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BIERMANN INTERPOLATION WITH HERMITE INFORMATION

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Abstract. If $P_1, P_2, ..., P_r$ and $Q_1, Q_2, ..., Q_r$ are Lagrange univariate projectors which form the chains i.e.

$$P_1 \le P_2 \le \cdots \le P_r, \quad Q_1 \le Q_2 \le \cdots \le Q_r$$

then the Biermann operator is defined by

$$B_r = P_1'Q_r'' \oplus P_2'Q_{r-1}'' \oplus \cdots \oplus P_r'Q_1''$$

where $P_1', \ldots, P_r', Q_1'', \ldots, Q_r''$ are the parametric extension (see [5])

In this paper, we construct projectors of Hermite type which form the chains. The representation of Biermann interpolation projector of Hermite type and the corresponding remainder term are given. Using Hermite information we increase the approximation order. We give some examples.

1. Preliminaries

LetX, Y be the liniar spaces on $\mathbb{R}$ or $\mathbb{C}$.

The liniar operator P defined on space X is called projector if and only if $P^2 = P$.

The operator $P^c = I - P$, where I is identity operator, is called the remainder projector of P.

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If P is projector on space X then the range space of projector P is denoted by

$$\mathcal{R}(P) = \{Pf|\ f \in X\} \tag{1}$$

The set of interpolation points of projector P is denoted by $\mathcal{P}(P)$.

Proposition 1.1. *If P, Q are comutative projectors*

$$\begin{aligned} &1) \quad \mathcal{R}(P \oplus Q) = \mathcal{R}(P) + \mathcal{R}(Q) \\ &2) \quad \mathcal{P}(P \oplus Q) = \mathcal{P}(P) \cup \mathcal{P}(Q). \end{aligned} \tag{2}$$

If P_1 and P_2 are projectors on space X, we define relation "$\leq$"

$$P_1 \leq P_2 \Leftrightarrow P_1 P_2 = P_1$$

Let be $f \in C(X \times Y)$ and $x \in X$. We define $f^x \in C(Y)$ by

$$f^x(t) = f(x,t), \quad t \in Y$$

For $y \in Y$ we define ${}^y f \in C(X)$ by

$${}^y f(s) = f(s,y), \quad s \in X$$

Let P be a liniar and bounded operator on $C(X)$. The parametric extension P' of P is defined by

$$(P'f)(x,y) = (P^y f)(x) \tag{3}$$

If P is a liniar and bounded operator on $C(Y)$ then the parametric extension Q'' of Q is defined by

$$(Q''f)(x,y) = (Qf^x)(y) \tag{4}$$

Proposition 1.2. *Let $r \in \mathbb{N}$, $P_1, \ldots, P_r$ be univariate interpolation projectors on $C(X)$ and $Q_1, \ldots, Q_r$ univariate interpolation projectors on $C(Y)$. Let $P_1', \ldots, P_r', Q_1'', \ldots, Q_r''$ be the corresponding parametric extension. We assume that*

$$P_1 \leq P_2 \leq \cdots \leq P_r, \quad Q_1 \leq Q_2 \leq \cdots \leq Q_r \tag{5}$$

Then

$$B_r = P_1' Q_r'' \oplus P_2' Q_{r-1}'' \oplus \cdots \oplus P_r' Q_1'' \tag{6}$$

is projector and it has representation

$$B_r = \sum_{m=1}^{r} P'_m Q''_{r+1-m} - \sum_{m=1}^{r-1} P'_m Q''_{r-m} \tag{7}$$

Moreover, we have

$$B_r^c = P_r'^c + P_{r-1}'^c Q_1''^c + \cdots + P_1'^c Q_{r-1}''^c + Q_r''^c - (P_r'^c Q_1''^c + \cdots + P_1'^c Q_r''^c) \tag{8}$$

where $P^c = I - P$, I the identity operator.

2. **Biermann interpolation**

In this section we present the Biermann interpolation operator and some of this properties from [5].

Let be the univariate projectors of polynomial interpolation

$$P_1, \ldots, P_r, Q_1, \ldots, Q_r$$

given by the folowing relations

$$(P_m f)(x) = \sum_{i=1}^{k_m} f(x_i)\phi_{i,m}(x)$$

$$(Q_n g)(y) = \sum_{j=1}^{l_n} g(y_j)\psi_{j,n}(y)$$

The sets of interpolation points are

$$\{x_1, \ldots, x_{k_m}\} \subseteq [a,b], \quad \{y_1, \ldots, y_{l_n}\} \subseteq [c,d]$$

with

$$1 \le k_1 < k_2 < \cdots < k_r, \quad 1 \le l_1 < l_2 < \cdots < l_r \tag{9}$$

The cardinal functions are given by

$$\phi_{i,m}(x) = \prod_{\substack{k=1\\k\neq i}}^{k_m} \frac{x - x_k}{x_i - x_k}, \quad 1 \le i \le k_m$$

$$\psi_{j,n}(y) = \prod_{\substack{l=1\\l\neq j}}^{l_n} \frac{y - y_l}{y_j - y_l}, \quad 1 \le j \le l_n$$

Then we have

$$\mathcal{R}(P_m) = \langle 1, x, \ldots, x^{k_m - 1} \rangle = \Pi_{k_m - 1}$$
$$\mathcal{R}(Q_n) = \langle 1, y, \ldots, y^{l_n - 1} \rangle = \Pi_{l_n - 1} \tag{10}$$

From (9) and (10) we have that parametric extensions

$$P_1', \ldots, P_r', Q_1'', \ldots, Q_r''$$

are bivariate interpolation projectors which form chains.

$$P_1' \leq \cdots \leq P_r', \quad Q_1'' \leq \cdots \leq Q_r'' \tag{11}$$

Moreover we have

$$P_m' Q_n'' = Q_n'' P_m', \quad 1 \leq m, n \leq r$$

$P_m' Q_n''$ is the tensor product projector of bivariate polynomial interpolation.

We have the representation

$$(P_m' Q_n'' f)(x, y) = \sum_{l=1}^{k_m} \sum_{j=1}^{l_n} f(x_i, y_j) \phi_{i,m}(x) \psi_{j,n}(y)$$

$P_m' Q_n''$ has the interpolation properties

$$(P_m' Q_n'' f)(x_i, y_j) = f(x_i, y_j), \quad 1 \leq i \leq k_m, \ 1 \leq j \leq l_n.$$

The range space defined by $P_m' Q_n''$ is

$$\mathcal{R}(P_m' Q_n'') = \Pi_{k_m - 1} \otimes \Pi_{l_n - 1}.$$

The projectors

$$P_1', \ldots, P_r', Q_1'', \ldots, Q_r''$$

generate a distributive lattice ξ of interpolation projectors on $C([a, b] \times [c, d])$. The interpolation projector B_r is defined by relation

$$B_r = P_1' Q_r'' \oplus P_2' Q_{r-1}'' \oplus \cdots \oplus P_r' Q_1''$$

where $r \in \mathbb{N}$. The projectors $B_1, \ldots, B_r$ form a finit chain

$$B_1 \leq B_2 \leq \cdots \leq B_r$$

Proposition 2.1. *The range space of Biermann interpolation projector is*

$$\mathcal{R}(B_r) = \Pi_{k_1-1} \otimes \Pi_{l_r-1} + \cdots + \Pi_{k_r-1} \otimes \Pi_{l_1-1} \tag{12}$$

Proposition 2.2. *The Biermann interpolation projector satisfies the interpolation properties*

$$(B_r f)(x_i, y_j) = f(x_i, y_j), \quad 1 \leq i \leq k_m,\ 1 \leq j \leq l_{r+1-m},\ 1 \leq m \leq r \tag{13}$$

The set of interpolation points possesses a disjoint representation

$$P(B_r) = \bigcup_{m=1}^{r} \bigcup_{n=0}^{r-m} \{(x_i, y_j):\ k_{m-1} < l \leq k_m,\ l_{r-m-n} < j \leq l_{r+1-m-n}\} \tag{14}$$

with $k_0 := 0,\ l_0 := 0$.

Using disjoint representation (14) of interpolation set $\mathcal{P}(B_r)$ of Biermann interpolation projector we obtain Lagrange representation of Biermann interpolant

$$B_r f = \sum_{m=1}^{r} \sum_{n=0}^{r-m} \sum_{i=1+k_{m-1}}^{k_m} \sum_{j=1+l_{r-m-n}}^{l_{r+1-m-n}} f(x_i, y_j)\Phi_{ij} \tag{15}$$

Proposition 2.3. *The cardinal function of Biermann interpolation is given by*

$$\Phi_{i,j}(x,y) = \sum_{s=m}^{m+n} \phi_{i,s}(x)\psi_{j,r+1-s}(y) - \sum_{s=m}^{m+n-1} \phi_{i,s}(x)\psi_{j,r-s}(y), \tag{16}$$

$$k_{m-1} < i \leq k_m,\ l_{r-m-n} < j \leq l_{r+1-m-n},\ 0 \leq n \leq r-m,\ 1 \leq m \leq r$$

Proposition 2.4. *The Cauchy form of remainder formula in bivariate Biermann interpolation is*

$$f(x,y) - (B_r f)(x,y) \tag{17}$$

$$= (x - x_1)\ldots(x - x_{k_r})\frac{f^{(k_r,0)}(\xi_r, y)}{k_r!} + (y - y_1)\ldots(y - y_{l_r})\frac{f^{(0,l_r)}(x, \eta_r)}{l_r!}$$

$$+ \sum_{m=1}^{r-1} \prod_{l=1}^{k_{r-m}} (x - x_i) \prod_{j=1}^{l_m} (y - y_j) \frac{f^{(k_{r-m},l_m)}(\xi_{r-m}, \eta_m)}{k_{r-m}!l_m!}$$

$$-\sum_{m=1}^{r}\prod_{l=1}^{k_{r+1-m}}(x-x_i)\prod_{j=1}^{l_m}(y-y_j)\frac{f^{(k_{r+1-m},l_m)}(\sigma_{r+1-m},\tau_m)}{k_{r+1-m}!l_m!}$$

where $\xi_i,\sigma_i \in [a,b],\ \eta_i,\tau_i \in [c,d]$ *with* $1 \le i \le r$.

Proposition 2.5. *Let be* $q = \min\{k_{r-m}+l_m :\ 0 \le m \le r\}$ *with* $k_0 = 0,\ l_0 = 0$. *Then* q *is the approximation order in bivariate Biermann interpolation, i.e.*

$$f(x,y)-(B_rf)(x,y) = O(h^q), \quad h \to 0 \tag{18}$$

Example Let be r=3 and triangular elements

$$x_i = \frac{ih}{3}, y_j = \frac{jh}{3}, 1 \le i,j \le 3, h > 0$$

$$k_m = m, l_n = n, 1 \le m,n \le 3$$

.

The cardinal functions are

$$\Phi_{13}(x,y) = \phi_{11}(x)\psi_{33}(y)$$

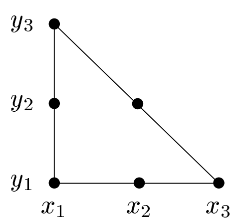
$$\Phi_{22}(x,y) = \phi_{22}(x)\psi_{22}(y)$$

$$\Phi_{31}(x,y) = \phi_{33}(x)\psi_{11}(y)$$

$$\Phi_{12}(x,y) = \phi_{11}(x)\psi_{23}(y)+\phi_{12}(x)\psi_{22}(y)-\phi_{11}(x)\psi_{22}(y)$$

$$\Phi_{21}(x,y) = \phi_{22}(x)\psi_{12}(y)+\phi_{23}(x)\psi_{11}(y)-\phi_{22}(x)\psi_{11}(y)$$

$$\Phi_{11}(x,y) = \phi_{11}(x)\psi_{13}(y)+\phi_{12}(x)\psi_{12}(y)+\phi_{13}(x)\psi_{11}(y)-\phi_{11}(x)\psi_{12}(y)-\phi_{12}(x)\psi_{11}(y)$$



The order of aproximation is 3.

3. Main result

Our goal is to construct Hermite projectors which form the chains and with their aid the Biermann operator of Hermite type.

Let be the univariate projectors of Hermite interpolation

$$P_1, \dots, P_r, Q_1, \dots, Q_r$$

given by relations

$$(P_m f)(x) = \sum_{i=1}^{k_m} \sum_{p=0}^{u_{i,m}} f^{(p)}(x_i) h_{ip}^m(x), \quad 1 \leq m \leq r$$

$$(Q_n g)(y) = \sum_{j=1}^{l_n} \sum_{q=0}^{v_{j,n}} g^{(q)}(y_j) \widetilde{h}_{jq}^n(y), \quad 1 \leq n \leq r$$

Assume that

$$\{x_1, \dots, x_{k_m}\} \subseteq [a, b]$$

$$\{y_1, \dots, y_{l_n}\} \subseteq [c, d]$$

with

$$\begin{gathered} 1 \leq k_1 < k_2 < \dots < k_r \\ 1 \leq l_1 < l_2 < \dots < l_r \end{gathered} \tag{19}$$

and

$$\begin{gathered} u_{i,m} \leq u_{i,m+1},\ i = \overline{1, k_m},\ m = \overline{1, r-1} \\ v_{j,n} \leq v_{j,n+1},\ i = \overline{1, l_n},\ n = \overline{1, r-1} \end{gathered} \tag{20}$$

The cardinal functions h_{ip}^m, $m = \overline{1, r}$ and $\widetilde{h}_{jq}^n$, $n = \overline{1, r}$ satisfy

$$\begin{cases} h_{ip}^{m\ (j)}(x_\nu) = 0,\ \nu \neq i,\ j = \overline{0, u_{\nu,m}} \\ h_{ip}^{m\ (j)}(x_i) = \delta_{jp},\ j = \overline{0, u_{i,m}} \end{cases}$$

for $p = \overline{0, u_{i,m}}$, $\nu, i = \overline{1, k_m}$ and respective

$$\begin{cases} \widetilde{h}_{jq}^{n\ (i)}(y_\nu) = 0,\ \nu \neq j,\ i = \overline{0, v_{\nu,n}} \\ \widetilde{h}_{jq}^{n\ (i)}(y_j) = \delta_{iq},\ i = \overline{0, v_{j,n}} \end{cases}$$

for $q=\overline{0,v_{j,n}}$, $\nu,j=\overline{1,l_n}$.

Theorem. 1. *The parametric extensions*

$$P'_1,\dots,P'_r,Q''_1,\dots,Q''_r$$

are bivariate interpolation projectors which form the chains

$$P'_1\le\cdots\le P'_r,\quad Q''_1\le\cdots\le Q''_r \tag{21}$$

Proof. Let be $1\le m_1\le m_2\le r$. Then

$$\begin{aligned} &k_{m_1}\le k_{m_2},\\ &u_{i,m_1}\le u_{i,m_2}, i\le k_{m_1}\end{aligned} \tag{22}$$

We have that

$$(P'_{m_1}P'_{m_2}f)(x,y)=\sum_{i_1=1}^{k_{m_1}}\sum_{p_1=0}^{u_{i_1m_1}}\left(\sum_{i_2=1}^{k_{m_2}}\sum_{p_2=0}^{u_{i_2m_2}}f^{(p_2,0)}(x_{i_2},y)h_{i_2p_2}^{m_2\ (p_1)}(x_{i_1})\right)h_{i_1p_1}^{m_1}(x) \tag{23}$$

But

$$h_{i_2p_2}^{m_2\ (p_1)}(x_{i_1})=\delta_{i_1i_2}\delta_{p_1p_2} \tag{24}$$

From (22), (23) and (24) we have

$$(P'_{m_1}P'_{m_2}f)(x,y)=\sum_{i_1=1}^{k_{m_1}}\sum_{p_1=0}^{u_{i_1m_1}}f^{(p_1,0)}(x_{i_1},y)h_{i_1p_1}^{m_1}(x)=(P'_{m_1}f)(x,y)$$

i.e.

$$P'_{m_1}\le P'_{m_2}$$

Thus $P'_1,P'_2,...,P'_r$ form a chain. Analogous $Q''_1,Q''_2,...,Q''_r$ are projectors which form a chain. □

Moreover we have

$$P'_mQ''_n=Q''_nP'_m,\quad 1\le m,n\le r$$

The tensor product projector $P'_mQ''_n$ of bivariate interpolation has representation

$$(P'_mQ''_nf)(x,y)=\sum_{i=1}^{k_m}\sum_{p=0}^{u_{i,m}}\sum_{j=1}^{l_n}\sum_{q=0}^{v_{j,n}}f^{(p,q)}(x_i,y_j)h_{ip}^m(x)\widetilde{h}_{jq}^n(y)$$

and it has the interpolation properties

$$(P'_m Q''_n f)^{(p,q)}(x_i, y_j) = f^{(p,q)}(x_i, y_j)$$

$$1 \le i \le k_m, \quad 1 \le j \le l_n, \quad 0 \le p \le u_{i,m}, \quad 0 \le q \le v_{j,n}$$

The projectors $P'_1, \ldots, P'_r, Q''_1, \ldots, Q''_r$ generate a distributive lattice ξ of projectors on $C([a,b] \times [c,d])$. A special element in this lattice is

$$B_r^H = P'_1 Q''_r \oplus P'_2 Q''_{r-1} \oplus \cdots \oplus P'_r Q''_1, \quad r \in \mathbb{N} \tag{25}$$

called Biermann projector of Hermite type

Let be $\alpha_i = u_{1,i} + \cdots + u_{k_i,i} + k_i, \quad \beta_i = v_{1,i} + \cdots + v_{l_i,i} + l_i, \quad 1 \le i \le r.$

Proposition 3.1. *The range space of projector B_r^H is given by*

$$\mathcal{R}(B_r^H) = \Pi_{\alpha_1 - 1} \otimes \Pi_{\beta_r - 1} + \cdots + \Pi_{\alpha_r - 1} \otimes \Pi_{\beta_1 - 1} \tag{26}$$

Proof. Using Proposition 1.1. we have

$$\mathcal{R}(B_r^H) = \mathcal{R}(P'_1 Q''_r) \cup \ldots \cup \mathcal{R}(P'_1 Q''_r)$$

Taking into account that

$$\mathcal{R}(P_m) = \Pi_{\alpha_m - 1}, 1 \le m \le r$$

$$\mathcal{R}(Q_n) = \Pi_{\beta_n - 1}, 1 \le m \le r$$

we get (26).□

Proposition 3.2. *Biermann projector B_r^H has the interpolation properties*

$$(B_r^H f)^{(p,q)}(x_i, y_j) = f^{(p,q)}(x_i, y_j) \tag{27}$$

$$1 \le i \le k_m, \; 1 \le j \le l_{r+1-m}, 1 \le m \le r,$$

$$u_{i,m-1} < p \le u_{i,m}, \quad 0 \le q \le v_{j,r-m+1},$$

where $u_{i,m-1} = -1$, $k_{m-1} < i \le k_m$, $1 \le m \le r$, with $k_0 = 0$

Proof. Using Proposition 1.1. we have

$$\mathcal{P}(B_r^H) = \mathcal{P}(P_1'Q_r'') \cup ... \cup \mathcal{P}(P_1'Q_r'')$$

If we denote

$$\mathcal{I}(P) = \{f^{(p,q)}(x_i.y_j) | (Pf^{(p,q)})(x_i.y_j) = f^{(p,q)}(x_i.y_j)\}$$

then

$$\mathcal{I}(B_r^H) = \mathcal{I}(P_1'Q_r'') \cup ... \cup \mathcal{I}(P_1'Q_r'')$$

We have

$$\begin{aligned} \mathcal{I}(P_1'Q_r'') &= \{f^{(p,q)}(x_i.y_j) | i = \overline{1,k_1},\ j = \overline{1,l_r},\ p = \overline{0,u_{i1}}, q = \overline{0,v_{jr}}\ \} \\ &= \{f^{(p,q)}(x_i.y_j) | i = \overline{1,k_1},\ j = \overline{1,l_r},\ p = \overline{u_{i0}+1,u_{i1}}, q = \overline{0,v_{jr}}\ \} \\ &= C_1 \end{aligned}$$

with $u_{i0} = -1$, $i = \overline{k_0+1,k_1}$

For $m = \overline{2,r}$ we have

$$\begin{aligned} &\mathcal{I}(P_m'Q_{r+1-m}'') = \{f^{(p,q)}(x_i.y_j) | i = \overline{1,k_m},\ j = \overline{1,l_{r+1-m}},\ p = \overline{0,u_{im}}, q = \overline{0,v_{j,r+1-m}}\} \\ &= \{f^{(p,q)}(x_i.y_j) | i = \overline{1,k_{m-1}},\ j = \overline{1,l_{r+1-m}},\ p = \overline{0,u_{i,m-1}}, q = \overline{0,v_{j,r+1-m}}\} \\ &\cup \{f^{(p,q)}(x_i.y_j) | i = \overline{1,k_{m-1}},\ j = \overline{1,l_{r+1-m}},\ p = \overline{u_{i,m-1}+1,u_{i,m}}, q = \overline{0,v_{j,r+1-m}}\} \\ &\cup \{f^{(p,q)}(x_i.y_j) | i = \overline{k_{m-1}+1,k_m},\ j = \overline{1,l_{r+1-m}},\ p = \overline{0,u_{i,m}}, q = \overline{0,v_{j,r+1-m}}\} \\ &= \{f^{(p,q)}(x_i.y_j) | i = \overline{1,k_{m-1}},\ j = \overline{1,l_{r+1-m}},\ p = \overline{0,u_{i,m-1}}, q = \overline{0,v_{j,r+1-m}}\} \\ &\cup \{f^{(p,q)}(x_i.y_j) | i = \overline{1,k_m},\ j = \overline{1,l_{r+1-m}},\ p = \overline{u_{i,m-1}+1,u_{i,m}}, q = \overline{0,v_{j,r+1-m}}\} \\ &= A_m \cup C_m \end{aligned}$$

with $u_{im} = -1$, $i = \overline{k_{m-1}+1,k_m}$.

As $A_m \subset \mathcal{I}(P_{m-1}'Q_{r+2-m}'')$, $m = \overline{2,r}$ it folows that

$$\mathcal{I}(B_r^H) = \mathcal{I}(P_1'Q_r'') \cup ... \cup \mathcal{I}(P_1'Q_r'') = C_1 \cup ... \cup C_r$$

q.e.d.□

Remark. 2. *The sets $C_i, i = \overline{1,r}$ are dizjoncts.*

From (7) we get the folowing representation for the projector B_r^H

$$(B_r^H f)(x,y) = \sum_{m=1}^{r}\sum_{i=1}^{k_m}\sum_{p=0}^{u_{i,m}}\sum_{j=1}^{l_{r+1-m}}\sum_{q=0}^{v_{j,r+1-m}} h_{ip}^m(x)\widetilde{h}_{jq}^{r+1-m}(y) f^{(p,q)}(x_i,y_j) \tag{28}$$

$$-\sum_{m=1}^{r-1}\sum_{i=1}^{k_m}\sum_{p=0}^{u_{i,m}}\sum_{j=1}^{l_{r-m}}\sum_{q=0}^{v_{j,r-m}} h_{ip}^m(x)\widetilde{h}_{jq}^{r-m}(y) f^{(p,q)}(x_i,y_j)$$

Taking into account (27) we obtain

$$(B_r^H f)(x,y) = \sum_{m=1}^{r}\sum_{i=1}^{k_m}\sum_{p=u_{i,m-1}+1}^{u_{im}}\sum_{j=1}^{l_{r-m+1}}\sum_{q=0}^{v_{j,r-m+1}} f^{(p,q)}(x_i,y_i)\Phi_{ij}^{pq}(x,y) \tag{29}$$

Proposition 3.3. *The cardinal functions* Φ_{ij}^{pq} *given by*

$$\Phi_{ij}^{pq}(x,y) = \sum_{m\in A_{ij}^{pq}} h_{ip}^m(x)\widetilde{h}_{jq}^{r-m+1}(y) - \sum_{m\in B_{ij}^{pq}} h_{ip}^m(x)\widetilde{h}_{jq}^{r-m}(y) \tag{30}$$

$$1 \le i \le k_m,\ 1 \le j \le l_{r+1-m}, 1 \le m \le r,$$

$$u_{i,m-1} < p \le u_{i,m}, \quad 0 \le q \le v_{j,r-m+1},$$

where

$A_{ij}^{pq} = \{m \in \{1,...,r\} | i \in X_m, p \le u_{i,m}, j \in Y_{r+1-m}, q \le v_{j,r+1-m}\}$
$B_{ij}^{pq} = \{m \in \{1,...,r-1\} | i \in X_m, p \le u_{i,m}, j \in Y_{r-m}, q \le v_{j,r-m}\}$
$X_m = \{1,...,k_m\}, Y_n = \{1,...,l_n\}$

Proof. For the function

$$f(x,y) = h_{ip}^r(x)\widetilde{h}_{jq}^r(y)$$

we have

$$B_r f = \Phi_{ij}^{pq}$$

Taking into accoumt relation (7) we get

$$\begin{aligned}
\Phi_{ij}^{pq} &= \sum_{m=1}^{r} P_m'(h_{ip}^r)\otimes Q''_{r+1-m}(\widetilde{h}_{jq}^r) - \sum_{m=1}^{r-1} P_m'(h_{ip}^r)\otimes Q''_{r-m}(\widetilde{h}_{jq}^r)\\
&= \sum_{\substack{m\in\{1,\dots,r\}\\ i\in X_m,\ p\geq u_{i,m}}} h_{ip}^m \otimes Q''_{r+1-m}(\widetilde{h}_{jq}^r) - \sum_{\substack{m\in\{1,\dots,r-1\}\\ i\in X_m,\ p\geq u_{i,m}}} h_{ip}^m \otimes Q''_{r-m}(\widetilde{h}_{jq}^r)\\
&= \sum_{\substack{m\in\{1,\dots,r\}\\ i\in X_m,\ p\geq u_{i,m}\\ j\in Y_{r+1-m},\ q\geq v_{j,r+1-m}}} h_{ip}^m \otimes \widetilde{h}_{jq}^{r+1-m} - \sum_{\substack{m\in\{1,\dots,r\}\\ i\in X_m,\ p\geq u_{i,m}\\ j\in Y_{r-m},\ q\geq v_{j,r-m}}} h_{ip}^m \otimes \widetilde{h}_{jq}^{r-m}
\end{aligned}$$

Proposition 3.4. *If $f \in C^{k_r,l_r}([a,b]\times[c,d])$ we have Cauchy form of remainder*

$$f(x,y)-(B_r^H f)(x,y) \tag{31}$$

$$= (x-x_1)^{u_{1,r}+1}\dots(x-x_{k_r})^{u_{k_r,r}+1}\frac{f^{(\alpha_r,0)}(\xi_r,y)}{\alpha_r!}$$

$$+(y-y_1)^{v_{1,r}+1}\dots(y-y_{l-r})^{v_{l_r,r}+1}\frac{f^{(0,\beta_r)}(x,\eta_r)}{\beta_r!}$$

$$+\sum_{m=1}^{r-1}\prod_{i=1}^{k_{r-m}}(x-x_i)^{u_{i,r-m}+1}\prod_{j=1}^{l_m}(y-y_j)^{v_{j,m}+1}\frac{f^{(\alpha_{r-m},\beta_m)}(\xi_{r-m},\eta_m)}{\alpha_{r-m}!\beta_m!}$$

$$-\sum_{m=1}^{r}\prod_{i=1}^{k_{r+1-m}}(x-x_i)^{u_{i,r+1-m}+1}\prod_{j=1}^{l_m}(y-y_j)^{v_{j,m}+1}\frac{f^{(\alpha_{r+1-m},\beta_m)}(\sigma_{r+1-m},\tau_m)}{\alpha_{r+1-m}!\beta_m}$$

$$\xi_i,\sigma_i\in[a,b],\quad \eta_i,\tau_i\in[c,d], 1\leq i\leq r$$

Proof. We have that

$$(P_m^c f)(x) = f(x)-(P_m f)(x) = \prod_{i=1}^{k_m}(x-x_i)^{u_{i,m}+1}\frac{f^{(\alpha_m)}(\xi_m)}{\alpha_m!}, \xi_m\in[a,b], f\in C^{k_m}([a,b])$$

$$(Q_n^c g)(y) = g(y)-(Q_m f)(y) = \prod_{j=1}^{l_n}(y-y_j)^{v_{j,n}+1}\frac{g^{(\beta_n)}(\eta_n)}{\beta_n!}, \eta_n\in[c,d], g\in C^{l_n}([c,d])$$

$$1\leq m,n\leq r$$

Taking into account relation (8) we get (31).□

Let be $h=b-a=d-c$ and $q=\min\{\alpha_{r-m}+\beta_m,\ 0\leq m\leq r\}$ with $\alpha_0=0$, $\beta_0=0$. Then we have

$$f(x,y)-(B_r^H f)(x,y) = O(h^q),\quad h\to 0. \tag{32}$$

Example. We determine the order of approximation of the Biermann interpolation projector B_r^H for triangular elements.We choice r=3 and

$$x_i = \frac{ih}{3}, y_j = \frac{jh}{3}, 1 \le i, j \le 3, h > 0$$

$$k_m = m, l_n = n, 1 \le m, n \le 3$$

Let be the Hermite interpolation projectors

$$P_1 = H^x \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \ P_2 = H^x \begin{pmatrix} x_1 & x_2 \\ 1 & 0 \end{pmatrix}, \ P_3 = H^x \begin{pmatrix} x_1 & x_2 & x_3 \\ 1 & 0 & 1 \end{pmatrix}$$

$$Q_1 = H^y \begin{pmatrix} y_1 \\ 1 \end{pmatrix}, \ Q_2 = H^y \begin{pmatrix} y_1 & y_2 \\ 1 & 1 \end{pmatrix}, \ Q_3 = H^y \begin{pmatrix} y_1 & y_2 & y_3 \\ 1 & 1 & 0 \end{pmatrix}$$

i.e.

$u_{11} = 0;$

$u_{12} = 1;\ u_{22} = 0;$

$u_{13} = 1;\ u_{23} = 0;\ u_{33} = 1;$

$v_{11} = 1;$

$v_{12} = 1;\ v_{22} = 1;$

$v_{13} = 1;\ v_{23} = 1;\ v_{33} = 0;$

It folows that parametric extension form the chains

$$P_1' \le P_2' \le P_3' \quad Q_1'' \le Q_2'' \le Q_3''$$

and we can define the Biermann operator of Hermite type

$$B_3^H = P_1' Q_3'' \oplus P_2' Q_2'' \oplus P_3' Q_2''.$$

The operator B_3^H has the interpolation properties

$(B_3^H f)^{(p,q)}(x_1, y_1) = f^{(p,q)}(x_1, y_1), (p,q) \in \{(0,0), (0,1), (1,0), (1,1)\}$

$(B_3^H f)^{(p,q)}(x_1, y_2) = f^{(p,q)}(x_1, y_2), (p,q) \in \{(0,0), (0,1), (1,0), (1,1)\}$

$(B_3^H f)^{(p,q)}(x_1, y_3) = f^{(p,q)}(x_1, y_3), (p,q) = (0,0)$

$(B_3^H f)^{(p,q)}(x_2, y_1) = f^{(p,q)}(x_2, y_1), (p,q) \in \{(0,0), (0,1)\}$

$(B_3^H f)^{(p,q)}(x_2, y_2) = f^{(p,q)}(x_2, y_2), (p,q) \in \{(0,0), (0,1)\}$

$(B_3^H f)^{(p,q)}(x_3, y_1) = f^{(p,q)}(x_3, y_1), (p,q) \in \{(0,0), (0,1), (1,0), (1,1)\}$

The cardinal functions of B_3^H are

$$\Phi_{11}^{pq}(x,y) = h_{1p}^1(x)\widetilde{h}_{1q}^3(y) + h_{1p}^2(x)\widetilde{h}_{1q}^2(y) + h_{1p}^3(x)\widetilde{h}_{1q}^1(y) - h_{1p}^1(x)\widetilde{h}_{1q}^2(y) - h_{1p}^2(x)\widetilde{h}_{1q}^1(y),\ (p,q) \in \{(0,0),(0,1)\}$$

$$\Phi_{11}^{pq}(x,y) = h_{1p}^2(x)\widetilde{h}_{1q}^2(y) + h_{1p}^3(x)\widetilde{h}_{1q}^1(y) - h_{1p}^2(x)\widetilde{h}_{1q}^1(y),\ (p,q) \in \{(1,0),(1,1)\}$$

$$\Phi_{12}^{pq}(x,y) = h_{1p}^1(x)\widetilde{h}_{2q}^3(y) + h_{1p}^2(x)\widetilde{h}_{2q}^2(y) - h_{1p}^1(x)\widetilde{h}_{2q}^2(y),\ (p,q) \in \{(0,0),(0,1)\}$$

$$\Phi_{12}^{pq}(x,y) = h_{1p}^2(x)\widetilde{h}_{2q}^2(y), (p,q) \in \{(1,0),(1,1)\}$$

$$\Phi_{13}^{pq}(x,y) = h_{1p}^1(x)\widetilde{h}_{3q}^3(y),\ (p,q) = (0,0)$$

$$\Phi_{21}^{pq}(x,y) = h_{2p}^2(x)\widetilde{h}_{1q}^2(y) + h_{2p}^3(x)\widetilde{h}_{1q}^1(y) - h_{2p}^2(x)\widetilde{h}_{1q}^1(y),\ (p,q) \in \{(0,0),(0,1)\}$$

$$\Phi_{22}^{pq}(x,y) = h_{2p}^2(x)\widetilde{h}_{2q}^2(y), (p,q) \in \{(0,0),(0,1)\}$$

$$\Phi_{31}^{pq}(x,y) = h_{3p}^3(x)\widetilde{h}_{1q}^1(y),\ (p,q) \in \{(0,0),(0,1),(1,0),(1,1)\}$$

We compute the approximation order

$\alpha_1 = u_{11} + 1 = 1;$

$\alpha_2 = u_{12} + u_{22} + 2 = 3;$

$\alpha_3 = u_{13} + u_{23} + u_{33} + 3 = 5;$

$\beta_1 = v_{11} + 1 = 2;$

$\beta_2 = v_{12} + v_{22} + 2 = 4;$

$\beta_3 = v_{13} + v_{23} + v_{33} + 3 = 5;$

$q = \min\{\alpha_3,\ \alpha_2 + \beta_1,\ \alpha_1 + \beta_2,\ \beta_3\} = 5;$

The order of approximation in this case is 5.

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