STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **L**, Number 3, September 2005

PROPERTIES OF SOME NEW SEMINORMED SEQUENCE SPACES DEFINED BY A MODULUS FUNCTION

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Abstract. In this paper we introduce the sequence spaces $\hat{c}_0(p,f,q,s)$, $\hat{c}(p,f,q,s)$ and $\hat{m}(p,f,q,s)$ using a modulus function f and defined over a seminormed space (X,q) seminormed by q. We study some properties of these sequence spaces and obtain some inclusion relations.

1. Introduction

Let m, c and c_0 be the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ with the usual norm $\|x\| = \sup_{k\geq 0} |x_k|$. Let D be the shift operator on s, that is, $Dx = (x_k)_{k=1}^{\infty}$, $D^2x = (x_k)_{k=2}^{\infty}$ and so on. It may be recalled that a Banach limit (see Banach [1]) L is a nonnegative linear functional on m such that L is invariant under shift operator (that is, $L(Dx) = L(x)$ for $x \in m$) and $L(e) = 1$, where $e = (1,1,...)$. A sequence $x \in m$ is almost convergent (see Lorentz [8]) if all Banach limits of x coincide. Let $\hat{c}$ denote the space of almost convergent sequences. It is proved by Lorentz [8] that

$$\hat{c} = \left\{x : \lim_{m\to\infty} t_{m,n}(x) \text{ exists uniformly in } n\right\}$$

where

$$t_{m,n}(x) = \frac{1}{m+1}\sum_{i=0}^{m} D^i x_n, \quad \left(D^0 = 1\right).$$

Several authors including Duran [5] , King [7] and Nanda ([12] ,[13]) have studied almost convergent sequences.

Received by the editors: 11.09.2005.
2000 *Mathematics Subject Classification.* 40A05.
Key words and phrases. Modulus function, seminorm, almost convergence, sequence spaces.

The notion of a modulus function was introduced by Nakano [11] in 1953. We recall that a modulus f is a function from $[0,\infty)$ to $[0,\infty)$ such that *(i)* $f(x)=0$ if and only if $x=0$, *(ii)* $f(x+y)\le f(x)+f(y)$, for all $x\ge 0$, $y\ge 0$, *(iii)* f is increasing, *(iv)* f is continuous from the right at 0.

Since $|f(x)-f(y)|\le f(|x-y|)$, it follows from condition (iv) that f is continuous on $[0,\infty)$. Furthermore, we have $f(nx)\le nf(x)$ for all $n\in\mathbb{N}$, from condition (ii), and so

$$f(x)=f\left(nx\frac{1}{n}\right)\le nf\left(\frac{x}{n}\right),$$

hence

$$\frac{1}{n}f(x)\le f\left(\frac{x}{n}\right) \text{ for all } n\in\mathbb{N}.$$

A modulus may be bounded or unbounded. For example, $f(x)=x^p$, $(0<p\le 1)$ is unbounded and $f(x)=\frac{x}{1+x}$ is bounded. Maddox [10] and Ruckle [14] used a modulus function to construct some sequence spaces.

After then some sequence spaces, defined by a modulus function, were introduced and studied by Bhardwaj [2], Bilgin [3], Connor [4], Esi [6], and many others.

Definition 1.1. *Let q_1, q_2 be seminorms on a vector space X. Then q_1 is said to be stronger than q_2 if whenever (x_n) is a sequence such that $q_1(x_n)\to 0$, then also $q_2(x_n)\to 0$. If each is stronger than the other q_1 and q_2 are said to be equivalent (one may refer to Wilansky* [15]*).*

Lemma 1.1. *Let q_1 and q_2 be seminorms on a linear space X. Then q_1 is stronger than q_2 if and only if there exists a constant M such that $q_2(x)\le Mq_1(x)$ for all $x\in X$ (see for instance Wilansky* [15]*).*

Let $p=(p_m)$ be a sequence of strictly positive real numbers and X be a seminormed space over the field $\mathbb{C}$ of complex numbers with the seminorm q. We define the sequence spaces as follows:

$$\begin{aligned}\hat{c}_0(p,f,q,s) &= \left\{x \in X : \lim_{m\to\infty} m^{-s}\left[\left(f\left(q\left(t_{m,n}(x)\right)\right)\right)\right]^{p_m} = 0 \text{ uniformly in } n\right\},\\ \hat{c}(p,f,q,s) &= \left\{x \in X : \lim_{m\to\infty} m^{-s}\left[\left(f\left(q\left(t_{m,n}(x-\ell e)\right)\right)\right)\right]^{p_m} = 0 \text{ for some } \ell,\right.\\ &\quad \text{uniformly in } n\},\\ \hat{m}(p,f,q,s) &= \left\{x \in X : \sup_{m,n} m^{-s}\left[\left(f\left(q\left(t_{m,n}(x)\right)\right)\right)\right]^{p_m} < \infty\right\}.\end{aligned}$$

where f is a modulus function.

The following inequalities will be used throughout the paper. Let $p = (p_m)$ be a bounded sequence of strictly positive real numbers with $0 < p_m \le \sup p_m = H$, $C = \max\left(1, 2^{H-1}\right)$, then

$$|a_m + b_m|^{p_m} \le C\left\{|a_m|^{p_m} + |b_m|^{p_m}\right\}, \tag{1.1}$$

where $a_m,\, b_m \in \mathbb{C}$.

2. Main results

Theorem 2.1. *Let $p = (p_m)$ be a bounded sequence, then $\hat{c}_0(p,f,q,s)$, $\hat{c}(p,f,q,s)$, $\hat{m}(p,f,q,s)$ are linear spaces.*

Proof. We give the proof for $\hat{c}_0(p,f,q,s)$ only. The others can be treated similarly. Let $x, y \in \hat{c}_0(p,f,q,s)$. For $\lambda, \mu \in \mathbb{C}$, there exist positive integers M_λ and N_λ such that $|\lambda| \le M_\lambda$ and $|\mu| \le N_\mu$. Since f is subadditive and q is a seminorm $m^{-s}\left[f\left(q\left(t_{m,n}(\lambda x + \mu y)\right)\right)\right]^{p_m} \le C\left(M_\lambda\right)^H m^{-s}\left[f\left(q\left(t_{m,n}(x)\right)\right)\right]^{p_m} + C\left(N_\mu\right)^H m^{-s}\left[f\left(q\left(t_{m,n}(y)\right)\right)\right]^{p_m} \to 0$,uniformly in n. This proves that $\hat{c}_0(p,f,q,s)$ is a linear space.

Theorem 2.2. *The space $\hat{c}_0(p,f,q,s)$ is a paranormed space, paranormed by*

$$g(x) = \sup_{m,n} m^{-s}\left(\left[f\left(q\left(t_{m,n}(x)\right)\right)\right]^{p_m}\right)^{\frac{1}{M}},$$

where $M = \max(1, \sup p_m)$. The spaces $\hat{c}(p,f,q,s)$, $\hat{m}(p,f,q,s)$ are paranormed by g, if $\inf p_m > 0$.

Proof. Omitted.

Theorem 2.3. *Let f be modulus function, then*

(i) $\hat{c}_0(p,f,q,s) \subseteq \hat{m}(p,f,q,s)$,

(ii) $\hat{c}(p,f,q,s) \subseteq \hat{m}(p,f,q,s)$.

Proof. We prove the second inclusion, since the first inclusion is obvious. Let $x \in \hat{c}(p,f,q,s)$, by definition of a modulus function (the inequality *(ii)*), we have

$$m^{-s}\left[f\left(q\left(t_{m,n}(x)\right)\right)\right]^{p_m} \leq Cm^{-s}\left[f\left(q\left(t_{m,n}(x-\ell)\right)\right)\right]^{p_m} + Cm^{-s}\left[f\left(q(\ell)\right)\right]^{p_m}.$$

Then there exists an integer K_ℓ such that $q(\ell) \leq K_\ell$. Hence, we have

$$m^{-s}\left[f\left(q\left(t_{m,n}(x)\right)\right)\right]^{p_m} \leq Cm^{-s}\left[f\left(q\left(t_{m,n}(x-\ell)\right)\right)\right]^{p_m} + Cm^{-s}\max(1,\left[(K_\ell)f(1)\right]^H), \tag{1}$$

so $x \in \hat{m}(p,f,q,s)$.

Theorem 2.4. *Let f, f_1, f_2 be modulus functions q, q_1, q_2 seminorms and $s, s_1, s_2 \geq 0$. Then*

(i) If $s > 1$ then $Z(f_1,q,s) \subseteq Z(f \circ f_1,q,s)$,

(ii) $Z(p,f_1,q,s) \cap Z(p,f_2,q,s) \subseteq Z(p,f_1+f_2,q,s)$,

(iii) $Z(p,f,q_1,s) \cap Z(p,f,q_2,s) \subseteq Z(p,f,q_1+q_2,s)$,

(iv) If q_1 is stronger than q_2 then $Z(p,f,q_1,s) \subseteq Z(p,f,q_2,s)$,

(v) If $s_1 \leq s_2$ then $Z(p,f,q,s_1) \subseteq Z(p,f,q,s_2)$,

(vi) If $q_1 \cong$(equivalent to) q_2, then $Z(p,f,q_1,s) = Z(p,f,q_2,s)$,

where $Z = \hat{m}$, $\hat{c}$ and $\hat{c}_0$.

Proof. (i) We prove this part for $Z = \hat{c}$ and the rest of the cases will follow similarly. Let $x \in \hat{c}(p,f,q,s)$, so that

$$S_m = m^{-s}\left[f_1\left(q\left(t_{m,n}(x-\ell)\right)\right)\right] \to 0.$$

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(c) < \varepsilon$ for $0 \leq t \leq \delta$. Now we write

$$I_1 = \{m \in \mathbb{N} : f_1 (q (t_{m,n} (x - \ell))) \leq \delta\}$$
$$I_2 = \{m \in \mathbb{N} : f_1 (q (t_{m,n} (x - \ell))) > \delta\}.$$

For $f_1 (q (t_{m,n} (x - \ell))) > \delta$,

$$f_1 (q (t_{m,n} (x - \ell))) < f_1 (q (t_{m,n} (x - \ell))) \delta^{-1} < 1 + \left[\left| f_1 (q (t_{m,n} (x - \ell))) \delta^{-1} \right|\right]$$

where $m \in I_2$ and $[|u|]$ denotes the integer part of u. By the definition of f we have for $f_1 (q (t_{m,n} (x - \ell))) > \delta$,

$$\begin{aligned} f (f_1 (q (t_{m,n} (x - \ell)))) &\leq \left(1 + \left[\left| f_1 (q (t_{m,n} (x - \ell))) \delta^{-1} \right|\right]\right) f (1) \\ &\leq 2f (1) f_1 (q (t_{m,n} (x - \ell))) \delta^{-1}. \end{aligned} \tag{2.1}$$

For $f_1 (q (t_{m,n} (x - \ell))) \leq \delta$,

$$f (f_1 (q (t_{m,n} (x - \ell)))) < \varepsilon \tag{2.2}$$

where $m \in I_1$. By (2.1) and (2.2) we have

$m^{-s} [f (f_1 (q (t_{m,n} (x - \ell))))] \leq m^{-s}\varepsilon + \left[2f (1) \delta^{-1}\right] S_m \to 0.$ as $m \to \infty$,uniformly n.

Hence $\hat{c} (p, f_1, q, s) \subseteq \hat{c} (p, f \circ f_1, q, s)$.

(ii) The proof follows from the following inequality

$$m^{-s} [(f_1 + f_2) (q (t_{m,n} (x)))]^{p_m} \leq Cm^{-s} [f_1 (q (t_{m,n} (x)))]^{p_m} + Cm^{-s} [f_2 (q (t_{m,n} (x)))]^{p_m}.$$

(iii), (iv) (v)and (vi) follow easily.

Corollary 2.1. *Let f be a modulus function, then we have*

(i) If $s > 1$, $Z (p, q, s) \subseteq Z (p, f, q, s)$,

(ii) $Z (p, f, q) \subseteq Z (p, f, q, s)$,

(iii) $Z (p, q) \subseteq Z (p, q, s)$,

(iv) $Z (f, q) \subseteq Z (f, q, s)$

where $Z = \hat{m}$, $\hat{c}$ *and* $\hat{c}_0$.

The proof is straightforward.

Theorem 2.5. *For any two sequences* $p = (p_k)$ *and* $r = (r_k)$ *of positive real numbers and for any two seminorms* q_1 *and* q_2 *on* X *we have* $Z(p, f, q_1, s) \cap Z(r, f, q_2, s) \neq \emptyset$.

Proof. The proof follows from the fact that the zero element $\bar{\theta}$ belongs to each of the classes of sequences involved in the intersection.

Theorem 2.6. *For any two sequences* $p = (p_m)$ *and* $r = (r_m)$, *we have* $\hat{c}_0(r, f, q, s) \subseteq \hat{c}_0(p, f, q, s)$ *if and only if* $\liminf \frac{p_m}{r_m} > 0$.

Proof. If we take $y_m = f(q(t_{m,n}(x)))$ for all $m \in \mathbb{N}$, then using the same technique of lemma 1 of Maddox [9], it is easy to prove the theorem.

Theorem 2.7. *For any two sequences* $p = (p_m)$ *and* $r = (r_m)$, *we have* $\hat{c}_0(r, f, q, s) = \hat{c}_0(p, f, q, s)$ *if and only if* $\liminf \frac{p_m}{r_m} > 0$ *and* $\liminf \frac{r_m}{p_m} > 0$.

Theorem 2.8. *Let* $0 < p_m \leq r_m \leq 1$. *Then* $\hat{m}(r, f, q, s)$ *is closed subspace of* $\hat{m}(p, f, q, s)$.

Proof. Let $x \in \hat{m}(r, f, q, s)$. Then there exists a constant $B > 1$ such that

$$k^{-s}[f(t_{m,n}(x))]^{r_m/M} \leq B \quad \text{for all } m, n$$

and so

$$k^{-s}[f(t_{m,n}(x))]^{p_m/M} \leq B \quad \text{for all } m, n.$$

Thus $x \in \hat{m}(p, f, q, s)$. To show that $\hat{m}(r, f, q, s)$ is closed, suppose that $x^i \in \hat{m}(r, f, q, s)$ and $x^i \to x \in \hat{m}(p, f, q, s)$. Then for every $0 < \varepsilon < 1$, there exists N such that for all m, n

$$k^{-s}\left[f\left(t_{m,n}\left(x^i - x\right)\right)\right]^{p_m/M} \leq B \quad \text{for all } i > N.$$

Now

$$k^{-s}\left[f\left(t_{m,n}\left(x^i - x\right)\right)\right]^{r_m/M} < k^{-s}\left[f\left(t_{m,n}\left(x^i - x\right)\right)\right]^{p_m/M} < \varepsilon \quad \text{for all } i > N.$$

Therefore $x \in \hat{m}(r, f, q, s)$. This completes the proof.

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