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THREE WAYS OF DEFINING THE BIVARIATE SHEPARD OPERATOR OF LIDSTONE TYPE

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Abstract. In this paper they are given three possible definitions of the bivariate Shepard operator of Lidstone type. Also, there are given error estimations for the corresponding interpolation formulas.

1. **First variant of the Shepard operator of Lidstone type**

Let f be a real-valued function defined on $X \subset \mathbb{R}^2$, $(x_i, y_i) \in X,\ i = 0, ..., N$ some distinct points and $r_i\,(x, y)$, the distances between a given point $(x, y) \in X$ and the points $(x_i, y_i)\,,\ i = 0, 1, ..., N$.

First, we consider the original bivariate operator introduced by Shepard in 1968. This operator is defined by:

$$(S_{N,\mu}f)(x, y) = \sum_{i=0}^{N} A_i(x, y) f(x_i, y_i), \tag{1}$$

where

$$A_i\,(x, y) = \frac{\prod\limits_{\substack{j=0 \\ j\neq i}}^{N} r_j^{\mu}\,(x, y)}{\sum\limits_{k=0}^{N} \prod\limits_{\substack{j=0 \\ j\neq k}}^{N} r_j^{\mu}\,(x, y)}, \tag{2}$$

with $\mu \in \mathbb{R}_+$.

The functions $A_i,\ i = 1, \ldots, N$ have the cardinality properties:

$$A_i(x_\nu, y_\nu) = \delta_{i\nu}, \quad i, \nu = 1, \ldots, N,$$

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and

$$\sum_{i=0}^{N} A_i\left(x,y\right)=1. \tag{3}$$

The main properties of $S_{N,\mu}$ are:

1. the interpolation property:

$$\left(S_{N,\mu}f\right)\left(x_i,y_i\right)=f\left(x_i,y_i\right),\quad i=0,1,\ldots,N$$

2. the degree of exactness is:

$$\operatorname{dex}\left(S_{N,\mu}\right)=0.$$

Consider $a,b,c,d\in\mathbb{R}$, $a<b$ and $c<d$ and let $\Delta: a=x_0<x_1<\ldots<x_{M+1}=b$ and $\Delta': c=y_0<y_1<\ldots<y_{N+1}=d$ denote uniform partitions of $[a,b]$ and $[c,d]$ with stepsizes $h=(b-a)/(M+1)$ and $l=(d-c)/(N+1)$, respectively. Further, let $\rho=\Delta\times\Delta'$ be a rectangular partition of $[a,b]\times[c,d]$.

In [4] it was introduced the bivariate Shepard operator of Lidstone type, using the classical definition of the Shepard operator (1).

For a function $f\in C^{2m-2}[a,b]$, according to [1], the Lidstone interpolant uniquely exists and it is of the form

$$(L_m^{\Delta}f)(x)=\sum_{i=0}^{M+1}\sum_{\mu=0}^{m-1}r_{m,i,\mu}(x)f^{(2\mu)}(x_i), \tag{4}$$

where $r_{m,i,j}$, $0\le i\le M+1$, $0\le j\le m-1$ are satisfying

$$D^{2v}r_{m,i,j}(x_\mu)=\delta_{i\mu}\delta_{2v,j},\quad 0\le\mu\le N+1, 0\le v\le m-1. \tag{5}$$

On the subinterval $[x_i,x_{i+1}]$, $0\le i\le M$, the polynomial $L_m^{\Delta}f$ can be explicitly expressed as

$$(L_m^{\Delta,i}f)(x):= \tag{6}$$

$$:=(L_m^{\Delta}f)|_{[x_i,x_{i+1}]}(x)=\sum_{k=0}^{m-1}\left[\Lambda_k\left(\tfrac{x_{i+1}-x}{h}\right)f^{(2k)}(x_i)+\Lambda_k\left(\tfrac{x-x_i}{h}\right)f^{(2k)}(x_{i+1})\right]h^{2k},$$

where Λ_k is the Lidstone polynomial of degree $2k+1$, $k\in\mathbb{N}$. In analogous way it is obtained the expression of $L_m^{\Delta',i}f$, corresponding to Δ'.

For a function $f \in C^{2m-2,2m-2}([a,b] \times [c,d])$, the bivariate Lidstone interpolant $L_m^{\rho} f$ uniquely exists and can be explicitly expressed as

$$(L_m^{\rho} f)(x,y) = \sum_{i=0}^{M+1} \sum_{\mu=0}^{m-1} \sum_{j=0}^{N+1} \sum_{\nu=0}^{m-1} r_{m,i,\mu}(x) r_{m,j,\nu}(y) f^{(2\mu,2\nu)}(x_i, y_j), \tag{7}$$

with $r_{m,i,j}$, $0 \leq i \leq M+1$, $0 \leq j \leq m-1$ given by (5).

Lemma 1. [1] *If* $f \in C^{2m-2,2m-2}([a,b] \times [c,d])$ *then*

$$(L_m^{\rho} f)(x,y) = (L_m^{\Delta} L_m^{\Delta'} f)(x,y) = (L_m^{\Delta'} L_m^{\Delta} f)(x,y).$$

Corollary 2. [1] *For a function* $f \in C^{2m-2,2m-2}([a,b] \times [c,d])$, *from Lemma 1, we have that*

$$\begin{aligned} f - L_m^{\rho} f &= (f - L_m^{\Delta} f) + L_m^{\Delta}(f - L_m^{\Delta'} f) \\ &= (f - L_m^{\Delta} f) + [L_m^{\Delta}(f - L_m^{\Delta'} f) - (f - L_m^{\Delta'} f)] + (f - L_m^{\Delta'} f). \end{aligned} \tag{8}$$

We recall that the $k-$th modulus of smoothness of $f \in L_p[a,b]$, $0 < p < \infty$, or of $f \in C[a,b]$, if $p = \infty$, is defined by (see, e.g., [11]):

$$\omega_k(f;t)_p = \sup_{0<h\leq t} \left\| \Delta_h^k f(x) \right\|_p,$$

where

$$\Delta_h^k f(x) = \sum_{i=0}^{k} (-1)^{k+i} \tbinom{k}{i} f(x+ih).$$

In what follows $\|\cdot\|$ detones the uniform norm over the corresponding interval.

We have some error bound for the bivariate Lidstone interpolation, that is useful in what follows. It is obtained based on some results from [5].

Theorem 3. *If* $f \in C^{2m-2,2m-2}([a,b] \times [c,d])$ *then*

$$\begin{aligned} \|f - L_m^{\rho} f\| \leq & \left(1 + \left\|L_m^{\Delta}\right\|\right) W_{2m} \max_{y \in [c,d]} \omega_{2m}\left(f(\cdot, y); \tfrac{b-a}{2m}\right) \\ & + \left(1 + \left\|L_m^{\Delta}\right\|\right) W_{2m} \max_{y \in [c,d]} \omega_{2m}\left((f - L_m^{\Delta'} f)(\cdot, y); \tfrac{b-a}{2m}\right) \\ & + \left(1 + \|L_m^{\Delta'}\|\right) W_{2m} \max_{x \in [a,b]} \omega_{2m}\left(f(x, \cdot); \tfrac{d-c}{2m}\right), \end{aligned} \tag{9}$$

where W_k *is Whitney's constant.*

The bivariate Shepard operator of Lidstone type is given by:

$$(S^{Li}f)(x,y) = \sum_{i=0}^{N} A_i(x,y)(L_m^{\rho,i}f)(x,y), \tag{10}$$

where $L_m^{\rho,i}f$ is the restriction of $L_m^{\rho}f$, given by (7), to the subrectangle $[x_i, x_{i+1}] \times [y_i, y_{i+1}]$, $0 \leq i \leq N$.

We have the bivariate Shepard-Lidstone interpolation formula,

$$f = S^{Li}f + R^{Li}f, \tag{11}$$

where $R^{Li}f$ is the remainder term.

Estimations of the remainder of this interpolation formula were obtained by us in [4] and [5].

2. Second variant of the Shepard operator of Lidstone type

For a function $f : [0,1]\times[0,1] \to R$ we consider the bivariate Shepard operator as a tensor product [13]:

$$(S_{M,N}f)(x,y) = \sum_{i=0}^{M}\sum_{j=0}^{N} s_{i,\lambda}(x)s_{j,\mu}(y)f\left(\tfrac{i}{M}, \tfrac{j}{N}\right), \tag{12}$$

where $\lambda, \mu > 1$ and

$$s_{i,\lambda}(x) = \frac{\left|x - \frac{i}{M}\right|^{-\lambda}}{\sum\limits_{k=0}^{M}\left|x - \frac{k}{M}\right|^{-\lambda}},$$

$$s_{j,\mu}(y) = \frac{\left|y - \frac{j}{N}\right|^{-\mu}}{\sum\limits_{k=0}^{N}\left|y - \frac{k}{N}\right|^{-\mu}}.$$

If we denote as in [13], by $S_{M,\lambda}(f,\cdot)$ the univariate Shepard operator regarding a univariate function f we have that

$$(S_{M,N}f)(x,y) = S_{M,\lambda}(f,x)S_{N,\mu}(f,y).$$

For a function $f \in C^{2m-2,2m-2}(D)$, $m \in \mathbb{N}$, the Shepard operator of Lidstone type corresponding to (12), is defined by

$$(S_{M,N}^{Li} f)(x,y) = \sum_{i=0}^{M} \sum_{j=0}^{N} s_{i,\lambda}(x) s_{j,\mu}(y) (L_m^{\rho,i,j} f) \left(\tfrac{i}{M}, \tfrac{j}{N} \right), \tag{13}$$

where $L_m^{\rho,i,j} f$ is the restriction of $L_m^{\rho} f$, given by (7), to the subrectangle $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, $0 \leq i \leq M$, $0 \leq j \leq N$.

The corresponding interpolation formula is

$$f = S_{M,N}^{Li} f + R_{M,N}^{Li} f,$$

where $R_{M,N}^{Li} f$ denotes the remainder.

Further we give some error bounds for this interpolation procedure. First, we recall some known results.

Theorem 4. [5] *If $f \in C^{2m-2}[a,b]$ then*

$$\| f - S_{M,\lambda}(f,x) \| \leq (1 + \left\| L_m^{\Delta} \right\|) W_{2m} \omega_{2m} \left(f; \tfrac{b-a}{2m} \right). \tag{14}$$

Theorem 5. *For any $f \in C^{2m-2,2m-2}(D)$ and $\mu > 2$ we have*

$$\begin{aligned} \left\| f - S_{M,N}^{Li} f \right\| &\leq \left(1 + \left\| L_m^{\Delta} \right\| \right) W_{2m} \max_{y \in [0,1]} \omega_{2m} \left(f(\cdot, y); \tfrac{1}{2m} \right) \\ &+ \left(1 + \left\| L_m^{\Delta} \right\| \right) W_{2m} \max_{y \in [0,1]} \omega_{2m} \left((f - L_m^{\Delta'} f)(\cdot, y); \tfrac{1}{2m} \right) \\ &+ \left(1 + \| L_m^{\Delta'} \| \right) W_{2m} \max_{x \in [0,1]} \omega_{2m} \left(f(x, \cdot); \tfrac{1}{2m} \right). \end{aligned} \tag{15}$$

Proof. By Corollary 2 and taking into account (13) it follows that

$$\begin{aligned} \left\| f - S_{M,N}^{Li} f \right\|_{C[0,1]} &= \| f - S_{M,\lambda} f \|_{C[0,1]} \\ &+ \left\| (f - L_m^{\Delta',i} f) - S_{M,\lambda}(f - L_m^{\Delta',i} f) \right\|_{C[0,1]} \\ &+ \| f - S_{N,\mu} f \|_{C[0,1]}. \end{aligned}$$

and from (14) we obtain (15). $\square$

3. Third variant of the Shepard operator of Lidstone type

In [13] was introduce another type of the bivariate Shepard operator which has good approximation properties and better global smoothness preservation properties then that defined by (1). It is defined by

$$S_{M,N}(f;x,y) = \tfrac{T_{M,N}(f;x,y)}{T_{M,N}(1;x,y)},$$

with $\mu > 0$, $f : D \to \mathbb{R}$, $D \subset \mathbb{R}^2$, $D = [0,1] \times [0,1]$, $x_i = i/M$, $i = 0,...,M$; $y_i = j/N$, $j = 0,...,N$ and

$$T_{M,N}(f;x,y) = \sum_{i=0}^{M} \sum_{j=0}^{N} \frac{f(x_i,y_j)}{[(x-x_i)^2 + (y-y_j)^2]^{\mu}}.$$

For a function $f \in C^{2m-2,2m-2}(D)$, $m \in \mathbb{N}$, the corresponding Shepard operator of Lidstone type is given by

$$(S^{Li}_{M,N}f)(x,y) = \tfrac{T^{Li}_{M,N}(f;x,y)}{T_{M,N}(1;x,y)}, \tag{16}$$

with

$$T^{Li}_{M,N}(f;x,y) = \sum_{i=0}^{M} \sum_{j=0}^{N} \tfrac{(L_m^{\rho,i,j}f)(x_i,y_j)}{[(x-x_i)^2+(y-y_j)^2]^{\mu}},$$

where $L_m^{\rho,i,j}f$ is the restriction of $L_m^{\rho}f$, given by (7), to the subrectangle $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, $0 \leq i \leq M$, $0 \leq j \leq N$.

Theorem 6. [13] *For any $f \in C(D)$ and $\mu > 3/2$ we have*

$$\|f - S_{M,N}(f)\| \leq c\omega(f; \tfrac{1}{M}, \tfrac{1}{N}), \tag{17}$$

where

$$\omega(f;\delta,\eta) = \sup\{|f(x+h,y+k) - f(x,y)| : 0 \leq h \leq \delta, 0 \leq k \leq \eta\}.$$

Using Theorem 6 we can give some error bounds.

Theorem 7. *For any $f \in C^{2m-2,2m-2}(D)$ and $\mu > 3/2$ we have*

$$\begin{aligned} \left\| f - S_{M,N}^{Li} f \right\| \leq & c\omega(L_m^\rho f; \tfrac{1}{M}, \tfrac{1}{N}) + \left(1 + \left\| L_m^\Delta \right\|\right) W_{2m} \max_{y\in[0,1]} \omega_{2m}\left(f(\cdot, y); \tfrac{1}{2m}\right) \\ & + \left(1 + \left\| L_m^\Delta \right\|\right) W_{2m} \max_{y\in[0,1]} \omega_{2m}\left((f - L_m^{\Delta'} f)(\cdot, y); \tfrac{1}{2m}\right) \\ & + \left(1 + \| L_m^{\Delta'} \|\right) W_{2m} \max_{x\in[0,1]} \omega_{2m}\left(f(x, \cdot); \tfrac{1}{2m}\right). \end{aligned} \tag{18}$$

Proof. We have

$$\left\| f - S_{M,N}^{Li} f \right\| \leq \| f - L_m^\rho f \| + \left\| L_m^\rho f - S_{M,N}^{Li} f \right\|$$

and by (16) and (17) we obtain

$$\left\| f - S_{M,N}^{Li} f \right\| \leq \| f - L_m^\rho f \| + c\omega(L_m^\rho f; \tfrac{1}{M}, \tfrac{1}{N}).$$

By (9) it follows (18). □

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