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ITERATES OF SOME MULTIVARIATE APPROXIMATION PROCESSES, VIA CONTRACTION PRINCIPLE

CLAUDIA BACOŢIU

Abstract. In this paper we study a general class of linear positive operators, using the theory of weakly Picard operators. The convergence of the iterates of the defined operators will be proven.

1. Introduction

In [2] and [1] Agratini and Rus applied the theory of weakly Picard operators to prove the convergence of iterates of a certain class of linear positive operators. In some particular cases, these operators are well known approximation operators, such as Bernstein or Stancu operators. In the above mentioned papers, the authors have considered the univariate, respectivelly the bivariate cases. In the present paper we give a generalization of these results to a class of linear positive operators defined on $C([0,1]^p)$, $p \in \mathbb{N}$.

2. Weakly Picard operators

Let $(X, \to)$ be an L-space and $A : X \to X$ an operator. In this paper we will use the following notations:

$$F_A := \{x \in X : A(x) = x\};$$

$$I(A) := \{Y \in P(X) : A(Y) \subset Y\};$$

$$A^0 := 1_X \, , \; A^{n+1} := A \circ A^n \;\forall n \in \mathbb{N}.$$

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Definition 2.1. *(Rus [7]) The operator A is said to be:*
(i) weakly Picard operator (WPO) if $\forall x_0 \in X\ A^n(x_0) \to x_0^*$, *and the limit* x_0^* *is a fixed point of* A, *which may depend on* x_0;
(ii) Picard operator (PO) if $F_A = \{x^*\}$ *and* $\forall x_0 \in X\ A^n(x_0) \to x^*$.
If A is an WPO, we consider the operator A^∞ defined by

$$A^\infty : X \to X, \quad A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

We have the next characterization theorem of WPOs:

Theorem 2.1. *(Rus [7]) The operator A is WPO if and only if there exists a partition of X,* $X = \bigcup_{\lambda\in\Lambda} X_\lambda$ *such that:*
(i) $X_\lambda \in I(A),\ \forall \lambda \in \Lambda$;
(ii) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ *is PO,* $\forall \lambda \in \Lambda$.

3. Main results

Let $p \geq 1$ be a fixed integer and

$$D := [0,1] \times [0,1] \times ... \times [0,1] = [0,1]^p.$$

$C(D) = \{f : D \to \mathbb{R}\ :\ f - \text{continuous}\}$.
We introduce the next notations: $\alpha^{\langle 0\rangle} := (0,0,...,0) = 0_{\mathbb{R}^p}$ is the null vector. For all $k \in \overline{1,p}$ and for all $1 \leq i_1 < i_2 < ... < i_k \leq p$, denote by $\alpha^{\langle k\rangle}_{i_1,i_2,...,i_k}$ the vector from $\mathbb{R}^p$ defined as follows: on positions $i_1, i_2, ..., i_k$ the value 1 appears and on all other positions the value 0 is displayed.

$$M_k := \{(i_1,i_2,...,i_k)\ :\ 1 \leq i_1 < i_2 < ... < i_k \leq p\} \subset \mathbb{N}^k \quad \forall k \in \overline{1,p}$$

$$\nu_D := \left\{\alpha^{\langle 0\rangle}\right\} \bigcup \left\{\alpha^{\langle k\rangle}_{i_1,i_2,...,i_k}\ :\ k \in \overline{1,p} \text{ and } (i_1,i_2,...,i_k) \in M_k\right\}.$$

Denote by e_α, $\alpha \in \nu_D$ the test functions

$$e_\alpha : D \to \mathbb{R}_+; \quad e_\alpha(x_1,x_2,...,x_p) := \prod_{k=1}^{p} x_k^{\alpha_k} \quad \forall (x_1,x_2,...,x_p) \in D,$$

with the convention that, if in a component, α_k is null, then $x_k^{\alpha_k}$ will be replaced by 1.

We notice that

$$Card(M_k) = \begin{pmatrix} p \\ k \end{pmatrix}, \ \forall k = \overline{1,p} \quad \text{and} \quad Card(\nu_D) = \sum_{k=0}^{p} \begin{pmatrix} p \\ k \end{pmatrix} = 2^p := N.$$

Remark 3.1. *Any* $\alpha \in \nu_D$ *is* $\alpha^{\langle 0 \rangle}$ *or there exist* $k \in \overline{1,p}$ *and* $(i_1, i_2, ..., i_k) \in M_k$ *such that* $\alpha = \alpha^{\langle k \rangle}_{i_1, i_2, ..., i_k}$.

Remark 3.2. *Because* $Card(\nu_D) = Card\{1, 2, ..., N\}$*, it follows that there exists a bijective function*

$$\omega : \nu_D \to \{1, 2, ..., N\}.$$

More precisely:

- for $k = 0$ *there exists a unique* $j \in \overline{1,N}$ *such that* $\omega(\alpha^{\langle 0 \rangle}) = j$ *and*

- for any $k \in \overline{1,p}$ *and for any* $(i_1, i_2, ..., i_k) \in M_k$ *there exists a unique* $j \in \overline{1,N}$ *such that* $\omega(\alpha^{\langle k \rangle}_{i_1, i_2, ..., i_k}) = j$.

For all $(m_1, m_2, ..., m_p) \in \mathbb{N}^p$ consider the next p-dimensional net

$$\Delta^k_{m_k} := (0 = x_{k,m_k,0} < x_{k,m_k,1} < ... < x_{k,m_k,m_k} = 1) \quad \forall k = \overline{1,p}.$$

We also consider the next systems of real positive functions

$$0 \le \psi_{k,m_k,i} \in C[0,1], \quad \forall i = \overline{0, m_k} \quad \forall k = \overline{1,p}.$$

Let the next assumptions be satisfied:

$$\sum_{i=0}^{m_k} \psi_{k,m_k,i}(x) = 1, \quad \forall x \in [0,1], \quad \forall k = \overline{1,p}; \tag{1}$$

$$\sum_{i=0}^{m_k} x_{k,m_k,i} \psi_{k,m_k,i}(x) = x, \quad \forall x \in [0,1], \quad \forall k = \overline{1,p}; \tag{2}$$

$$\psi_{k,m_k,0}(0) = \psi_{k,m_k,m_k}(1) = 1, \quad \forall k = \overline{1,p}. \tag{3}$$

We also introduce the next notation:

$$K := \{0, 1, ..., m_1\} \times \{0, 1, ..., m_2\} \times ... \times \{0, 1, ..., m_p\}.$$

Clearly,

$$\partial K = \{(0,0,...,0),(m_1,0,...,0),...,(0,0,...,m_p),...,(m_1,m_2,...,m_p)\} \subset \mathbb{R}^p.$$

Notice that $Card\partial K = N$ and

$$(x_{1,m_1,i_1},...,x_{p,m_p,i_p}) \in \nu_D, \quad \forall (i_1,...,i_p) \in \partial K. \tag{4}$$

Let $u_{m_1,...,m_p} : D \to \mathbb{R}$ be the function given by

$$u_{m_1,...,m_p}(x_1,...,x_p) := \sum_{(i_1,...,i_p)\in\partial K} \psi_{1,m_1,i_1}(x_1)...\psi_{p,m_p,i_p}(x_p) \tag{5}$$

and

$$\sigma_{m_1,...,m_p} := \inf\left\{u_{m_1,...,m_p}(x_1,...,x_p) \, : \, (x_1,...,x_p) \in D\right\}. \tag{6}$$

We define now the operators:

$$L_{m_1,m_2,...,m_p} : C(D) \to C(D)$$

by

$$(L_{m_1,m_2,...,m_p} f)(x_1,x_2,...,x_p) :=$$

$$= \sum_{i_1=0}^{m_1} ... \sum_{i_k=0}^{m_k} ... \sum_{i_p=0}^{m_p} \psi_{1,m_1,i_1}(x_1)...\psi_{k,m_k,i_k}(x_k)...\psi_{p,m_p,i_p}(x_p)\cdot \tag{7}$$

$$\cdot f(x_{1,m_1,i_1},...,x_{k,m_k,i_k},...,x_{p,m_p,i_p})$$

for all $f \in C(D)$, $\forall (x_1,x_2,...,x_p) \in D$.

Proposition 3.1. *The operators $L_{m_1,m_2,...,m_p}$ have the next properties:*

(i) $L_{m_1,m_2,...,m_p}(e_\alpha) = e_\alpha$, for all $\alpha \in \nu_D$;

(ii) $(L_{m_1,m_2,...,m_p} f)(\alpha) = f(\alpha)$, for all $f \in C(D)$, $\forall \alpha \in \nu_D$;

(iii) $L_{m_1,m_2,...,m_p}$ are linear and positive.

Proof: The first statement follows from (1) and (2). The second follows from (1) and (3). The last statement is obvious.□

For all $\Lambda = (\lambda_1,\lambda_2,...,\lambda_N) \in \mathbb{R}^N$, consider the sets

$$X_\Lambda := \left\{f \in C(D) \, : \, f(\alpha) := \lambda_{\omega(\alpha)}, \, \forall \alpha \in \nu_D\right\} \tag{8}$$

Lemma 3.1. *(i) For all $\Lambda \in \mathbb{R}^N$, the sets X_Λ are closed in $C(D)$;*
(ii) $X_\Lambda \in I\Big(L_{m_1,m_2,...,m_p}\Big)$;
(iii) $C(D) = \bigcup_{\Lambda\in\mathbb{R}^N} X_\Lambda$ is a partition of the space $C(D)$.

The main result is given by the next theorem.

Theorem 3.1. *If $\sigma_{m_1,...,m_p}$ given by (6) is non-zero, then the operators $L_{m_1,m_2,...,m_p}$ defined by (7) are WPOs and for all $(m_1, m_2, ..., m_p) \in \mathbb{N}^p$, we have:*

$$L^{\infty}_{m_1,m_2,...,m_p}(f) = \varphi^*_f, \quad \forall f \in C(D).$$

The function φ^*_f is defined by

$$\varphi^*_f(x_1, x_2, ..., x_p) = C^0_0 + \sum_{i_1\in M_1} C^1_{i_1} x_{i_1} + \sum_{(i_1,i_2)\in M_2} C^2_{i_1,i_2} x_{i_1} x_{i_2} + ...+$$

$$+ \sum_{(i_1,i_2,...,i_k)\in M_k} C^k_{i_1,i_2,...,i_k} x_{i_1} x_{i_2} ... x_{i_k} + ... + C^p_{1,2,...,p} x_1 x_2 ... x_p \quad \forall f \in C(D)\ \forall (x_1, x_2, ..., x_p) \in D$$

where C^0_0 and $C^k_{i_1,i_2,...,i_k}$, $\forall k \in \overline{1,p}$, $\forall (i_1, i_2, ..., i_k) \in M_k$ are real numbers which depend of f, given by

$$C^0_0 := f(\alpha^{\langle 0\rangle});$$

$$C^k_{i_1,i_2,...,i_k} := (-1)^k f(\alpha^{\langle 0\rangle}) + (-1)^{k-1} \sum_{s_1=1}^{k} f(\alpha^{\langle 1\rangle}_{i_{s_1}}) + (-1)^{k-2} \sum_{1\le s_1<s_2\le k} f(\alpha^{\langle 2\rangle}_{i_{s_1},i_{s_2}})+$$

$$+... + (-1)^{k-l} \sum_{1\le s_1<s_2<...<s_l\le k} f(\alpha^{\langle l\rangle}_{i_{s_1},i_{s_2},...,i_{s_l}}) + ... + (-1)^0 f(\alpha^{\langle k\rangle}_{i_1,i_2,...,i_k});$$

for all $k \in \overline{1,p}$, $\forall (i_1, i_2, ..., i_k) \in M_k$.

Proof: By virtue of Lemma 3.1, the sets X_Λ are closed, $X_\Lambda \in I\Big(L_{m_1,m_2,...,m_p}\Big)$, and $C(D) = \bigcup_{\Lambda\in\mathbb{R}^N} X_\Lambda$ is a partition of the space $C(D)$.
Denote by $\|\cdot\|_{C(D)}$ the Cebysev norm in $C(D)$, i.e.

$$\|v\|_{C(D)} := \sup_{(x_1,...,x_p)\in D} |v(x_1, ..., x_p)|, \quad \forall v \in C(D).$$

For all $\Lambda \in C(D)$ and for all $f, g \in X_\Lambda$ we have:

$$|(L_{m_1,m_2,...,m_p} f)(x_1, x_2, ..., x_p) - (L_{m_1,m_2,...,m_p} g)(x_1, x_2, ..., x_p)| =$$

$$= \Big| \sum_{i_1=0}^{m_1} ... \sum_{i_k=0}^{m_k} ... \sum_{i_p=0}^{m_p} \psi_{1,m_1,i_1}(x_1) ... \psi_{k,m_k,i_k}(x_k) ... \psi_{p,m_p,i_p}(x_p) \cdot$$

$$\cdot (f-g)(x_{1,m_1,i_1},...,x_{k,m_k,i_k},...,x_{p,m_p,i_p})\Big| =$$

$$= \Big| \sum_{(i_1,...,i_p)\in K} \psi_{1,m_1,i_1}(x_1)...\psi_{p,m_p,i_p}(x_p)\cdot(f-g)(x_{1,m_1,i_1},...,x_{p,m_p,i_p})\Big| \le$$

$$\le \Big| \sum_{(i_1,...,i_p)\in K-\partial K} \psi_{1,m_1,i_1}(x_1)...\psi_{p,m_p,i_p}(x_p)\cdot(f-g)(x_{1,m_1,i_1},...,x_{p,m_p,i_p})\Big| +$$

$$+ \Big| \sum_{(i_1,...,i_p)\in \partial K} \psi_{1,m_1,i_1}(x_1)...\psi_{p,m_p,i_p}(x_p)\cdot(f-g)(x_{1,m_1,i_1},...,x_{p,m_p,i_p})\Big| \overset{(4)}{=}$$

$$= \Big| \sum_{(i_1,...,i_p)\in K-\partial K} \psi_{1,m_1,i_1}(x_1)...\psi_{p,m_p,i_p}(x_p)\cdot(f-g)(x_{1,m_1,i_1},...,x_{p,m_p,i_p})\Big| \le$$

$$\le \left[\sum_{(i_1,...,i_p)\in K-\partial K} \psi_{1,m_1,i_1}(x_1)...\psi_{p,m_p,i_p}(x_p)\right]\cdot \|f-g\|_{C(D)} =$$

$$= \left[\sum_{(i_1,...,i_p)\in K} \psi_{1,m_1,i_1}(x_1)...\psi_{p,m_p,i_p}(x_p) - \sum_{(i_1,...,i_p)\in \partial K} \psi_{1,m_1,i_1}(x_1)...\psi_{p,m_p,i_p}(x_p)\right]\cdot$$

$$\cdot \|f-g\|_{C(D)} \overset{(1)}{=} \left[1 - \sum_{(i_1,...,i_p)\in \partial K} \psi_{1,m_1,i_1}(x_1)...\psi_{p,m_p,i_p}(x_p)\right]\cdot \|f-g\|_{C(D)} \overset{(5)}{=}$$

$$= \left[1-u_{m_1,...,m_p}(x_1,...,x_p)\right]\cdot \|f-g\|_{C(D)} \overset{(6)}{\le} (1-\sigma_{m_1,...,m_p})\cdot \|f-g\|_{C(D)}.$$

Because $\sigma_{m_1,...,m_p}$ in non-zero, the restrictions $L_{m_1,m_2,...,m_p}|_{X_\Lambda}$ are contractions with the same constant $1-\sigma_{m_1,...,m_p} \in [0,1[$. Consequently, they are POs.

It can be proven that for all $\Lambda \in \mathbb{R}^N$, $\varphi_f^* \in X_\Lambda$ $\forall f \in X_\Lambda$. For any $\Lambda \in \mathbb{R}^N$, the restriction $L_{m_1,m_2,...,m_p}|_{X_\Lambda}$ has a unique fixed point which is φ_f^* (it follows from Proposition 3.1).

From Theorem 2.1 it follows that $L_{m_1,m_2,...,m_p} : C(D) \to C(D)$ are WPOs. Besides, for all $f \in C(D)$, the limit operator is φ_f^*. □

Remark 3.3. *In the case $p = 2$ we have $D = [0,1]\times[0,1]$, $N = 4$,*

$$\alpha^{\langle 0\rangle} = (0,0), \quad \alpha_1^{\langle 1\rangle} = (1,0), \quad \alpha_2^{\langle 1\rangle} = (0,1), \quad \alpha_{1,2}^{\langle 2\rangle} = (1,1)$$

and

$$\nu_D = \{(0,0),(1,0),(0,1),(1,1)\}$$

There exists a bijective function $\omega : \nu_D \to \{1, 2, 3, 4\}$.

For all $\Lambda := (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{R}^4$ *consider the sets:*

$$X_\Lambda := \{f \in C(D) \,:\, f(0,0) = \lambda_{\omega(0,0)}, f(1,0) = \lambda_{\omega(1,0)}, f(0,1) = \lambda_{\omega(0,1)}, f(1,1) = \lambda_{\omega(1,1)}\}$$

For all $m_1 := m \in \mathbb{N}$, $m_2 := n \in \mathbb{N}$, *the operators* $L_{m,n}$ *are WPOs and*

$$L^\infty_{m,n}(f) = \varphi^*_f, \quad \forall f \in C(D)$$

where

$$\varphi^*_f(x,y) = \underbrace{f(\alpha^{\langle 0\rangle})}_{C^0_0} + \left(\underbrace{[f(\alpha^{\langle 1\rangle}_1) - f(\alpha^{\langle 0\rangle})]}_{C^1_1} x + \underbrace{[f(\alpha^{\langle 1\rangle}_2) - f(\alpha^{\langle 0\rangle})]}_{C^1_2} y \right) +$$

$$+ \underbrace{\left(f(\alpha^{\langle 0\rangle}) - [f(\alpha^{\langle 1\rangle}_1) + f(\alpha^{\langle 1\rangle}_2)] + f(\alpha^{\langle 2\rangle}_{1,2}) \right)}_{C^2_{1,2}} xy$$

So, we reobtain [1; Remark 1 - Theorem 9] in the particular case $a_1 = a_2 = 0$ *and* $b_1 = b_2 = 1$.

4. Applications

4.1. Bernstein operators of $(m_1, ..., m_p)$ order.

For all $(m_1, ..., m_p) \in \mathbb{N}^p$ consider the next system of points:

$$\Delta^k_{m_k} := \left(0 = \frac{0}{m_k} < \frac{1}{m_k} < ... < \frac{m_k}{m_k} = 1 \right) \quad \forall k = \overline{1,p}.$$

Let the functions $\psi_{k,m_k,i}$ be the fundamental polynomials of Bernstein

$$\psi_{k,m_k,i}(x) := b_{m_k,i}(x) = \begin{pmatrix} m_k \\ i \end{pmatrix} x^i (1-x)^{m_k - i} \quad \forall x \in [0,1]$$

for all $i = \overline{0, m_k}$, $k = \overline{1,p}$.

Then the polynomials $L_{m_1,m_2,...,m_p} : C(D) \to C(D)$ from (7) are the Bernstein polynomials of $(m_1, ..., m_p)$ order, given by

$$(L_{m_1,m_2,...,m_p} f)(x_1, x_2, ..., x_p) :=$$

$$= \sum_{i_1=0}^{m_1} ... \sum_{i_p=0}^{m_p} b_{m_1,i_1}(x_1)...b_{m_p,i_p}(x_p) \cdot f\left(\frac{i_1}{m_1}, ..., \frac{i_p}{m_p} \right).$$

The next theorem states the convergence of the iterates of the generalized Bernstein operator.

Theorem 4.1. *The Bernstein operators* $L_{m_1,m_2,...,m_p}$ *are WPOs and*

$$L^{\infty}_{m_1,m_2,...,m_p}(f)=\varphi^*_f,\quad \forall f\in C(D)$$

with φ^*_f *as in Theorem 3.1.*

Remark 4.1. *In the particular case* $p=2$, $m_1:=m$, $m_2:=n$ *we reobtain the estimation*

$$\lambda_{m,n}=\frac{1}{2^{m+n-2}}$$

(see [1; §4.1.])

4.2. **Stancu modified operators of** $(m_1,...,m_p)$ **order.** For all $(m_1,...,m_p)\in\mathbb{N}^p$ consider the systems of points: $\Delta^k_{m_k}, k=\overline{1,p}$ as in the previous application.
The functions $\psi_{k,m_k,i}$ are the fundamental polynomials of Stancu:

$$\psi_{k,m_k,i}(x):=w_{m_k,i,\alpha_k}(x)=\frac{\begin{pmatrix} m_k\\ i\end{pmatrix}x^{[i,-\alpha_k]}(1-x)^{[m_k-i,-\alpha_k]}}{1^{[m_k,-\alpha_k]}}\qquad \forall x\in[0,1]$$

for all $i=\overline{0,m_k}$, $k=\overline{1,p}$. α_k are real positive numbers.
Then $L_{m_1,m_2,...,m_p}$ from (7) are the Stancu modified polynomials of $(m_1,...,m_p)$ order, given by

$$(L_{m_1,m_2,...,m_p}f)(x_1,x_2,...,x_p):=(S^{\langle\alpha_1,\alpha_2,...,\alpha_p\rangle}_{m_1,m_2,...,m_p}f)(x_1,x_2,...,x_p)=$$

$$=\sum_{i_1=0}^{m_1}...\sum_{i_p=0}^{m_p}w_{m_1,i_1,\alpha_1}(x_1)...w_{m_p,i_p,\alpha_p}(x_p)\cdot f\left(\frac{i_1}{m_1},...,\frac{i_p}{m_p}\right)$$

The next theorem states the convergence of the iterates of the generalized Stancu operators:

Theorem 4.2. *The Stancu operators* $S^{\langle\alpha_1,\alpha_2,...,\alpha_p\rangle}_{m_1,m_2,...,m_p}$ *are WPOs and*

$$\left(S^{\langle\alpha_1,\alpha_2,...,\alpha_p\rangle}_{m_1,m_2,...,m_p}\right)^{\infty}(f)=\varphi^*_f,\quad \forall f\in C(D)$$

where φ^* *is as in Theorem 3.1.*

Remark 4.2. *In the particular case $p = 2$, $m_1 := m$, $m_2 := n$ we obtain:*

$$\lambda_{m,n} \geq \frac{1}{2^{m+n-2} \cdot 1^{[m,-\alpha_1]} \cdot 1^{[n,-\alpha_2]}}$$

which is the estimation given in [1; §4.2.].

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"Samuel Brassai" High School,
Cluj-Napoca, Romania
E-mail address: `Claudia.Bacotiu@clujnapoca.ro`