [0,1]\}, \qquad (1.3)\\
&\Lambda = \{\lambda \in [0,1] : H(x,\lambda) = x \text{ for some } x \in D\}.
\end{aligned}$$

**Theorem 1.1.** *Let $X$ be a set endowed with the separating gauge structures $\mathcal{P} = \{p_\alpha\}_{\alpha \in A}$ and $\mathcal{Q}^\lambda = \{q_\beta^\lambda\}_{\beta \in B}$ for $\lambda \in [0,1]$. Let $D \subset X$ be $\mathcal{P}$-sequentially closed, $H : D \times [0,1] \to X$ a map, and assume that the following conditions are satisfies:*

*(i) for each $\lambda \in [0,1]$, there exists a function $\varphi_\lambda : B \to B$ and $a^\lambda \in [0,1)^B$, $a^\lambda = \{a_\beta^\lambda\}_{\beta \in B}$ such that*

$$q_\beta^\lambda(H(x,\lambda), H(y,\lambda)) \le a_\beta^\lambda q_{\varphi_\lambda(\beta)}^\lambda(x,y), \tag{1.4}$$

$$\sum_{n=1}^{\infty} a_\beta^\lambda a_{\varphi_\lambda(\beta)}^\lambda a_{\varphi_\lambda^2(\beta)}^\lambda \ldots a_{\varphi_\lambda^{n-1}(\beta)}^\lambda q_{\varphi_\lambda^n(\beta)}^\lambda(x,y) < \infty \tag{1.5}$$

*for every $\beta \in B$ and $x, y \in D$;*

*(ii) there exists $\rho > 0$ such that for each $(x,\lambda) \in \Sigma$, there is a $\beta \in B$ with*

$$\inf\{q_\beta^\lambda(x,y) : y \in X \setminus D\} > \rho; \tag{1.6}$$

*(iii) for each $\lambda \in [0,1]$, there is a function $\psi : A \to B$ and $c \in (0,\infty)^A$, $c = \{c_\alpha\}_{\alpha \in A}$ such that*

$$p_\alpha(x,y) \le c_\alpha q_{\psi(\alpha)}^\lambda(x,y) \qquad \text{for all } \alpha \in A \text{ and } x, y \in X; \tag{1.7}$$

*(iv) $(X,\mathcal{P})$ is a sequentially complete gauge space;*

*(v) if $\lambda \in [0,1]$, $x_0 \in D$, $x_n = H(x_{n-1},\lambda)$ for $n = 1,2,\ldots$, and $\mathcal{P}$-$\lim_{n\to\infty} x_n = x$, then $H(x,\lambda) = x$;*

*(vi) for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ with*

$$q_{\varphi_\lambda^n(\beta)}^\lambda(x, H(x,\lambda)) \le (1 - a_{\varphi_\lambda^n(\beta)}^\lambda)\varepsilon$$

*for $(x,\mu) \in \Sigma$, $|\lambda - \mu| \le \delta$, all $\beta \in B$, and $n \in \mathbb{N}$.*

*In addition, assume that $H_0 := H(.,0)$ has a fixed point. Then, for each $\lambda \in [0,1]$, the map $H_\lambda := H(.,\lambda)$ has a unique fixed point.*

**Remark 1.2.** *Notice that, by condition (ii) we have: for each* $(x, \lambda) \in \Sigma$*, there is a* $\beta \in B$ *such that the set*

$$B(x, \lambda, \beta) = \{y \in X : q^{\lambda}_{\varphi^n_\lambda(\beta)}(x, y) \leq \rho,\ \forall n \in \mathbb{N}\} \subset D. \tag{1.8}$$

*The proof of Theorem 1.1, in* [1]*, shows that the contraction condition (1.4) given on* $D$*, can be asked only on sets of the form (1.8), more exactely for* $(x, \lambda) \in \Sigma$ *and* $y \in B(x, \lambda, \beta)$*.*

## 2. **Existence Results**

This section contains existence results for the equations (1.1) and (1.2).

**Theorem 2.1.** *Let* $E$ *be a locally convex space, Hausdorff separated, complete by sequences, with the topology defined by the saturated and sufficient set of semi-norms* $\{|.|_\alpha, \alpha \in A\}$ *and let* $\delta > 0$ *be a fixed number. Assume that the following conditions are satisfied:*

*(1)* $K : [0, 1]^2 \times E \to E$ *is continuous;*

*(2) there exists* $r = \{r_\alpha\}_{\alpha \in A}$ *such that, any solution* $x$ *of the equation*

$$x(t) = \lambda \int_0^1 K(t, s, x(s))ds, \qquad t \in [0, 1], \tag{2.9}$$

*for some* $\lambda \in [0, 1]$ *satisfies* $|x(t)|_\alpha \leq r_\alpha$*, for all* $t \in [0, 1]$ *and* $\alpha \in A$*;*

*(3) there exists* $\{L_\alpha\}_{\alpha \in A} \in [0, 1)^A$ *such that*

$$|K(t, s, x) - K(t, s, y)|_\alpha \leq L_\alpha |x - y|_{f(\alpha)} \tag{2.10}$$

*whenever* $\alpha \in A$*,for all* $t, s \in [0, 1]$ *and* $x, y \in E_r$ *where* $E_r = \{x \in E :$ *there exists* $\alpha \in A$ *such that* $|x|_\alpha \leq r_\alpha + \delta\}$*;*

*(4)*

$$\sum_{n=0}^{\infty} L_\alpha L_{f(\alpha)} ... L_{f^n(\alpha)} < \infty \tag{2.11}$$

*for every* $\alpha \in A$*;*

*(5) for every* $\alpha \in A$ *and for each continuous function* $g : [0, 1] \to E$ *one has*

$$\sup\{|g(t)|_{f^n(\alpha)} : t \in [0, 1],\ n = 0, 1, 2, ...\} < \infty;$$

*(6) there exists $C$ with $0 < C \leq \dfrac{1 - L_{f^n(\alpha)}}{M_{f^n(\alpha)}}$ for all $\alpha \in A$ and $n \in \mathbb{N}$, where* $M_\alpha := \sup\limits_{\substack{t,s\in[0,1],\\ |x|_{f(\alpha)} \leq r_{f(\alpha)}}} |K(t,s,x)|_\alpha$.

*Then problem (1.1) has a solution.*

Notice that $M_\alpha < \infty$. Indeed, from (2.10) we have

$$\begin{aligned} |K(t,s,x)|_\alpha &\leq |K(t,s,x) - K(t,s,0)|_\alpha + |K(t,s,0)|_\alpha \\ &\leq L_\alpha r_{f(\alpha)} + \max_{t,s\in[0,1]} |K(t,s,0)|_\alpha < \infty \end{aligned}$$

for all $t, s \in [0,1]$ and $x \in E$ with $|x|_{f(\alpha)} \leq r_{f(\alpha)}$.

*Proof.* We shall apply Theorem 1.1. Let $X = C([0,1], E)$. For each $\alpha \in A$ we define the map $d_\alpha : X \times X \to \mathbb{R}_+$, by

$$d_\alpha(x,y) = \max_{t\in[0,1]} |x(t) - y(t)|_\alpha .$$

It is easy to show that $d_\alpha$ is a pseudo-metric on $X$ and the family $\{d_\alpha\}_{\alpha\in A}$ defines on $X$ a gauge structure, separated and complete by sequences.

Here $\mathcal{P} = \mathcal{Q}^\lambda = \{d_\alpha\}_{\alpha\in A}$ for every $\lambda \in [0,1]$. Let $D$ be the closure in X of the set

$$\{x \in X : \ d_\alpha(x,0) \leq r_\alpha + \delta \text{ for some } \alpha \in A\}.$$

We define $H : D \times [0,1] \to X$, by $H(x,\lambda) = \lambda A(x)$ , where

$$A(x)(t) = \int_0^1 K(t,s,x(s))ds.$$

In what follows we shall check conditions (i)-(vi) in Theorem 1.1. We shall start with condition (ii) by technical reason.

*Condition (ii)* becomes: there exists $\rho > 0$ such that for each solution $(x,\lambda) \in D \times [0,1]$, of $x = H(x,\lambda)$, there is an $\alpha \in A$ with

$$\inf\{d_\alpha(x,y) : y \in X\backslash D\} > \rho.$$

To prove this, let us note that if $y \in X \backslash D$, one has $d_\alpha(y,0) > r_\alpha + \delta$ for every $\alpha \in A$. Consequently, for at least one $t \in [0,1]$,

$$|x(t) - y(t)|_\alpha \geq |y(t)|_\alpha - |x(t)|_\alpha > r_\alpha + \delta - r_\alpha = \delta.$$

Then $d_\alpha(x,y) > \delta$. Hence (ii) holds for any $\rho \in (0,\delta)$.

*Condition (i)* becomes: for each $\alpha \in A$ there exists $f(\alpha) \in A$ and $L_\alpha \in [0,1)$ such that

$$d_\alpha(H(x,\lambda), H(y,\lambda)) \leq L_\alpha d_{f(\alpha)}(x,y), \tag{2.12}$$

$$\sum_{n=1}^{\infty} L_\alpha L_{f(\alpha)} ... L_{f^{n-1}(\alpha)} d_{f^n(\alpha)}(x,y) < \infty, \tag{2.13}$$

for all $x, y \in D$.

According to Remark 1.2, it suffices to have (2.12) on sets of the form (1.8). Let $(x,\lambda) \in D \times [0,1]$, such that $H(x,\lambda) = x$, and let $\beta \in A$. The set $B(x,\lambda,\beta) := \{y \in X : d_{f^n(\beta)}(x,y) \leq \rho, \forall\, n \in \mathbb{N}\}$ is included in $D$. From the fact that $H(x,\lambda) = x$ it follows that $|x(t)|_\alpha \leq r_\alpha$, for every $t \in [0,1]$ and $\alpha \in A$; from $y \in B(x,\lambda,\beta)$ it follows that $|y(t)|_\beta \leq r_\beta + \delta$, for every $t \in [0,1]$.

Then for $x$ with $H(x,\lambda) = x$ and $y \in B(x,\lambda,\beta)$ we have

$$\begin{aligned}
|H(x,\lambda)(t) - H(y,\lambda)(t)|_\alpha &= \lambda \left| \int_0^1 (K(t,s,x(s)) - K(t,s,y(s)))\, ds \right|_\alpha \\
&\leq \lambda \int_0^1 |K(t,s,x(s)) - K(t,s,y(s))|_\alpha\, ds \\
&\leq \lambda \int_0^1 L_\alpha\, |x(s) - y(s)|_{f(\alpha)}\, ds \\
&\leq \lambda L_\alpha \max_{t\in[0,1]} |x(s) - y(s)|_{f(\alpha)} \\
&= \lambda L_\alpha d_{f(\alpha)}(x,y) \\
&\leq L_\alpha d_{f(\alpha)}(x,y).
\end{aligned}$$

Then $\max_{t\in[0,1]} |H(x,\lambda)(t) - H(y,\lambda)(t)|_\alpha \leq L_\alpha d_{f(\alpha)}(x,y)$, that is (2.12).

Now (2.13) follows from (4) and (5).

*Condition (iii)* is trivial since $\mathcal{P} = \mathcal{Q}^\lambda$.

*Condition (iv)*: $(X, \{d_\alpha\}_{\alpha\in A})$ is a sequentially complete gauge space since $E$ is complete by sequences.

*Condition (v)*: Let $\lambda \in [0,1],\ x_0 \in D,\ x_n = H(x_{n-1},\lambda)$ for $n = 1, 2, ...$ and assume $\mathcal{P}\text{-}\lim\limits_{n\to\infty} x_n = x$. We wish to obtain that $H(x,\lambda) = x$.

We have

$$\begin{aligned}
\left|H(x,\lambda)(t) - x(t)\right|_\alpha &= \left|H(x,\lambda)(t) - x_n(t) + x_n(t) - x(t)\right|_\alpha \\
&\leq \left|H(x,\lambda)(t) - x_n(t)\right|_\alpha + \left|x_n(t) - x(t)\right|_\alpha \\
&= \left|H(x,\lambda)(t) - H(x_{n-1},\lambda)(t)\right|_\alpha + \left|x_n(t) - x(t)\right|_\alpha \\
&\leq \int_0^1 L_\alpha \left|x(s) - x_{n-1}(s)\right|_{f(\alpha)} ds + \left|x_n(t) - x(t)\right|_\alpha \\
&\leq L_\alpha \max_{s\in[0,1]} \left|x(s) - x_{n-1}(s)\right|_{f(\alpha)} + \sup_{t\in[0,1]} \left|x_n(t) - x(t)\right|_\alpha \\
&= L_\alpha d_{f(\alpha)}(x_{n-1}, x) + d_\alpha(x_n, x).
\end{aligned}$$

Passing to the supremum we obtain

$$d_\alpha(H(x,\lambda), x) \leq L_\alpha d_{f(\alpha)}(x_{n-1}, x) + d_\alpha(x_n, x).$$

Letting $n \to \infty$, we deduce $d_\alpha(H(x,\lambda), x) = 0$.Since this equality is true for all $\alpha \in A$ and $\{d_\alpha\}_{\alpha\in A}$ is separated, we have $H(x,\lambda) = x$ as we wished.

*Condition (vi)* becomes: for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$d_{f^n(\alpha)}(x, H(x,\lambda)) \leq (1 - L_{f^n(\alpha)})\varepsilon$$

whenever $(x,\mu) \in D \times [0,1]$, $H(x,\mu) = x$, $|\lambda - \mu| \leq \delta$, $\alpha \in A$ and $n \in \mathbb{N}$.

Indeed, using (2) and (6) we obtain

$$\begin{aligned}
|x(t)-H(x,\lambda)(t)|_{f^n(\alpha)} &= |H(x,\mu)(t)-H(x,\lambda)(t)|_{f^n(\alpha)} \\
&= |\mu-\lambda| \left| \int_0^1 K(t,s,x(s))ds \right|_{f^n(\alpha)} \\
&\le |\mu-\lambda| \int_0^1 |K(t,s,x(s))|_{f^n(\alpha)}\, ds \\
&\le |\mu-\lambda|\, M_{f^n(\alpha)} \\
&\le |\mu-\lambda|\, \frac{1-L_{f^n(\alpha)}}{C}.
\end{aligned}$$

So condition (vi) is true with $\delta(\varepsilon)=C\varepsilon$.

In addition $H(.,0)=0\cdot A(.)=0$. Hence $H(.,0)$ has a fixed point.

Thus all the assumptions of Theorem 1.1 are satisfied and the proof is completed. □

In Banach space, Theorem 2.1 becomes the following well-known result.

**Corollary 2.2.** *Let $(E,|.|)$ be a Banach space. Assume that the following conditions are satisfied:*

*(1) $K:[0,1]^2\times E\to E$ is continuous;*

*(2) there exists $r>0$ such that, any solution $x$ of the equation*

$$x(t)=\lambda\int_0^1 K(t,s,x(s))ds, \qquad t\in[0,1], \tag{2.14}$$

*for some $\lambda\in[0,1]$ satisfies $|x(t)|<r$, for all $t\in[0,1]$ and any $\lambda\in[0,1]$;*

*(3) there exists $L\in[0,1)$ such that*

$$|K(t,s,x)-K(t,s,y)|\le L\,|x-y| \tag{2.15}$$

*for all $t,s\in[0,1]$ and $x,y\in E$ with $|x|,|y|\le r$.*

*Then problem (1.1) has a solution.*

Notice that an analogue result is true for Volterra integral equation (1.2).

In particular, we obtain an existence principle for the initial value problem

$$\begin{cases} x'(t) = K(t, x(t)) & t \in [0,1] \\ x(0) = 0 \end{cases} \tag{2.16}$$

is equivalent to the integral equation

$$x(t) = \int_0^t K(s, x(s))ds, \qquad t \in [0,1] \tag{2.17}$$

for which the following result holds.

**Theorem 2.3.** *Let $E$ be a locally convex space, Hausdorff separated, complete by the sequences, with the topology defined by the saturated and sufficient set of semi-norms $\{|.|_\alpha, \alpha \in A\}$ and let $\delta > 0$ be a fixed number. Assume that the following conditions are satisfied:*

*(1) $K : [0,1] \times E \to E$ is continuous;*

*(2) there exists $r = \{r_\alpha\}_{\alpha \in A}$ such that, any solution $x$ of the equation*

$$x(t) = \lambda \int_0^t K(s, x(s))ds \qquad t \in [0,1]$$

*for some $\lambda \in [0,1]$ satisfies $|x(t)|_\alpha \leq r_\alpha$, for all $t \in [0,1]$ and $\alpha \in A$;*

*(3) there exists $\{L_\alpha\}_{\alpha \in A} \in [0,1)^A$ such that*

$$|K(t,x) - K(t,y)|_\alpha \leq L_\alpha |x - y|_{f(\alpha)}$$

*whenever $\alpha \in A$, for all $t \in [0,1]$ and $x, y \in E_r$;*

*(4) $\sum_{n=0}^\infty L_\alpha L_{f(\alpha)} \ldots L_{f^n(\alpha)} < \infty, \qquad$ for every $\alpha \in A$;*

*(5)for every $\alpha \in A$ and for each continuous function $g : [0,1] \to E$, one has*

$$\sup\{|g(t)|_{f^n(\alpha)} : t \in [0,1],\ n = 0,1,2,\ldots\} < \infty;$$

*(6) there exists $C$ with $0 < C \leq \dfrac{1 - L_{f^n(\alpha)}}{M_{f^n(\alpha)}}$, for all $\alpha \in A$ and $n \in \mathbb{N}$, where* $M_\alpha := \sup\limits_{\substack{t\in[0,1], \\ |x|_{f(\alpha)} \leq r_{f(\alpha)}}} |K(t,x)|_\alpha$.

*Then, the problem (2.17) has a solution.*

The next theorem is concerning with the "a priori" boundedness condition (2).

**Theorem 2.4.** *Assume $K : [0,1] \times E \to E$ is continuous. In addition assume that for each $\alpha \in A$, there exists $\beta_\alpha \in C([0,1], \mathbb{R}_+)$ and $\psi_\alpha : \mathbb{R}_+ \to (0, \infty)$ nondecreasing with $\frac{1}{\psi_\alpha} \in L^1_{loc}(\mathbb{R}_+)$ such that*

$$\left|K(t,x)\right|_\alpha \leq \beta_\alpha(t)\psi_\alpha(\left|x\right|_\alpha), \text{ for } x \in E, t \in [0,1] \tag{2.18}$$

*and*

$$\int_0^\infty \frac{d\tau}{\psi_\alpha(\tau)} > \int_0^1 \beta_\alpha(s)ds. \tag{2.19}$$

*Then condition (2) in Theorem 2.3 is satisfied..*

*Proof.* Let $x$ be any solution of the problem

$$\begin{cases} x'(t) = \lambda K(t, x(t)), & t \in [0,1] \\ x(0) = 0 \end{cases}$$

for some $\lambda \in [0,1]$, and let $\alpha \in A$ by arbitrary. Then

$$x(t) = \lambda \int_0^t K(s, x(s))ds, \qquad t \in [0,1]$$

and so

$$\left|x(t)\right|_\alpha \leq \lambda \int_0^t \left|K(s, x(s))\right|_\alpha ds = \lambda \int_0^t \left|x'(s)\right|_\alpha ds.$$

Let $w_\alpha(t) = \int_0^t \left|x'(s)\right|_\alpha ds$. Then $\left|x(t)\right|_\alpha \leq w_\alpha(t)$ on $[0,1]$. Using (2.18) we obtain

$$w'_\alpha(t) = \left|x'(t)\right|_\alpha = \lambda \left|K(t, x(t))\right|_\alpha \leq \lambda\beta_\alpha(t)\psi_\alpha(\left|x(t)\right|_\alpha) \leq \lambda\beta_\alpha(t)\psi_\alpha(w_\alpha(t))$$

on $[0,1]$. Next

$$\frac{w'_\alpha(t)}{\psi_\alpha(w_\alpha(t))} \leq \lambda\beta_\alpha(t) \leq \beta_\alpha(t)$$

and

$$\int_0^t \frac{w'_\alpha(s)}{\psi_\alpha(w_\alpha(s))}ds \leq \int_0^t \beta_\alpha(s)ds \leq \int_0^1 \beta_\alpha(s)ds.$$

Make the following change of variable $w_\alpha(s) = \tau$ and use (2.19) to derive

$$\int_0^{w_\alpha(t)} \frac{d\tau}{\psi_a(\tau)} \leq \int_0^1 \beta_\alpha(s)ds < \int_0^\infty \frac{d\tau}{\psi_\alpha(\tau)}.$$

The last inequality implies that there exists $r_\alpha < \infty$ such that $w_\alpha(t) \leq r_\alpha$ for every $t \in [0,1]$. Hence $\left|x(t)\right|_\alpha \leq r_\alpha$, for every $t \in [0,1]$. Therefore (2) holds. □

A better existence result is true for the Volterra integral equation

$$x(t) = \int_0^t K(t,s,x(s))ds, \qquad t \in [0,1]. \tag{2.20}$$

**Theorem 2.5.** *Let $E$ be a locally convex space, Hausdorff separated, complete by the sequences, with the topology defined by the saturated and sufficient set of semi-norms $\{|.|_\alpha : \alpha \in A\}$ and let $\delta > 0$ a fixed number. Assume that the following conditions are satisfied:*

*(1) $K : [0,1]^2 \times E \to E$ is continuous;*

*(2) there exists $r = \{r_\alpha\}_{\alpha \in A}$ such that each solution $x$ of the equation*

$$x(t) = \lambda \int_0^t K(t,s,x(s))ds, \qquad t \in [0,1]$$

*for some $\lambda \in [0,1]$ satisfies $|x(t)|_\alpha \leq r_\alpha$, for all $t \in [0,1]$ and $\alpha \in A$;*

*(3) there exists $\{L_\alpha\}_{\alpha \in A} \in (0,\infty)^A$ such that*

$$|K(t,s,x) - K(t,s,y)|_\beta \leq L_\alpha |x-y|_{f(\beta)} \qquad \text{for every } \beta \in \mathcal{O}_\alpha$$

*whenever $\alpha \in A$; $t, s \in [0,1]$ and $x, y \in E_r$; here $\mathcal{O}_\alpha := \{\alpha, f(\alpha), f^2(\alpha), ...\}$;*

*(4) for every $\alpha \in A$ and for each continuos function $g : [0,1] \to E$ one has*

$$\sup\{|g(t)|_{f^n(\alpha)} : t \in [0,1],\ n = 0,1,2,...\} < \infty;$$

*(5) $\sup\limits_n M_{f^n(\alpha)} < \infty$, for every $\alpha \in A$.*

*Then problem (2.20) has a solution.*

*Proof.* We also apply Theorem 1.1. Let $X = C([0,1], E)$. We define the applications $\|.\|_\alpha : X \to \mathbb{R}_+$ by

$$\|x\|_\alpha = \max_{t\in[0,1]} \left(|x(t)|_\alpha e^{-\theta_\alpha t}\right)$$

where $\theta_\alpha > 0$ will we precised in what follows. This applications are semi-norms on the linear space $X$, and the family $\{\|.\|_\alpha\}_{\alpha\in A}$ defines on $X$ a structure of a locally convex space, separated, complete by sequences.

Let $a < 1$. For each $\alpha \in A$ and $\theta_\alpha > 0$, we define the pseudo-metric $d_\alpha : X \times X \to \mathbb{R}_+$, by

$$d_\alpha(x,y) = \|x-y\|_\alpha .$$

Here again $\mathcal{P} = \mathcal{Q}^\lambda = \{d_\alpha\}_{\alpha \in A}$ for all $\lambda \in [0,1]$. Let $D$ be the closure of

$$\{x \in X : \text{there is } \alpha \in A \text{ with } d_\alpha(x,0) \le r_\alpha + \delta\}.$$

We define $H : D \times [0,1] \to X$, by $H(x,\lambda) = \lambda A(x)$, where

$$A(x)(t) = \int_0^t K(t,s,x(s))ds.$$

Now we check conditions (i)-(vi) from Theorem 1.1.

First we check *condition (ii)*: For any $y \in X \setminus D$ one has $d_\alpha(y,0) > r_\alpha + \delta$ for every $\alpha \in A$. Then for at least one $t \in [0,1]$, we have

$$\begin{aligned} |x(t) - y(t)|_\alpha e^{-\theta_\alpha t} &\ge (|y(t)|_\alpha - |x(t)|_\alpha) e^{-\theta_\alpha t} \\ &= |y(t)|_\alpha e^{-\theta_\alpha t} - |x(t)|_\alpha e^{-\theta_\alpha t} \\ &\ge d_\alpha(y,0) - d_\alpha(x,0) \\ &> r_\alpha + \delta - r_\alpha = \delta. \end{aligned}$$

Then $d_\alpha(x,y) > \delta$ for all $y \in X \setminus D$. So $\inf\{d_\alpha(x,y) : y \in X \setminus D\} > \rho$ for any $\rho \in (0,\delta)$.

*Condition (i)*: Using the statements made in Remark 1.2,we will check the condition (1.4) on sets of the form (1.8). Let $(x,\lambda) \in D \times [0,1]$, such that $H(x,\lambda) = x$, and let $\beta \in A$. The set $B(x,\lambda,\beta) := \{y \in X : d_{f^n(\beta)}(x,y) \le \rho, \ \forall\ n \in \mathbb{N}\}$ is included in $D$. From the fact that $H(x,\lambda) = x$ it follows that $|x(t)|_\alpha e^{-\theta_\alpha t} \le r_\alpha$, for every $t \in [0,1]$, every $\alpha \in A$ and $\theta_\alpha > 0$; from $y \in B(x,\lambda,\beta)$ it follows that $|y(t)|_\beta e^{-\theta_\beta t} \le r_\beta + \delta$, for every $t \in [0,1]$ and $\theta_\beta > 0$.

Let $x$ with $H(x,\lambda)=x$ and $y\in B(x,\lambda,\beta)$. Then for $\gamma\in\mathcal{O}_\beta$ we have

$$\begin{aligned}
|H(x,\lambda)(t)-H(y,\lambda)(t)|_\gamma &= \lambda\left|\int_0^t (K(t,s,x(s))-K(t,s,y(s)))\,ds\right|_\gamma\\
&\le \lambda\int_0^t |K(t,s,x(s))-K(t,s,y(s))|_\gamma\,ds\\
&\le \lambda\int_0^t L_\beta\,|x(s)-y(s)|_{f(\gamma)}\,e^{-\theta_\beta s}e^{\theta_\beta s}ds\\
&\le \lambda L_\beta \max_{t\in[0,1]}\left(|x(s)-y(s)|_{f(\gamma)}\,e^{-\theta_\beta s}\right)\int_0^t e^{\theta_\beta s}ds\\
&= \lambda L_\beta d_{f(\gamma)}(x,y)\int_0^t e^{\theta_\beta s}ds\\
&\le \frac{L_\beta}{\theta_\beta}d_{f(\gamma)}(x,y)e^{\theta_\beta t}.
\end{aligned}$$

So we have

$$|H(x,\lambda)(t)-H(y,\lambda)(t)|_\beta\,e^{-\theta_\beta t}\le\frac{L_\beta}{\theta_\beta}d_{f(\gamma)}(x,y).$$

Consequently

$$d_\gamma(H(x,\lambda),H(y,\lambda))\le\frac{L_\beta}{\theta}d_{f(\gamma)}(x,y).$$

We choose $\theta_\alpha>0$ large enough that

$$\frac{L_\alpha}{\theta_\alpha}\le a$$

and

$$L_\alpha+\sup_n M_{f^n(\alpha)}\le\theta_\alpha \tag{2.21}$$

for all $\alpha\in A$.

For each $\alpha\in A$ series (1.5) is dominated by the convergent series $\sum\limits_{n=0}^{\infty}a^n$ which obviously is convergent. This together with condition (4) guarantees condition (i) from Theorem 1.1.

For *condition (iii)* and *condition (iv)* see the proff of Theorem 2.1.

*Condition (v)*: We have

$$\begin{aligned}
\left|H(x,\lambda)(t)-x(t)\right|_\alpha &= \left|H(x,\lambda)(t)-x_n(t)+x_n(t)-x(t)\right|_\alpha \\
&\le \left|H(x,\lambda)(t)-x_n(t)\right|_\alpha + \left|x_n(t)-x(t)\right|_\alpha \\
&= \left|H(x,\lambda)(t)-H(x_{n-1},\lambda)(t)\right|_\alpha + \left|x_n(t)-x(t)\right|_\alpha \\
&\le \int_0^t L_\alpha \left|x(s)-x_{n-1}(s)\right|_{f(\alpha)} e^{-\theta_\alpha s} e^{\theta_\alpha s} ds + \left|x_n(t)-x(t)\right|_\alpha \\
&\le L_\alpha \max_{s\in[0,1]} \left(\left|x(s)-x_{n-1}(s)\right|_{f(\alpha)} e^{-\theta_\alpha s}\right) \int_0^t e^{\theta_\alpha s} ds + \\
&+ \left|x_n(t)-x(t)\right|_\alpha \le \frac{L_\alpha}{\theta_\alpha} d_{f(\alpha)}(x_{n-1},x) e^{\theta_\alpha t} + \left|x_n(t)-x(t)\right|_\alpha .
\end{aligned}$$

Hence

$$\left|H(x,\lambda)(t)-x(t)\right|_\alpha \le \frac{L_\alpha}{\theta_\alpha} d_{f(\alpha)}(x_{n-1},x) e^{\theta_\alpha t} + \left|x_n(t)-x(t)\right|_\alpha .$$

If we multiply by $e^{-\theta_\alpha t}$, we obtain

$$\left|H(x,\lambda)(t)-x(t)\right|_\alpha e^{-\theta_\alpha t} \le d_{f(\alpha)}(x_{n-1},x) + \left|x_n(t)-x(t)\right|_\alpha e^{-\theta_\alpha t}.$$

Taking the supremum into the above inequality, we obtain

$$d_\alpha(H(x,\lambda),x) \le d_{f(\alpha)}(x_{n-1},x) + d_\alpha(x_n,x).$$

Letting $n\to\infty$, we deduce that $d_\alpha(H(x,\lambda),x)=0$ and so $H(x,\lambda)=x$.

*Condition (vi)* From

$$\begin{aligned}
\left|x(t)-H(x,\lambda)(t)\right|_{f^n(\alpha)} &= \left|H(x,\mu)(t)-H(x,\lambda)(t)\right|_{f^n(\alpha)} \\
&= \left|\mu-\lambda\right| \left|\int_0^t K(t,s,x(s))ds\right|_{f^n(\alpha)} \\
&\le \left|\mu-\lambda\right| \int_0^t \left|K(t,s,x(s))\right|_{f^n(\alpha)} e^{-\theta_\alpha s} e^{\theta_\alpha s} ds \\
&\le \left|\mu-\lambda\right| M_{f^n(\alpha)} \int_0^t e^{\theta_\alpha s} ds.
\end{aligned}$$

we obtain

$$\left|x(t)-H(x,\lambda)(t)\right|_{f^n(\alpha)} \le \left|\mu-\lambda\right| \frac{M_{f^n(\alpha)}}{\theta_\alpha} e^{\theta_\alpha t},$$

and using (2.21) we deduce

$$|x(t) - H(x,\lambda)(t)|_{f^n(\alpha)} e^{-\theta_\alpha t} \leq |\mu - \lambda| \frac{M_{f^n(\alpha)}}{\theta_\alpha} \leq |\mu - \lambda| \left(1 - \frac{L_\alpha}{\theta_\alpha}\right).$$

So condition(vi) is true for $\delta = \varepsilon$.

In addition $H(.,0) = 0 \cdot A(.) = 0$. Hence $H(.,0)$ has a fixed point. Thus Theorem (1.1), applies. □

In case that $f : A \to A$ is the identity map, Theorem 2.5 reduces to the following result.

**Theorem 2.6.** *Let $E$ be a locally convex space, Hausdorff separated ,complete by the sequences, with the topology defined by the saturated and sufficient set of semi-norms $\{|.|_\alpha, \alpha \in A\}$ and $\delta > 0$ a fixed number. Assume that the following conditions are satisfied:*

*(1) $K : [0,1]^2 \times E \to E$ is continuous;*

*(2) there exists $r = \{r_\alpha\}_{\alpha \in A}$ such that, each solution $x$ of the problems*

$$x(t) = \lambda \int_0^t K(t,s,x(s))ds$$

*has the property $|x(t)|_\alpha \leq r_\alpha$, for all $t \in [0,1], \alpha \in A$ and every $\lambda \in [0,1]$;*

*(3)there exists $L_\alpha > 0$ such that*

$$|K(t,s,x) - K(t,s,y)|_\alpha \leq L_\alpha |x - y|_{f\alpha}$$

*whenever $\alpha \in A$, for all $t,s \in [0,1]$, and $x,y \in E_r$;*

*Then, the problem (2.20) has a solution.*

## References


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